The basics of rigidity

Lectures I and II

Session on Granular Matter

Institut Henri Poincaré

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What determines rigidity?

What determines rigidity?

- The physics of the materials.

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- The physics of the materials.
- The external forces on the structure.

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- The combinatorics/topology of the structure.

What determines rigidity?

- The physics of the materials.
- The external forces on the structure.
- The combinatorics/topology of the structure.
- THE GEOMETRY OF THE STRUCTURE.

What model?

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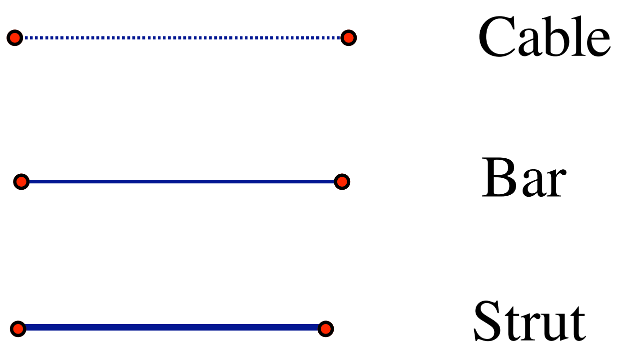
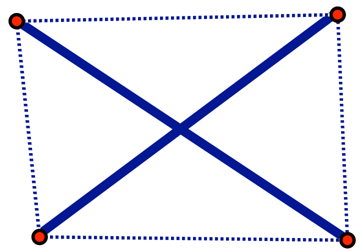
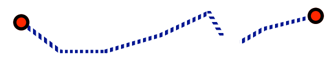
My favorite is a tensegrity.



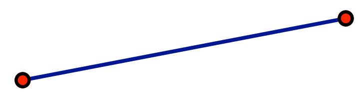


The constraints

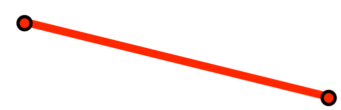
Cables can decrease in length or stay the same length, but NOT increase in length.



Bars must stay the same length.



Struts can increase in length or stay the same length, but NOT decrease in length.

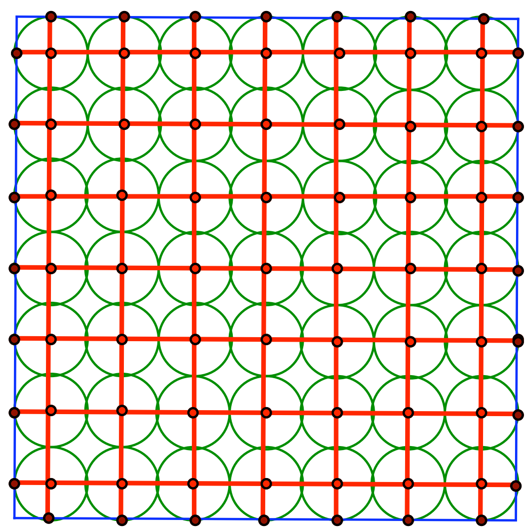


An example: Packings of circles

Place a vertex at the center of each circle.

Place a strut between the centers of every pair of touching circles, and from the center of a circle to the point on the boundary of the container that holds the circles.

The boundary vertices are pinned.



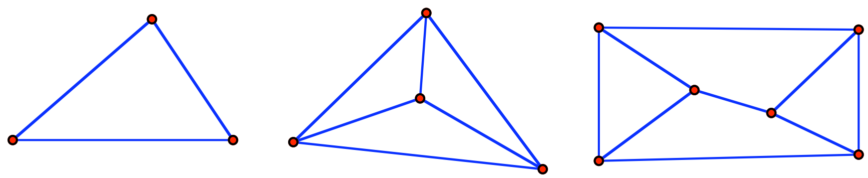
What sort of rigidity/stablility?

What sort of rigidity/stablility?

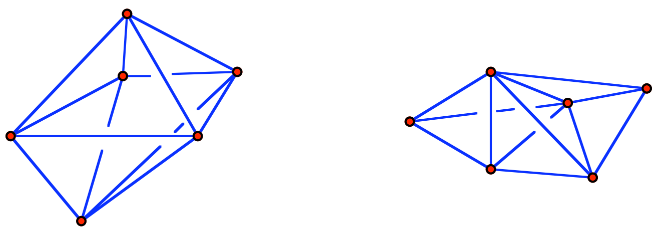
Two configurations **p** and **q** are *congruent* if every distance between vertices of **p** is the same for the corresponding distance for corresponding vertices of **q**.

A tensegrity structure with configuration **p** is *rigid* if every other configuration **q** sufficiently close to **p** satisfying the member (i.e. cable, bar, strut) constraints is congruent to **p**.

Examples of rigid structures

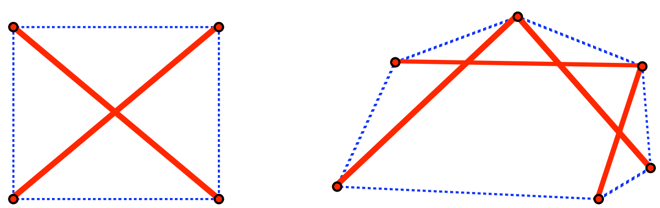


Bar frameworks
in the plane.



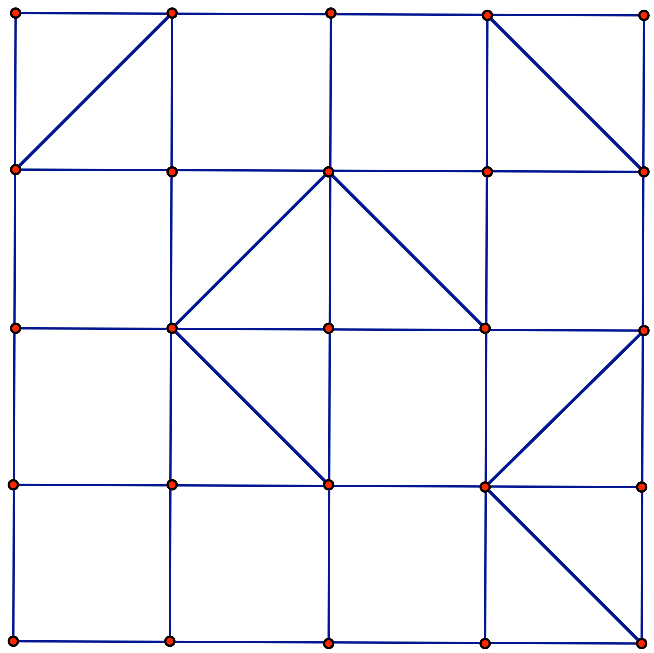
The edges of a convex triangulated
polyhedral surface.

Bar frameworks
in space.



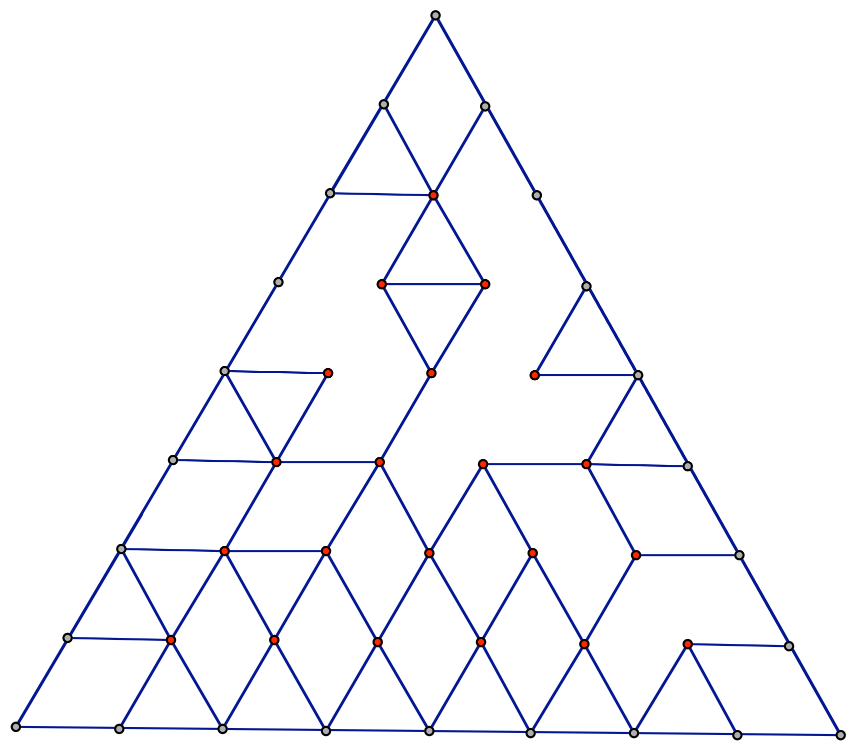
Tensegrities in
the plane, but
rigid in space.

Examples of rigid structures



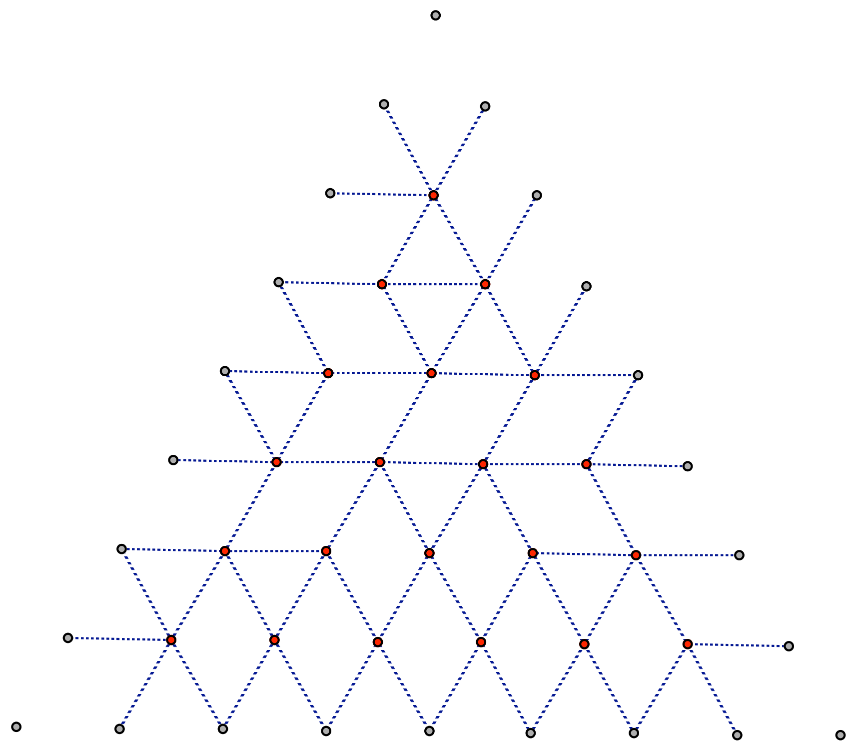
A square grid of bars with some diagonal bracing.
(Bolker, Crapo 1979)

