

# Percolation

## Lecture V

Session on Granular Matter

Institut Henri Poincaré

R. Connelly

Cornell University

Department of Mathematics

# What is Percolation?

# What is Percolation?

- A classic example: Immerse a large stone in a bucket of water. What is the probability that the center of the stone is wetted?
- In the plane, model the stone as a square lattice, and declare each edge to be open with probability  $p > 0$ .
- There is a critical probability  $p_c$  such that for  $p > p_c$  a point in the middle is connected to the outside with high probability, and for  $p < p_c$  it isn't.

# Rigidity Percolation

Start with a planar triangular lattice. Leave each edge with probability  $p > 0$ . As a generic bar-and-joint framework, what is the probability that two vertices will be in the same rigid component? Is there a critical probability  $p_c$  as in the classical case?

# Tension Percolation

Start with the planar triangular lattice, clamp the boundary and again leave in an edge with probability  $p$ . But now regard the graph as having all cables, with pinned vertices on the boundary, a kind of spider web. Is there critical probability  $p_c < 1$  as before?

But first some basics.

# Generic configurations

A configuration  $\mathbf{p}$  is called *generic* if the coordinates of all of its points are algebraically independent over the rational numbers. In other words, any non-zero polynomial with rational coefficients will not vanish when the variables are replaced by the coordinates of  $\mathbf{p}$ .

For example,  $\pi, e, \sqrt{2}, \dots$  are algebraically independent.

# Generic rigidity

**Theorem:** If a bar framework is rigid at one generic configuration then it is infinitesimally rigid at all generic configurations. If it is infinitesimally rigid at some, possibly non-generic, configuration, then it is infinitesimally rigid at all generic configurations.

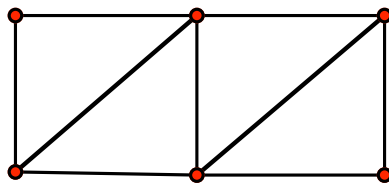
This means that generic rigidity is a combinatorial property of the graph  $G$  only.

**Theorem** (Laman 1972): Suppose that  $G$  has  $n$  vertices and  $e = 2n - 3$  edges. Then  $G$  is generically rigid in the plane if and only if for every subgraph of  $G$  with  $n'$  vertices and  $e'$  edges  $e' \leq 2n' - 3$ .

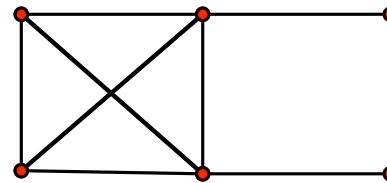
# Algorithms

**Corollary** (Sugihara 1979, Lovász and Yemini 1982): Given a graph  $G$  with  $n$  vertices and  $e$  edges, there is an algorithm to determine whether  $G$  is generically rigid in the plane in at most  $O(ne)$  steps. (Variations: 2-tree condition, the pebble game, etc.) For the following description, see Jacobs and Hendrickson 1997.

In both cases  $9 = e = 2n - 3 = 2 \cdot 6 - 3$ .