Examples of rigid structures



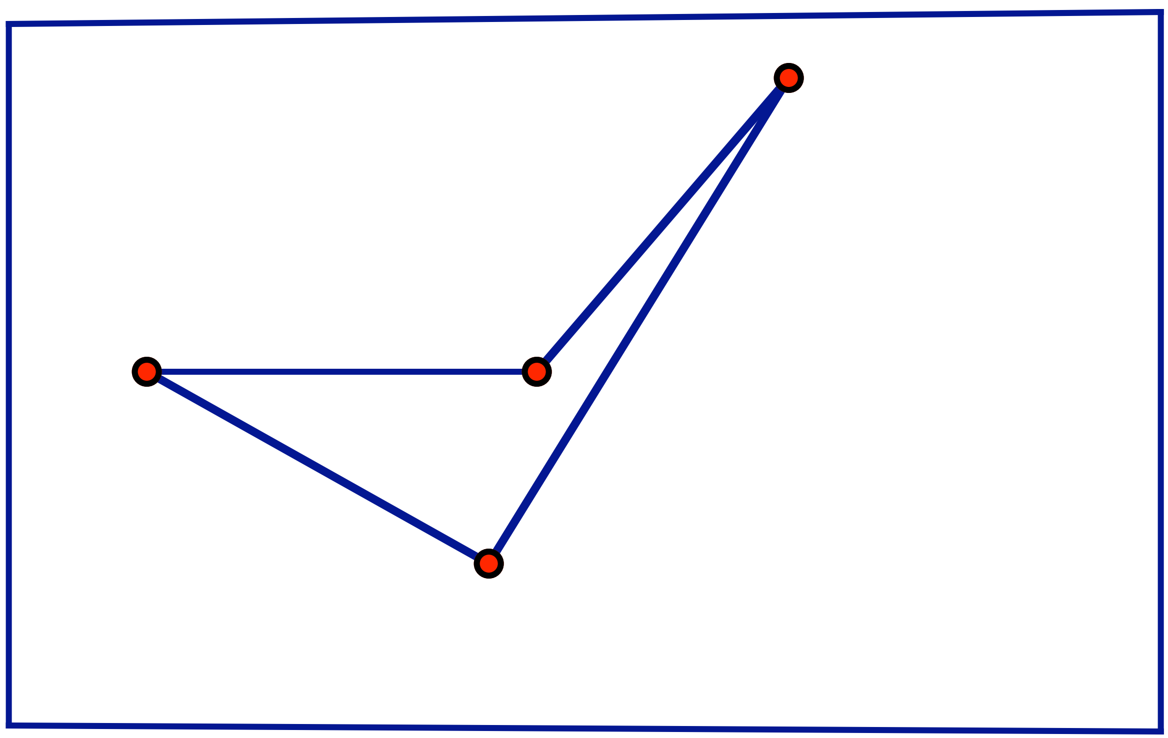
A bar framework in the plane with the boundary vertices pinned.
Internal bars are deleted with a certain probability p.

Examples of rigid structures

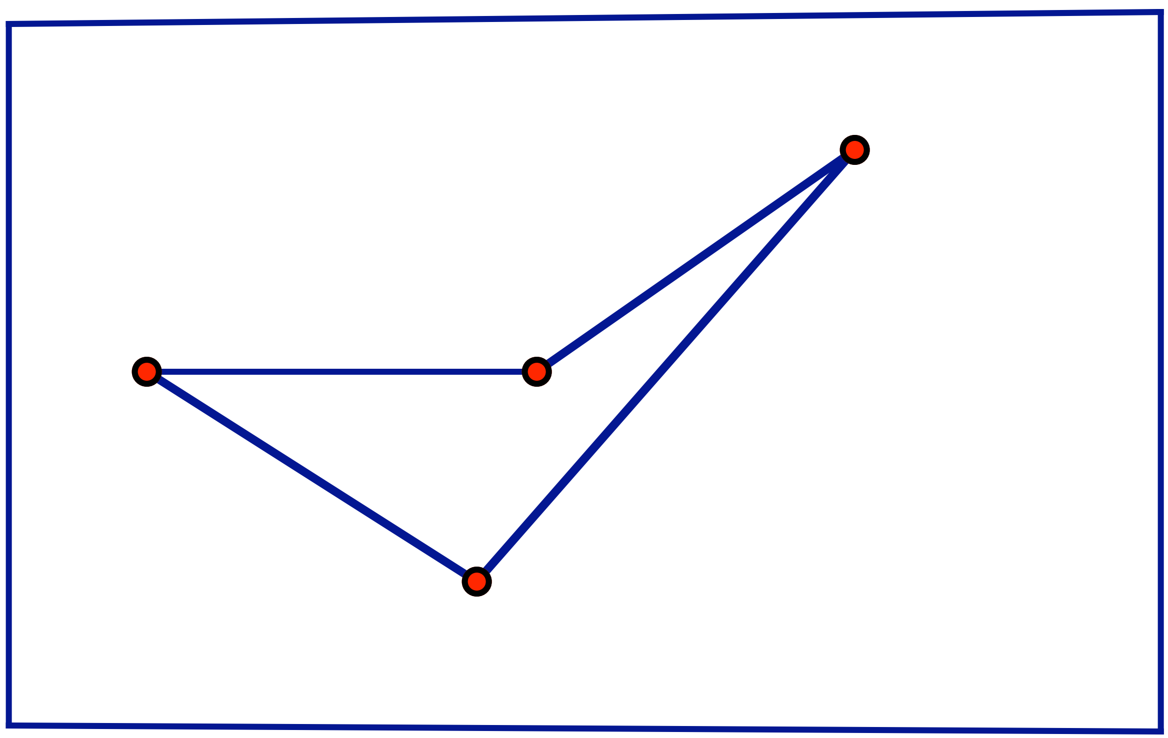


A cable framework in the plane with the boundary vertices pinned
Internal bars are deleted with a certain probability p.

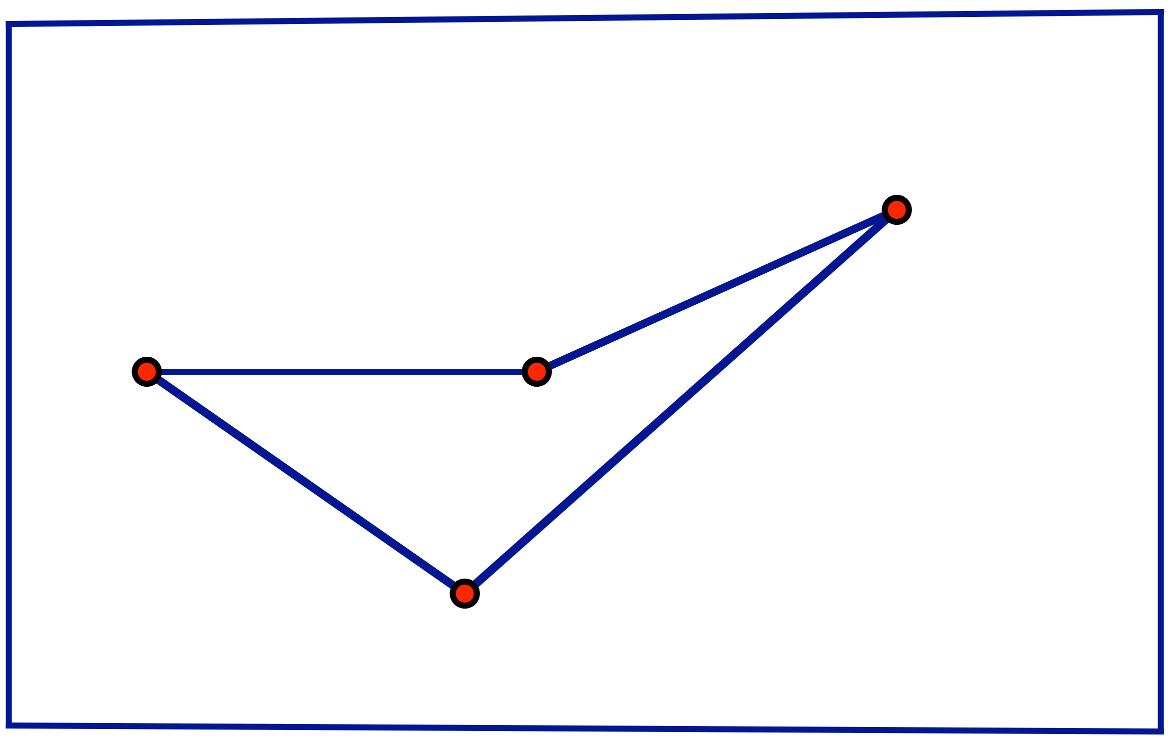
Examples of flexible structures



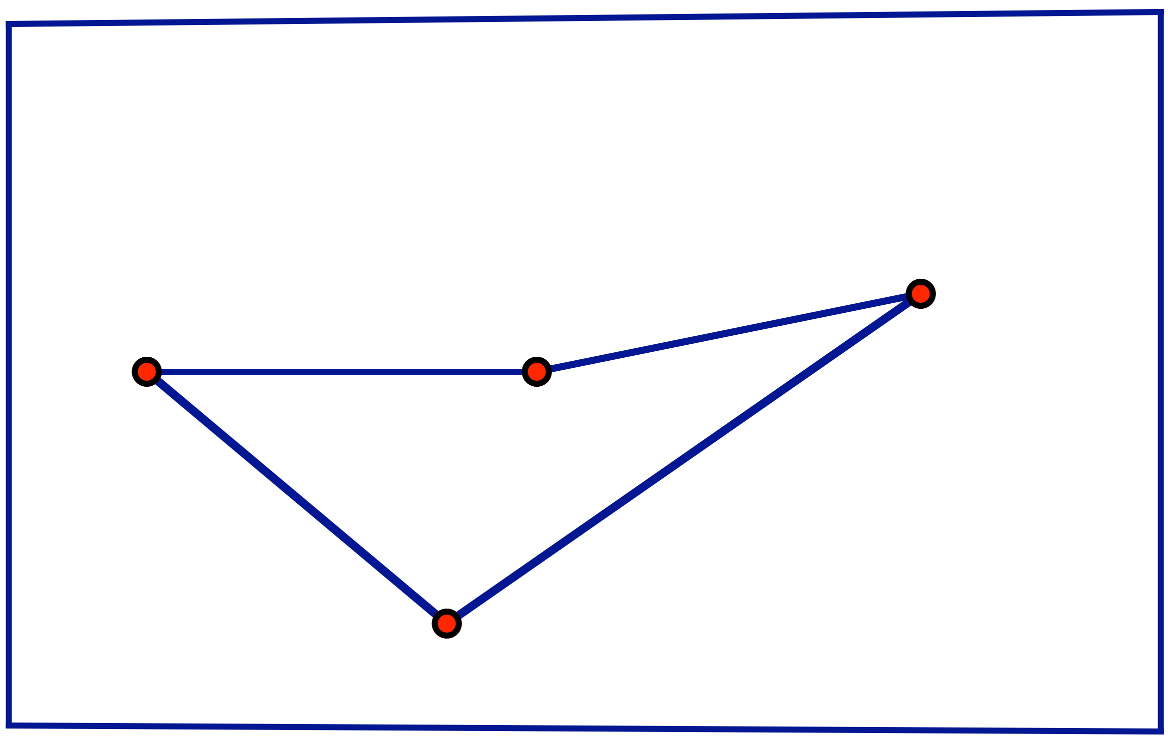
Examples of flexible structures



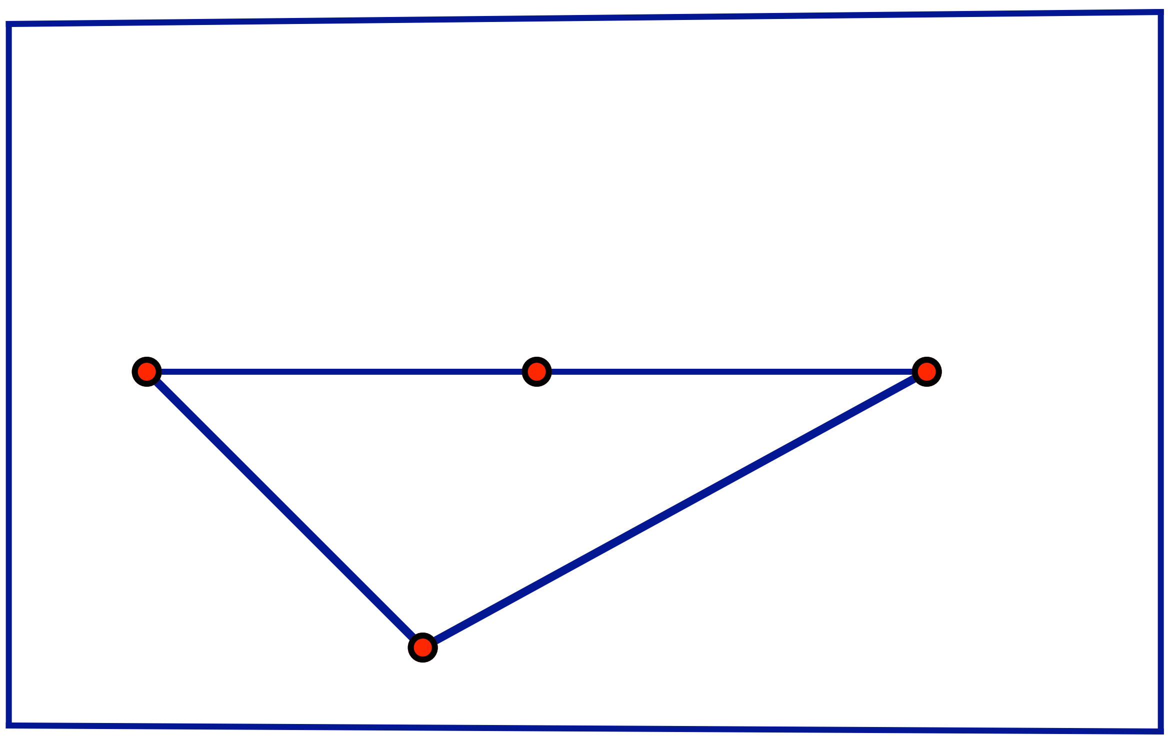
Examples of flexible structures



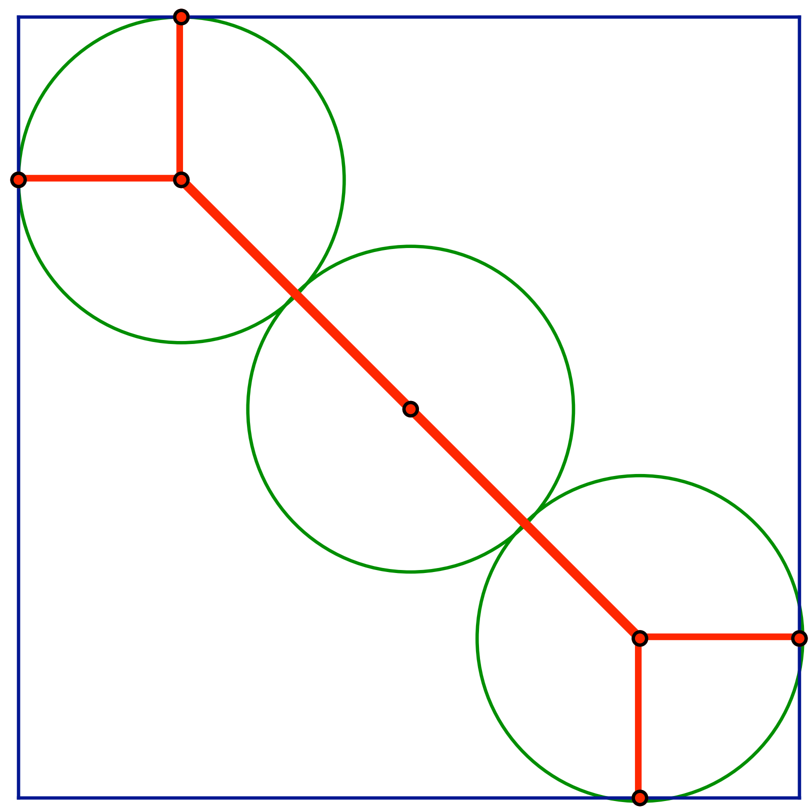
Examples of flexible structures



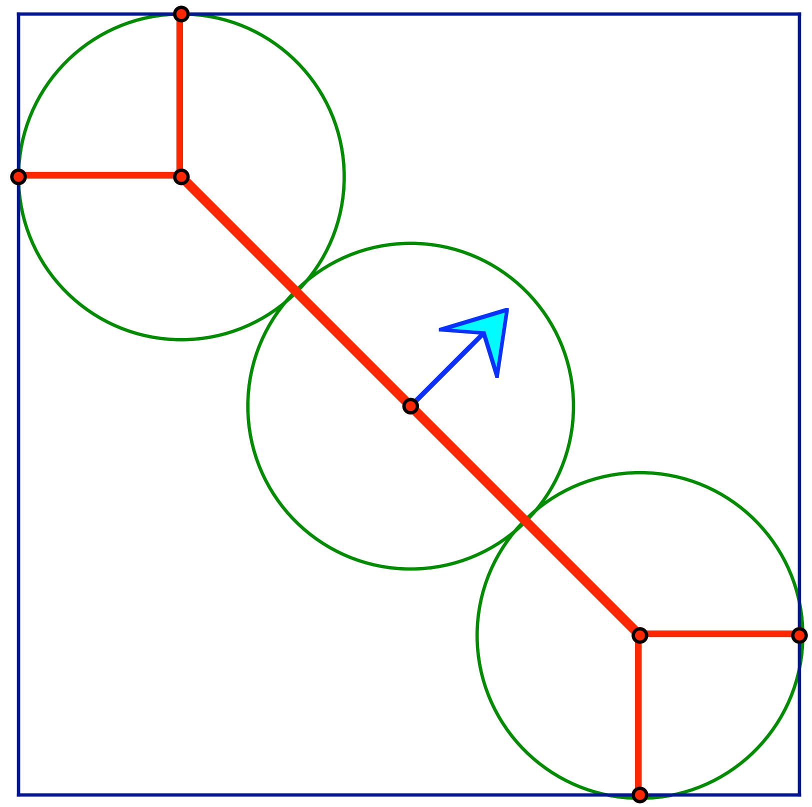
Examples of flexible structures



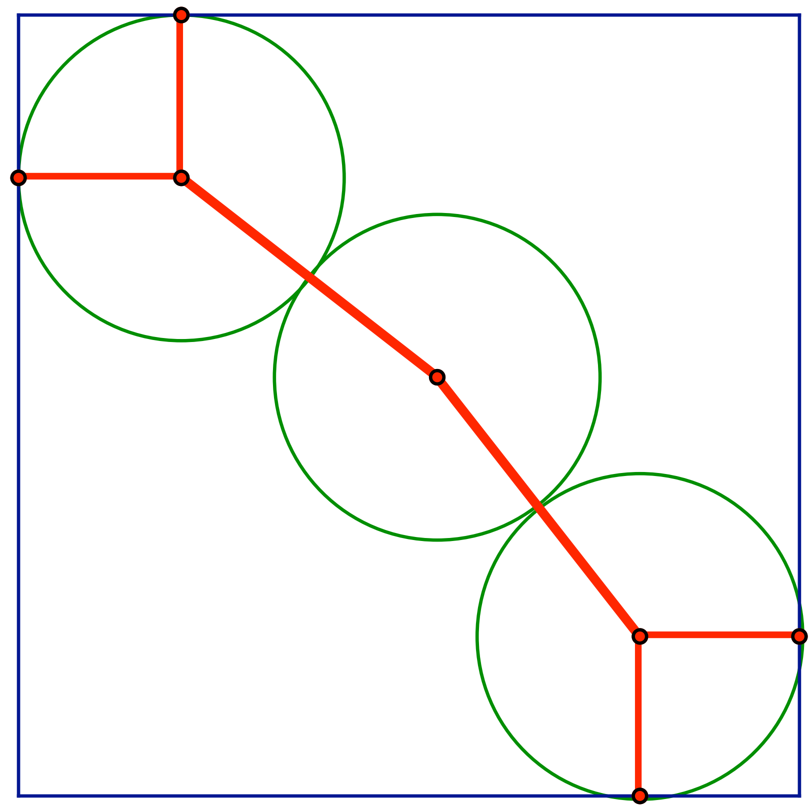
Examples of flexible structures



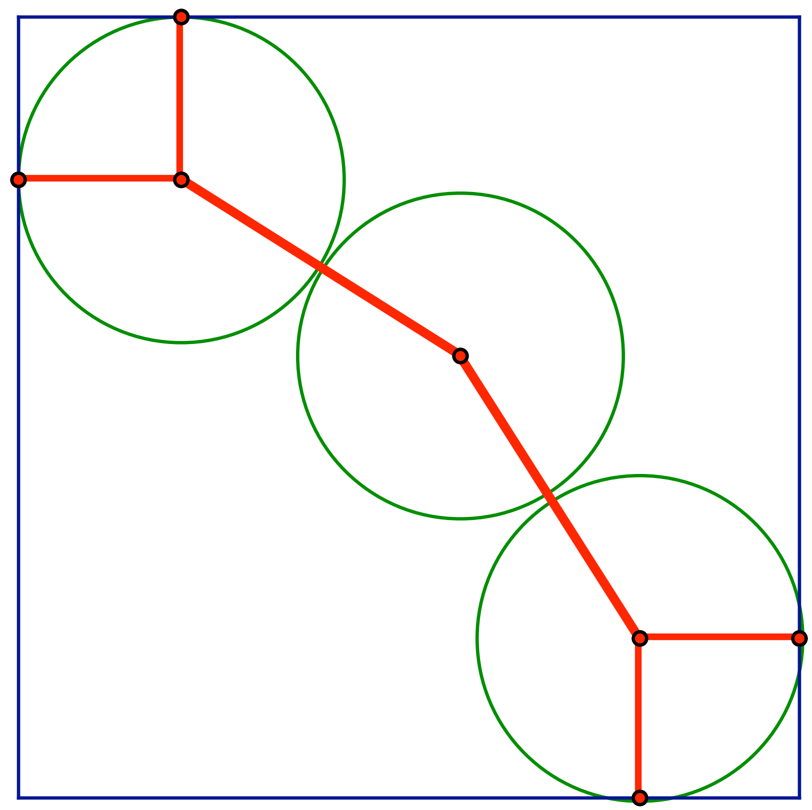
Examples of flexible structures



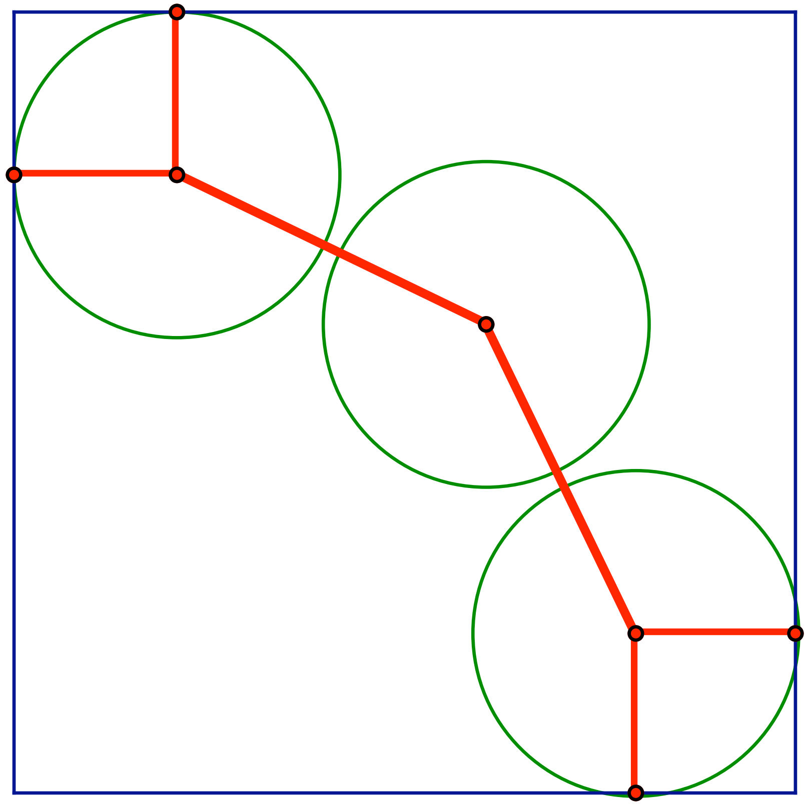
Examples of flexible structures



Examples of flexible structures



Examples of flexible structures



What sort of rigidity/stablility?

There are two equivalent concepts of rigidity that are a natural beginning first step.

- *Infinitesimal rigidity*, which thinks in terms of infinitesimal displacements, i.e. velocity vectors, and
- *Static rigidity*, which thinks in terms of forces and loads on the structure.

Infinitesimal Flexes (or Motions)

An *infinitesimal flex* $\mathbf{p}'$ of a (tensegrity) structure is a vector p_i' assigned to each vertex p_i of the tensegrity such that:

$(p_i - p_j)(p_i' - p_j') \leq 0$, when {i, j} is a cable.

$(p_i - p_j)(p_i' - p_j') = 0$, when {i, j} is a bar.

$(p_i - p_j)(p_i' - p_j') \geq 0$, when {i, j} is a strut.

Trivialities

An infinitesimal flex $\mathbf{p}' = (p_1', p_2', \ldots p_n')$ is *trivial* if it is the derivative at t=0 of smooth family of congruence of the ambient space.

In 3-space this means that there are vectors r and T such that, for all i = 1, 2, …, n

$$p_i' = r \times p_i + T.$$

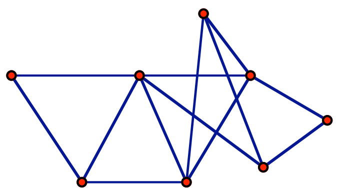
Taking the cross product with r is an infinitesimal rotation, and adding T is an infinitesimal translation. It is easy to check that such a $\mathbf{p}'$ is always an infinitesimal flex.

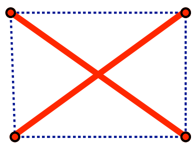
Infinitesimal rigidity

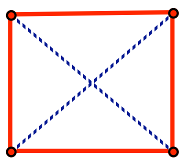
A tensegrity framework is *infinitesimally rigid* if every infinitesimal flex is trivial.

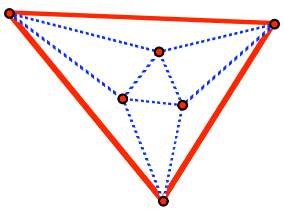
- This depends on the ambient dimension.
- There is always a minimum number of constraints that must be satisfied.
- An alternative is to pin some of the vertices, so that the only trivial infinitesimal flex is the 0 infinitesimal flex.

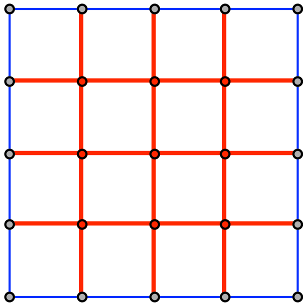
Examples of infinitesimally rigid structures in the plane.



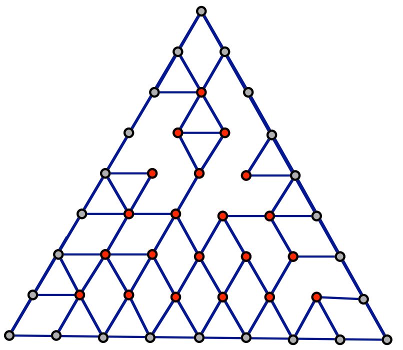




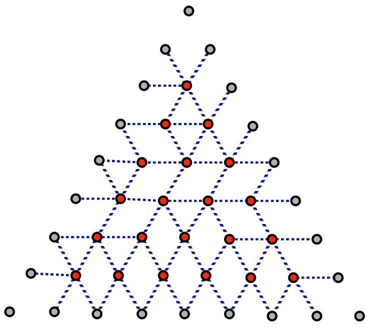




A strut framework in the plane with the boundary vertices pinned.



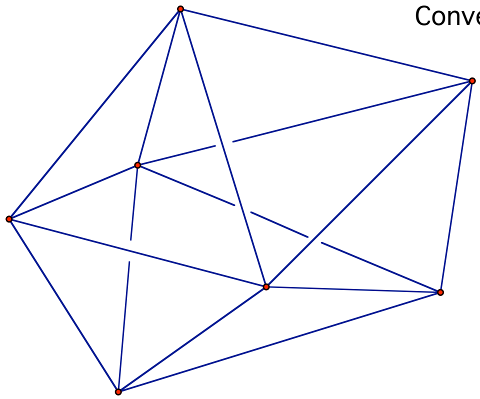
A bar framework in the plane with the boundary vertices pinned. Internal bars are deleted with a certain probability p.



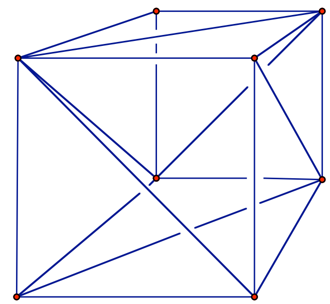
A cable framework in the plane with the boundary vertices pinned.
Internal bars are deleted with a certain probability p.

Examples of infinitesimally rigid structures in space

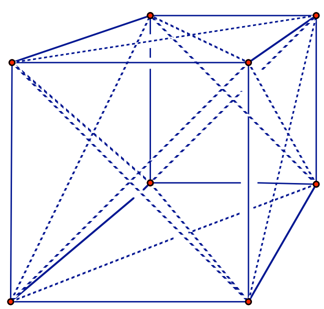
Convex polyhedral surfaces



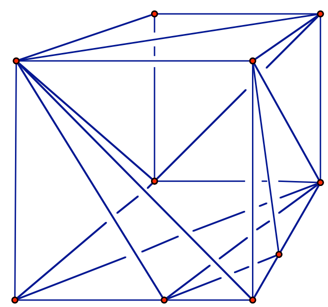
Each face is a triangle
(Max Dehn 1916)



Each face is triangulated
with no new vertices.
(A. D. Alexandrov 1958)



Each face has cables so that it is
infinitesimally rigid in its plane.
(Connelly, Whiteley, Roth 1980's)



Each face is triangulated
with no vertices inside a
face.
(A. D. Alexandrov 1958)

Infinitesimally flexible structures in the plane

Mathematical language.

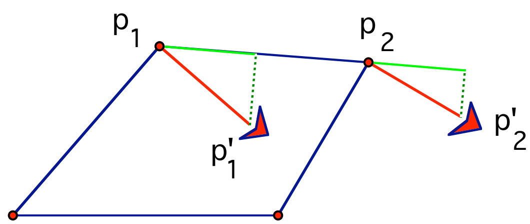
A flexible bar framework with an infinitesimal flex.

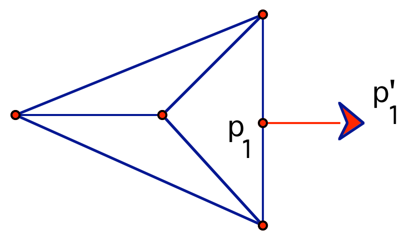
A rigid bar framework with an infinitesimal flex.

Engineering language.

An infinitesimal mechanism that is a "finite" mechanism.

An infinitesimal mechanism that is NOT a finite mechanism.





The vectors of the infinitesimal flex are in red and attached to the corresponding vertex.

If the vector is not shown, it is assumed that it is the 0 vector, and effectively that vertex is pinned.

If one end of a bar is pinned, then the vector of the infinitesimal flex at the other end must be perpendicular to the bar.

For a bar, in general the projection of the vector at the ends of the bar onto the line of the bar (shown in green above) must be the same length and direction.

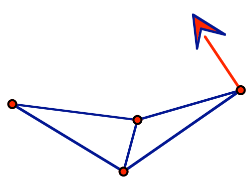
Infinitesimally flexible structures in space

Mathematical language.

Engineering language.

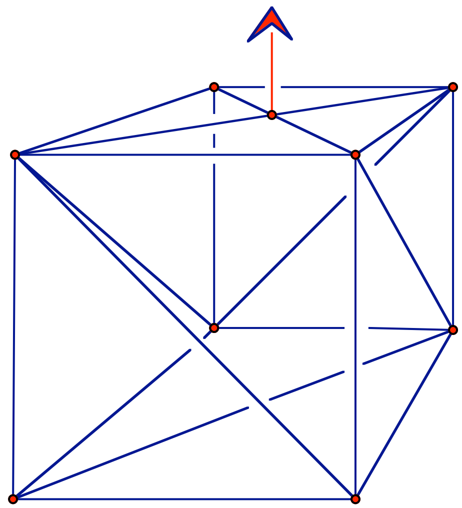
A flexible bar framework with an infinitesimal flex.

An infinitesimal mechanism that is a "finite" mechanism.



A rigid bar framework with an infinitesimal flex.

An infinitesimal mechanism that is NOT a finite mechanism.



Any triangulated polyhedral surface that has a vertex in the relative interior of a face will have an infinitesimal flex as indicated above.

Calculating infinitesimal rigidity for bar frameworks

When $\{i, j\}$ is a bar, we have

$$(p_i - p_j)(p_i' - p_j') = 0.$$

Think of $\mathbf{p}'$ as the unknown and solve:

$$R(\mathbf{p})\mathbf{p}' = 0,$$

where

$$R(\mathbf{p}) = \begin{bmatrix} & \overset{i}{} & & \overset{j}{} & \\ & & & & \\ (p_i - p_j)^t & \cdots & 0 & \cdots & (p_j - p_i)^t \\ & & & & \end{bmatrix}_{e \times nd} \{i,j\} \qquad \mathbf{p}' = \begin{bmatrix} p_i' \\ p_j' \end{bmatrix}_{nd \times 1}$$

$(\ )^t$ is the transpose taking a column vector to a row vector.

Counting

Suppose that the bar graph G has e bars and n vertices in dimension d, and that the configuration $\mathbf{p} = (p_1, p_2, \ldots p_n)$ does not lie in a (d-1)-dimensional hyperplane. Then the space of trivial infinitesimal flexes is d(d+1)/2 dimensional.

So if G($\mathbf{p}$) is infinitesimally rigid in $\mathbf{E}^d$, the rank of the rigidity matrix R($\mathbf{p}$) must be nd-d(d+1)/2, and the number of rows

$$e \geq nd - d(d+1)/2.$$

For the plane d=2, $e \geq 2n-3$.

For space d=3, $e \geq 3n-6$.

Counting for tensegrities

If G(**p**) is a tensegrity framework with n vertices and e members that is infinitesimally rigid in $\mathbf{E}^d$, then some constraints are given by inequalities instead of equality constraints. So we need at least one more member. That is

$$e \geq nd - d(d+1)/2 + 1.$$

For the plane d=2, $e \geq 2n-2$.

For space d=3, $e \geq 3n-5$.

Counting for pinned frameworks

When the framework has some pinned vertices, the trivial infinitesimal flexes are just $\mathbf{p}' = 0$. So for bar frameworks n non-pinned vertices and e members,

$$e \geq nd.$$

For tensegrity frameworks,

$$e \geq nd + 1.$$

The rigidity map

For a graph G, the *rigidity map* f: E^{nd} -> E^e is the function that assigns to each configuration **p** of n vertices in d-space, the squared lengths of edges of G, f(**p**)=(. . ., $|p_i - p_j|^2$, . . .), where e is the number of edges of G.

The *rigidity matrix* R(**p**) = df is the differential of f.

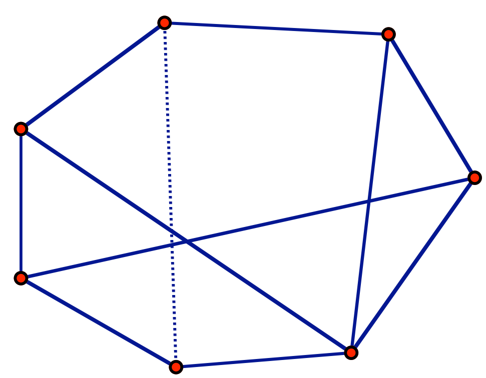
Basic general theorem: If a (bar) framework is infinitesimally rigid in E^d, then it is rigid in E^d.

Proof: Apply the inverse function theorem to f. //

We have seen examples where the converse of this theorem is false.

An application to mechanisms

Suppose that an infinitesimally rigid bar framework in the plane has e bars, n vertices, and e = 2n -3. If you remove one bar, then it becomes a mechanism, by applying the inverse function theorem.



n = 7, 2n - 3 = 11 = e, and replacing a bar by a cable creates a flexible framework.

Forces

A *force* $\mathbf{F}=(F_1, F_2, \ldots, F_n)$ is a row vector F_i assigned to each vertex i of a configuration $\mathbf{p}=(p_1, p_2, \ldots p_n)$.

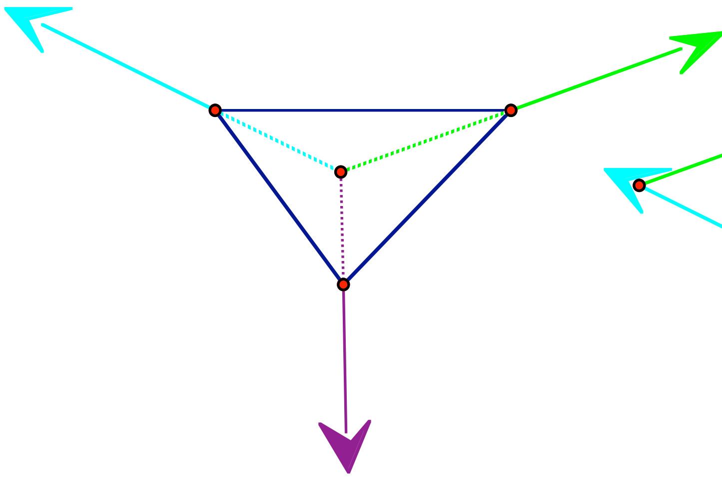
F is called an *equilibrium* force if as a vector in E^{nd}, it is orthogonal to the linear subspace of trivial infinitesimal flexes.

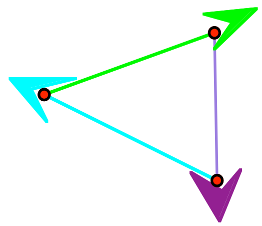
In physics this means that **F** has no linear or angular momentum. In E^3 it satisfies the following equations:

$$\sum_i F_i = 0,$$

$$\sum_i F_i \times p_i = 0.$$

Example of equilibrium forces





For 3 forces at applied at 3 points, the angular momentum condition implies that the line extending the 3 vectors must go through a point.

The linear momentum condition implies is just that the vector sum is 0.

Note that in dimension 3 the equilibrium condition is 6 linear equations. In dimension 2 it is 3 linear equations.

Stresses

A *stress* defined for a tensegrity framework is a scalar $\omega_{ij}=\omega_{ji}$ assigned to each member {i,j} (=cable, bar, strut). We write $\omega=(\ldots,\omega_{ij},\ldots)$ as a single row vector. We say ω is *proper* when

$$\omega_{ij} \geq 0 \text{ for } \{i,j\} \text{ a cable,}$$

$$\omega_{ij} \leq 0 \text{ for } \{i,j\} \text{ a strut.}$$

The stress for a bar can be either sign. (These should be properly called *stress coefficients*. A stress is normally a force, but for brevity we stay with calling these simply stresses.)

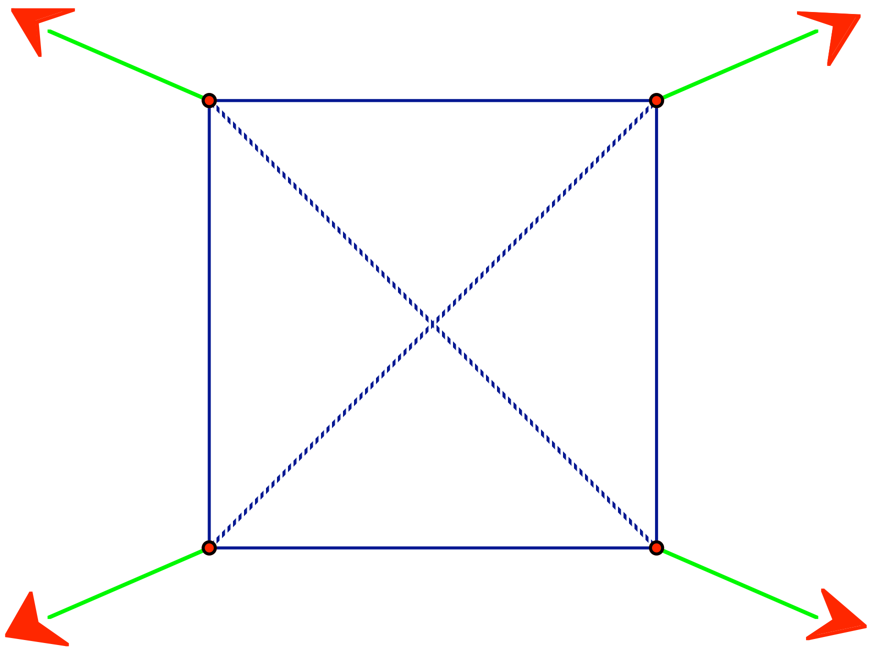
Resolution of forces

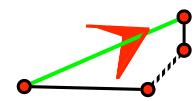
Suppose a force **F**=(F_1, F_2, …, F_n) is assigned to a configuration **p**=(p_1, p_2, … p_n) in $\mathbf{E}^d$. (**F** is often called a *load* as well.) For a given tensegrity graph G, we say that a (proper) stress ω=(…,ω_{ij},…) *resolves* **F**, if the following *equilibrium equation* holds at every vertex i.

$$F_i + \sum_j \omega_{ij} (p_j - p_i) = 0.$$

Note that if F is resolved by the stress ω, then **F** is necessarily an equilibrium force.

An example of a resolution





A force diagram demonstrating the equilibrium condition at one vertex.

Each segment, except the force F_i, represents $\square_{ij}(p_j - p_i)$.

Static rigidity

A tensegrity framework G(**p**) is called *statically rigid* if every equilibrium force **F** can be resolved by a proper stress □.

In terms of the rigidity matrix this says that for every equilibrium force **F** there is a proper stress □, such that

$$\mathbf{F} + \square R(\mathbf{p}) = 0.$$

Theorem: A tensegrity framework G(**p**) is statically rigid if and only if it is infinitesimally rigid.

Comments

- When the configuration $\mathbf{p}=(p_1, p_2, \ldots p_n)$ in $\mathbf{E}^d$ does not lie in a hyperplane, a bar framework is statically and infinitesimally rigid if and only if the rank of the rigidity matrix $R(\mathbf{p})$ is $nd-d(d+1)/2$.
- If a statically rigid tensegrity framework has at least one cable or strut, it requires at least $nd-d(d+1)/2+1$ members altogether. Thus there must be at least $2n-2$ members in the plane and $3n-5$ members in 3-space.

More Comments

- If a stress □ resolves the 0 force, i.e. □R(**p**) = 0, it is called a *self stress* or an *equilibrium* stress.
- When there is exactly one solution to the equilibrium equations **F** + □R(**p**) = 0, the framework is called statically determinant, otherwise it is called statically indeterminant.

Convex surfaces with all faces triangles

Consider a bar framework G(**p**) composed of all the vertices and edges of a convex polytope P with all faces triangles. Let n be the number of vertices, e the number of edges (i.e. bars), and f the number of faces of P. Then

$$n - e + f = 2 \quad \text{(Euler's formula)}$$

$$2e = 3f \quad \text{(All faces triangles).}$$

This implies that $e = 3n - 6$.

Convex surfaces with all faces triangles

Recall that a bar framework G(**p**) is infinitesimally rigid in $\mathbf{E}^3$ if and only if the rank of the rigidity matrix R(**p**) is 3n-6, the number of rows of R(**p**) in this case. This means that this G(**p**) is infinitesimally rigid in $\mathbf{E}^3$ if and only if the only self stress for G(**p**) is 0. This is the case:

Theorem (M. Dehn 1916): The bar framework G(**p**) composed of all the vertices and edges of a convex polytope P with all faces triangles is statically rigid in $\mathbf{E}^3$.

Static rigidity for Tensegrities

When G(**p**) does not consist just of bars, the determination of static and infinitesimal rigidity is a linear programming feasibility problem:

Solve:

$(p_i - p_j)(p_i' - p_j') \leq 0$, when $\{i, j\}$ is a cable.

$(p_i - p_j)(p_i' - p_j') = 0$, when $\{i, j\}$ is a bar.

$(p_i - p_j)(p_i' - p_j') \geq 0$, when $\{i, j\}$ is a strut.

For $\mathbf{p}' = (p_1', p_2', \ldots p_n')$ non-trivial.

Static rigidity for Tensegrities

There is a useful insight to understand tensegrity frameworks in terms bar frameworks:

Theorem (B. Roth-W. Whiteley 1981): A tensegrity framework G(**p**) is infinitesimally rigid in $\mathbf{E}^d$ if and only if G_0(p) is infinitesimally rigid, where G_0 replaces every member with a bar, and G(**p**) has a proper self stress □, where $□_{ij}$ is not 0 for all cables and struts {i,j}.

Proof of the Roth-Whiteley Thm.

Suppose that a tensegrity framework G(**p**) is statically rigid (in $\mathbf{E}^d$), and {i,j} is a cable. Let

$$\mathbf{F}(i,j)=(0\ldots,\ p_j-p_i,\ 0\ldots,0,\ p_i-p_j,\ 0\ldots)$$

be the equilibrium force obtained by applying p_j-p_i at p_i, and p_i-p_j at p_j. Then there is a proper stress □(i,j) resolving **F**(i,j). But adding 1 to the stress $□_{ij}$ in **F**(i,j) creates a self stress for G(**p**) that is non-zero for the member {i,j}. Doing this for all the cables, similarly for the struts, and adding these self stresses all together creates a self stress for G(**p**) that is non-zero for all the cables and struts.

Proof of the Roth-Whiteley Thm.

Suppose that $\omega=(\ldots,\omega_{ij},\ldots)$ is a proper self stress that is non-zero for all cables and struts, and that the underlying bar framework $G_0(\mathbf{p})$ is infinitesimally rigid in $\mathbf{E}^d$. Let $\mathbf{p}' = (p_1', p_2', \ldots p_n')$ be an infinitesimal flex of $G(\mathbf{p})$. Then $\omega R(\mathbf{p}) = 0$, by the equilibrium condition. Furthermore,

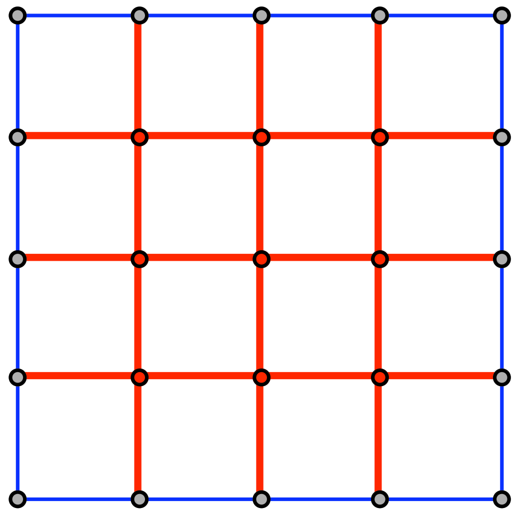
$$\omega R(\mathbf{p})\mathbf{p}' = \sum_{i<j}\omega_{ij}(p_i - p_j)(p_i' - p_j') < 0$$

unless $(p_i - p_j)(p_i' - p_j')=0$ for each $\{i,j\}$ a cable or strut. So $\mathbf{p}'$ is an infinitesimal flex of $G_0(\mathbf{p})$, the underlying bar framework, and so must be trivial.

More Comments

- If a framework is such that it is statically rigid and statically determinant then it is called *isostatic*.
- Any convex triangulated polyhedral surface in 3-space is isostatic as a bar framework.
- If any tensegrity framework has a strut or a cable, then it must NOT be isostatic by the Roth-Whiteley theorem.
- For example, if G has a cable or strut, and **F** is any equilibrium force, F can be resolved with a proper stress that is 0 on some cable or strut.

An application



The grey vertices are pinned.

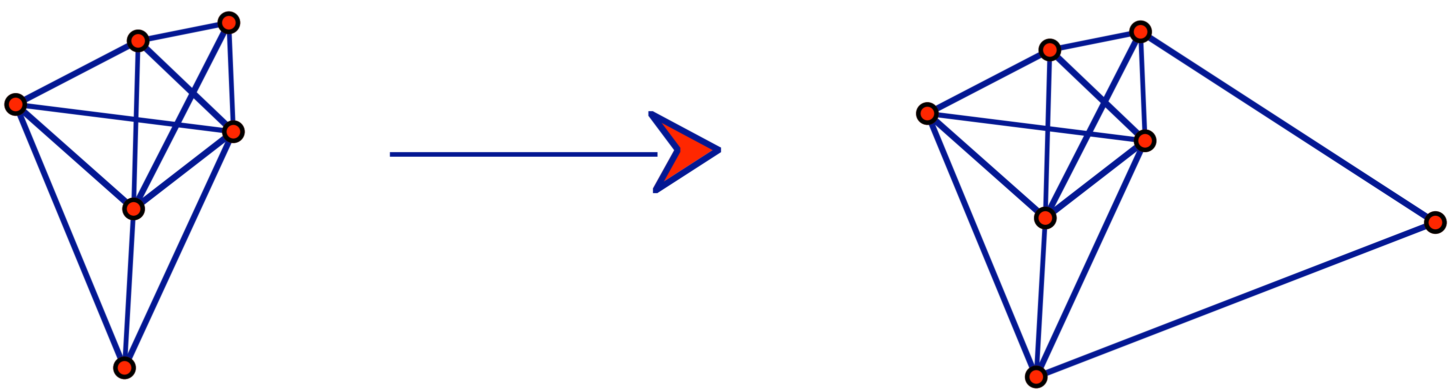
As a strut tensegrity framework, this is statically rigid.

Replacing all the struts by bars results in a statically rigid bar framework.

Assigning a stress of -1 on all the members is an equilibrium self stress.
There is no need for equilibrium at the pinned vertices.

A Handy Tool

Suppose you have an infinitesimally rigid bar framework in the plane with two distinct vertices p_1 and p_2. Attach another vertex p_3 with two bars to p_1 and p_2 so that p_3 is not on the line connecting p_1 and p_2. Then the new bar framework is infinitesimally rigid.

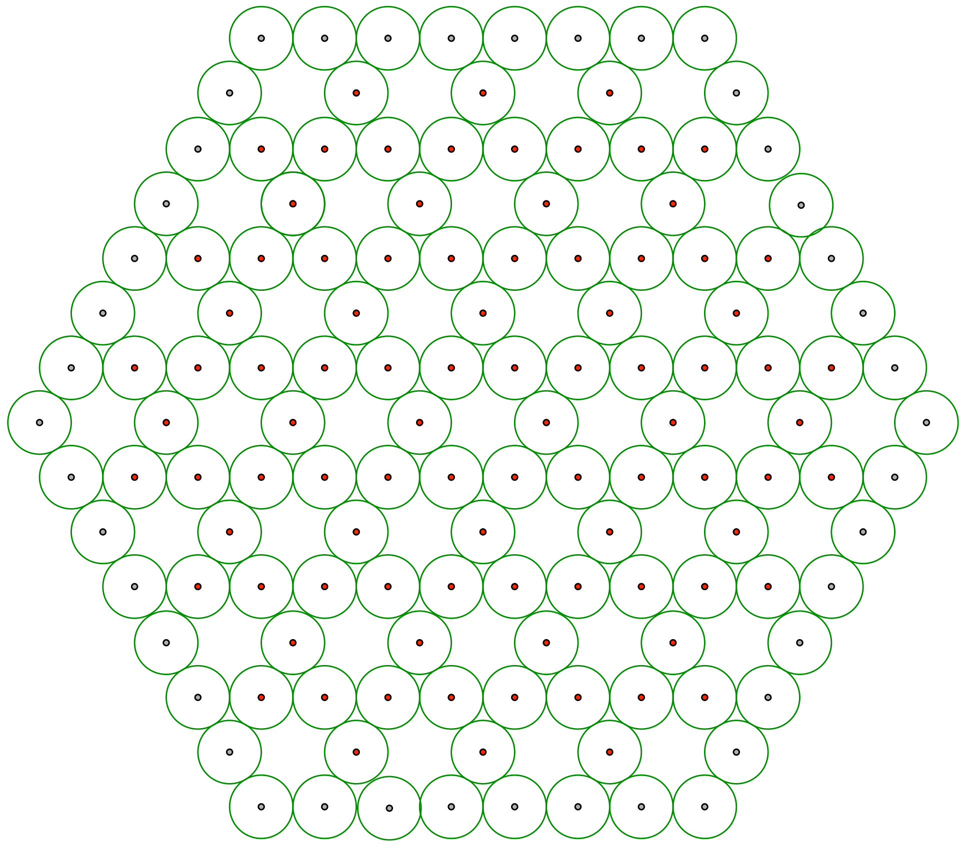


Statically rigid

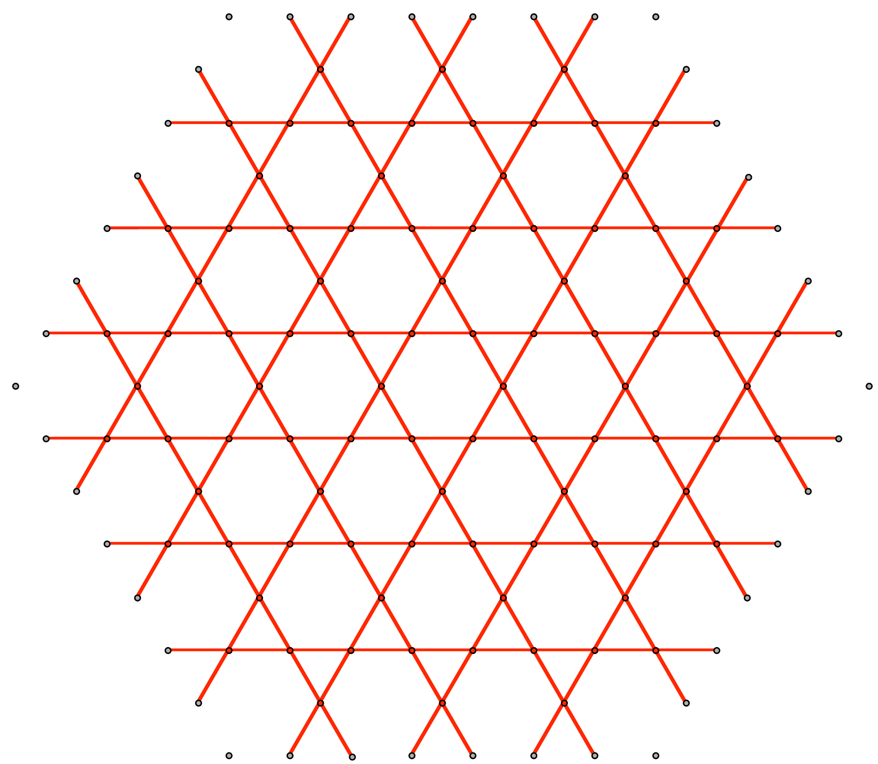
Also statically rigid

Another application

The Kagome lattice

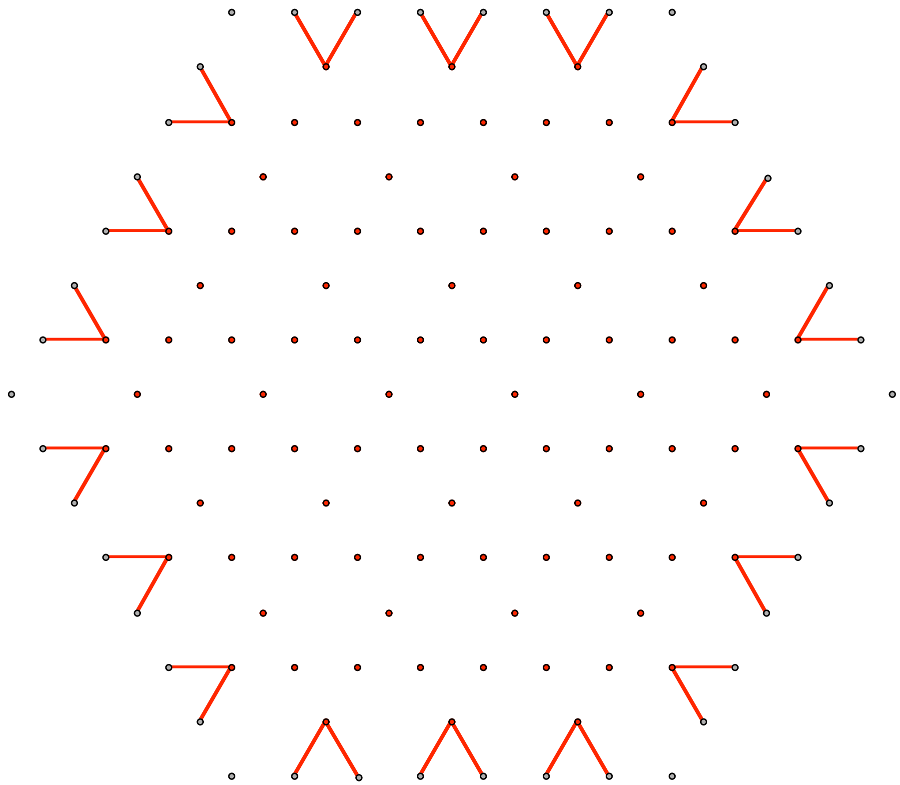


Another application



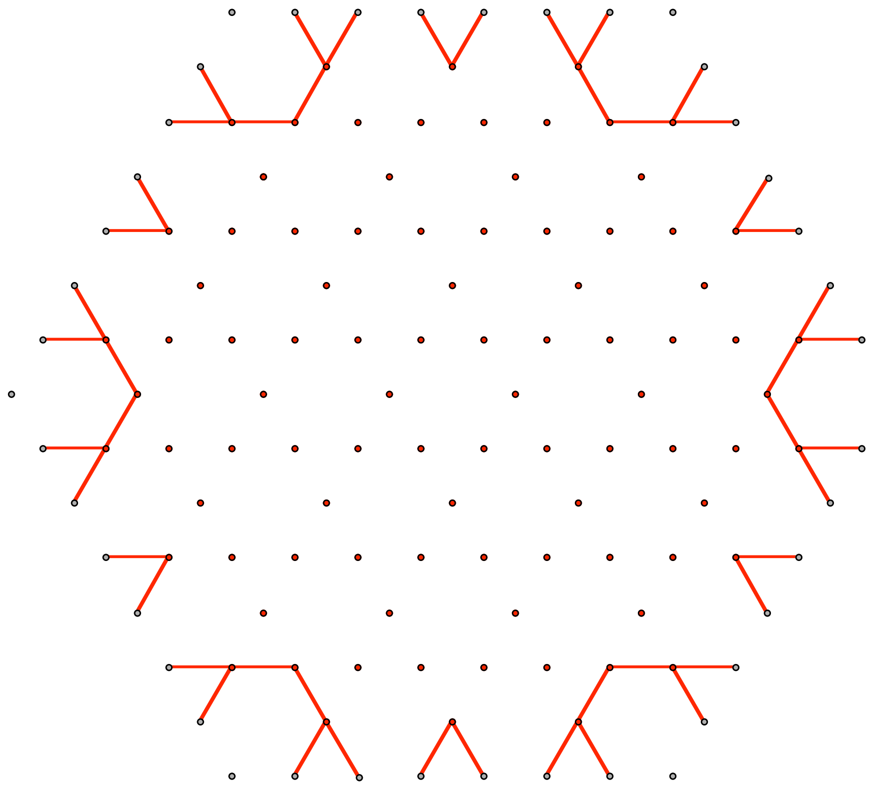
The associated strut framework is infinitesimally rigid because . . .

Another application



The bar framework can be constructed from the outside in ...

Another application



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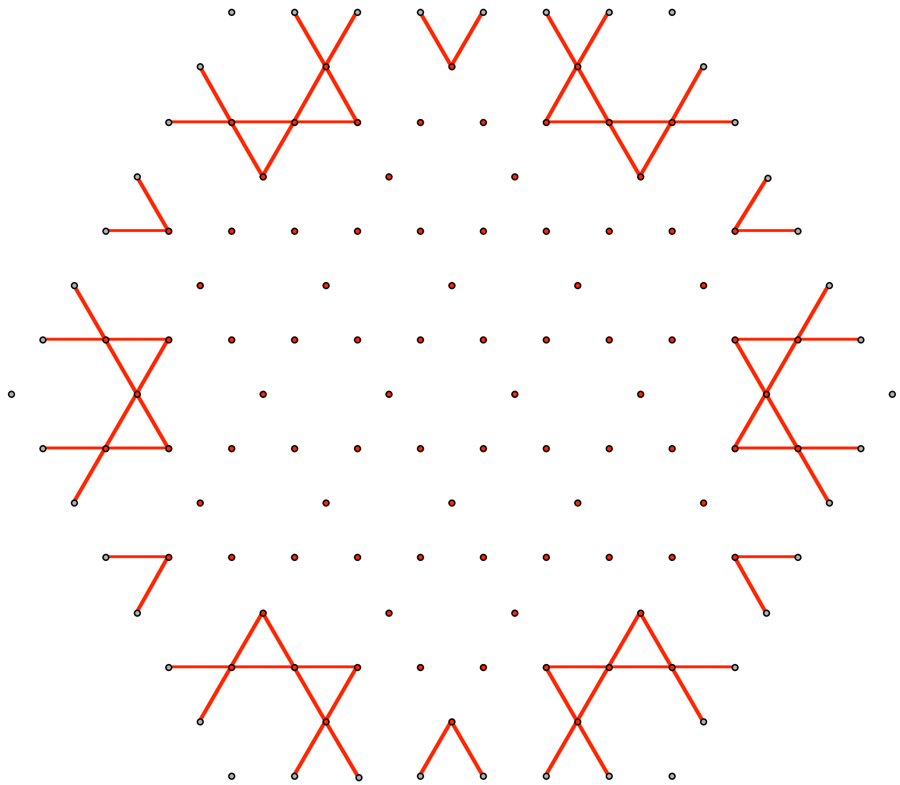
Another application



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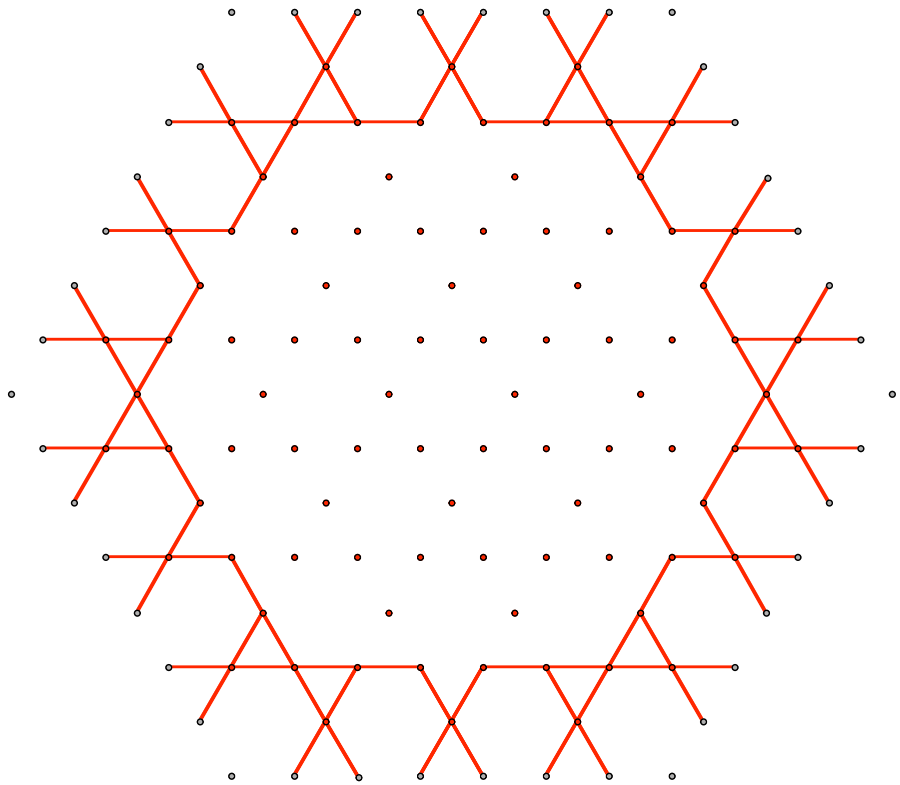
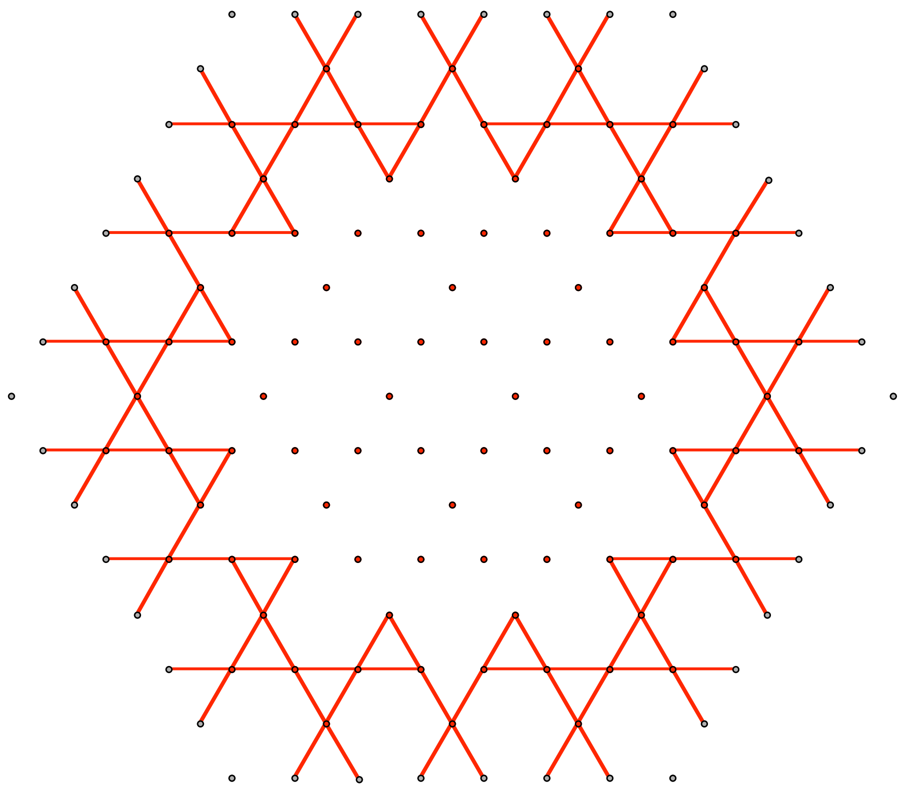
Another application



The bar framework can be constructed from the outside in ...

Another application



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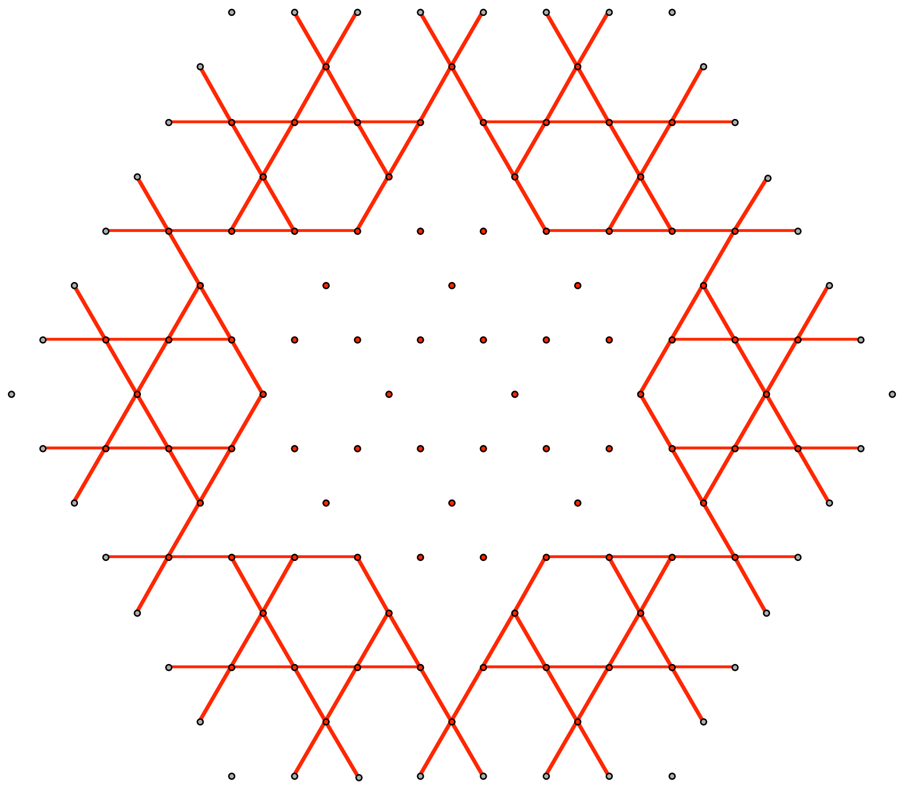
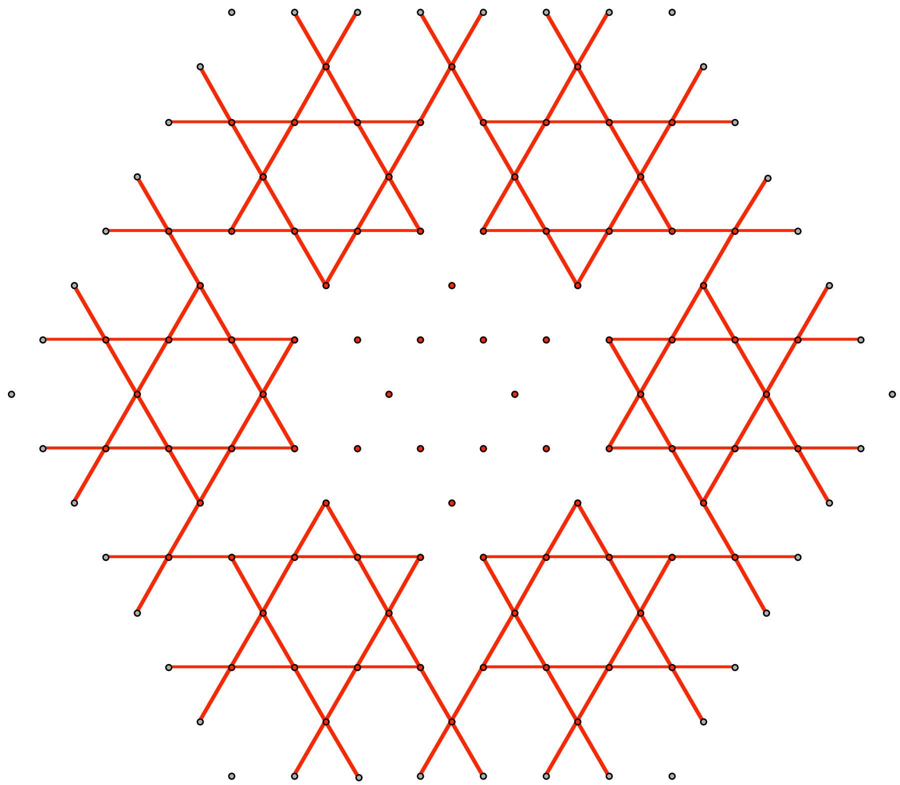
Another application



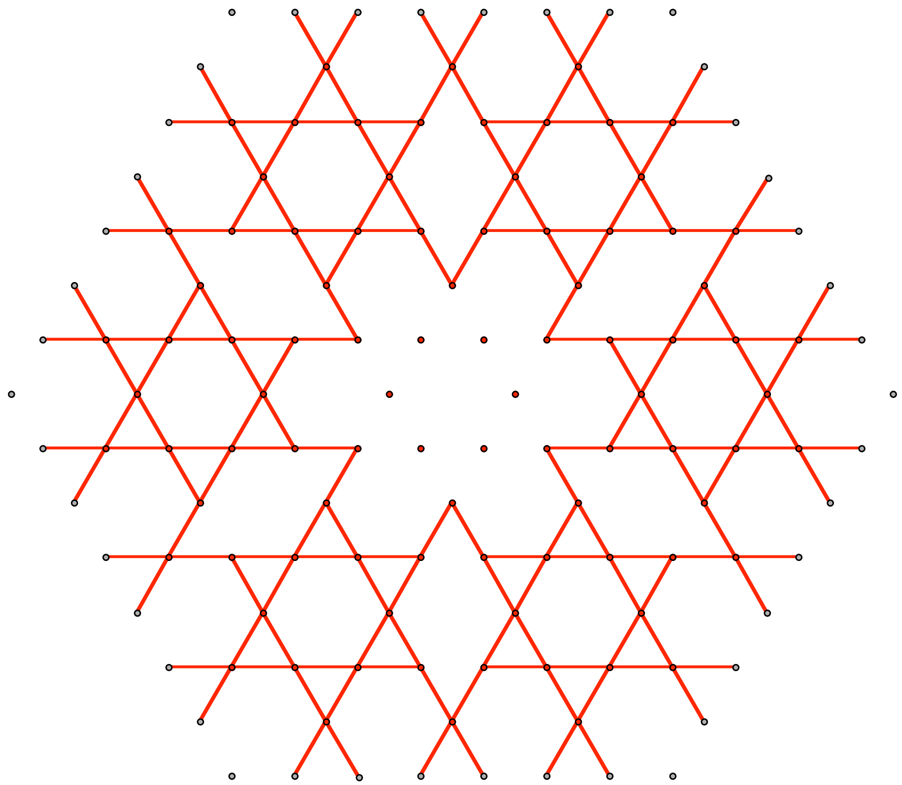
The bar framework can be constructed from the outside in ...

Another application



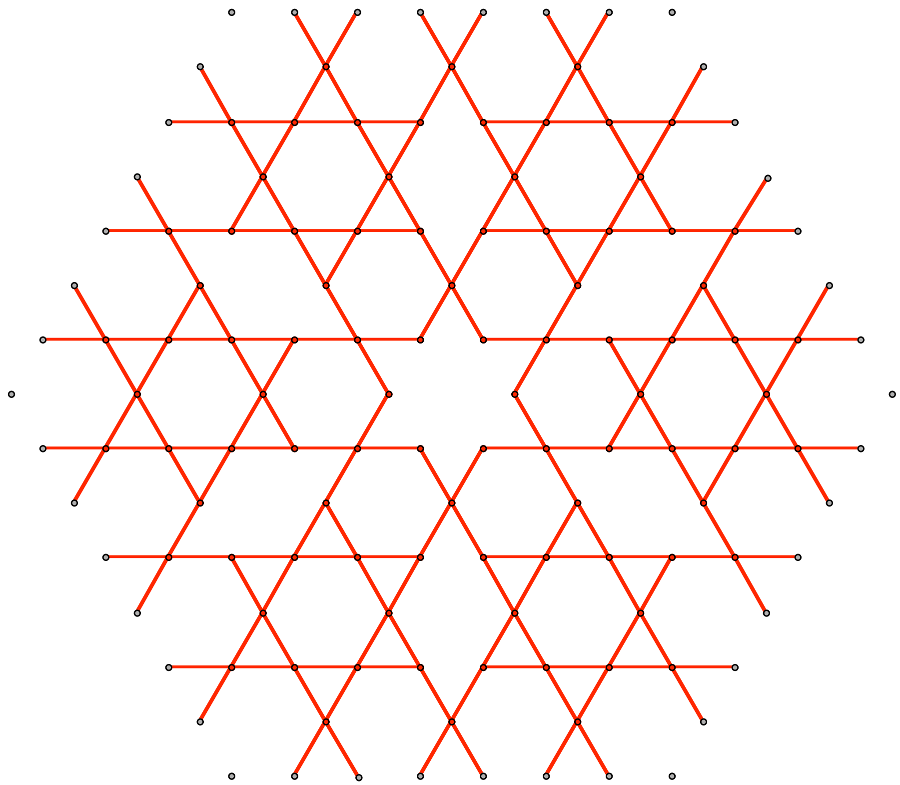
The bar framework can be constructed from the outside in ...

Another application



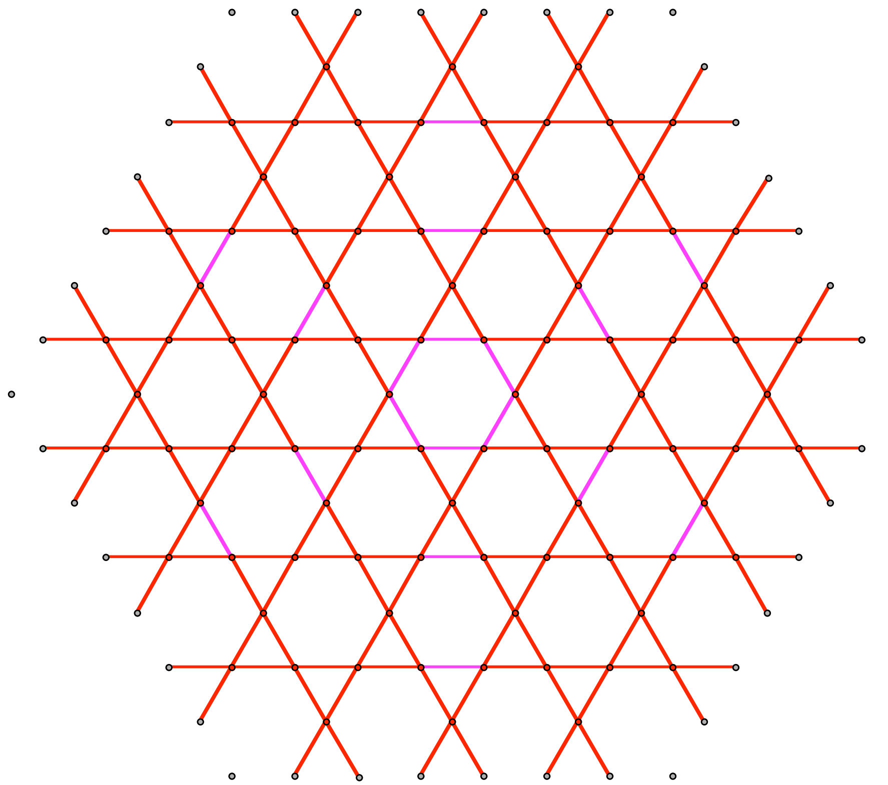
The bar framework can be constructed from the outside in ...

Another application



This bar framework is infinitesimally rigid in the plane.

Another application



When the purple members are inserted, the strut framework has a stress where all stresses are -1.