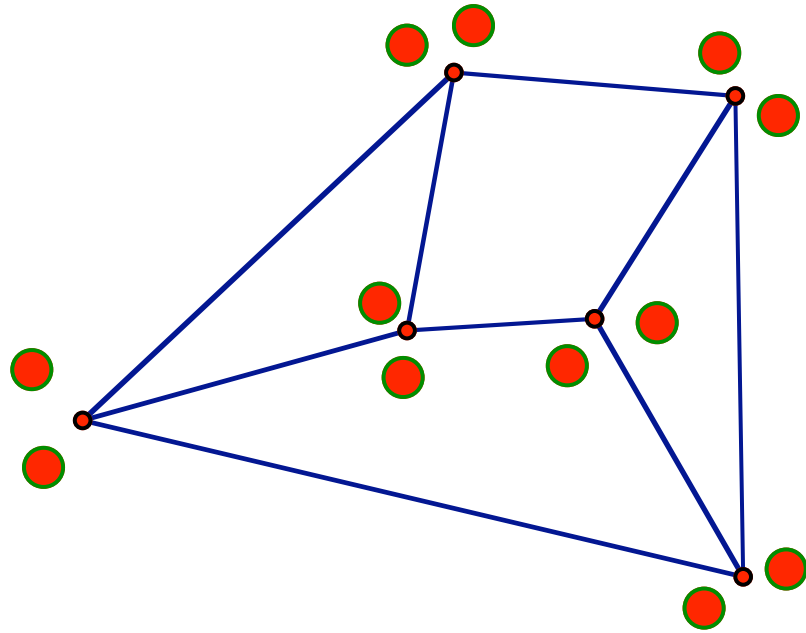
Generically rigid



Not generically rigid

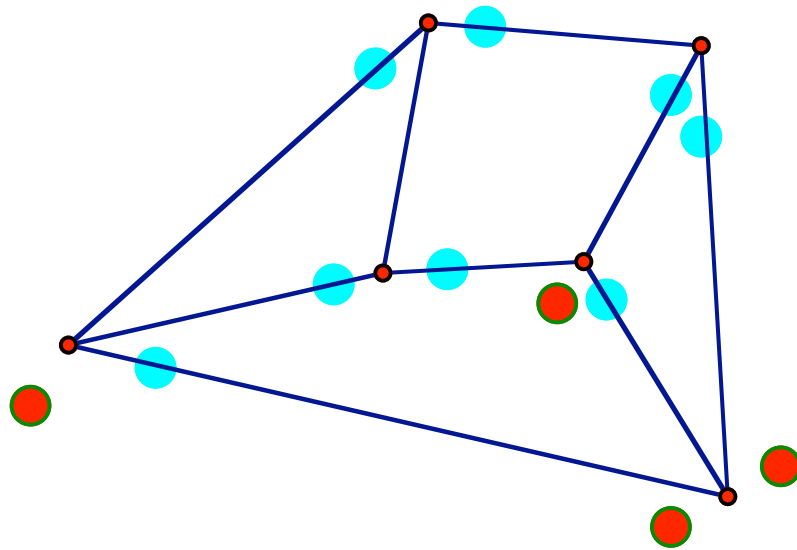


# The pebble game



Start with 2 pebbles  
assigned to each vertex.

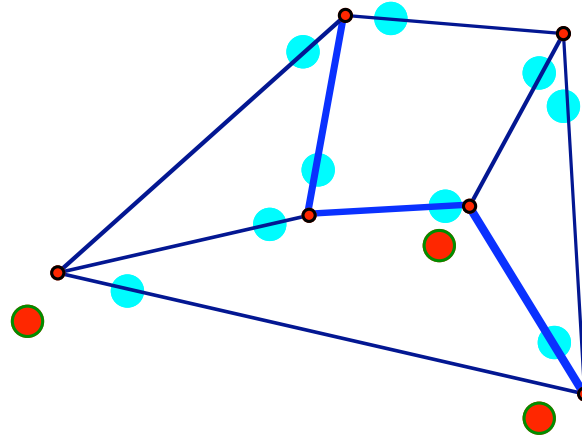
# The pebble game



Then try to assign each pebble to an adjacent edge, but do not assign more than one pebble to each edge.

# The pebble game

If all edges are covered by pebbles from adjacent vertices, this is a certificate that Laman's condition holds and the graph is generically rigid in the plane.

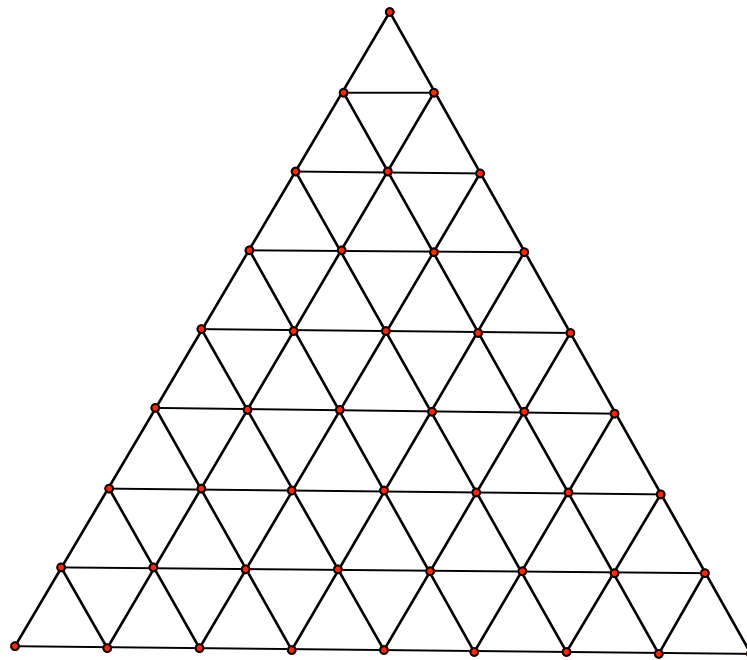


If there are more than 3 pebbles left, backtrack along other edges until there is an edge without a pebble.

# Rigidity Percolation

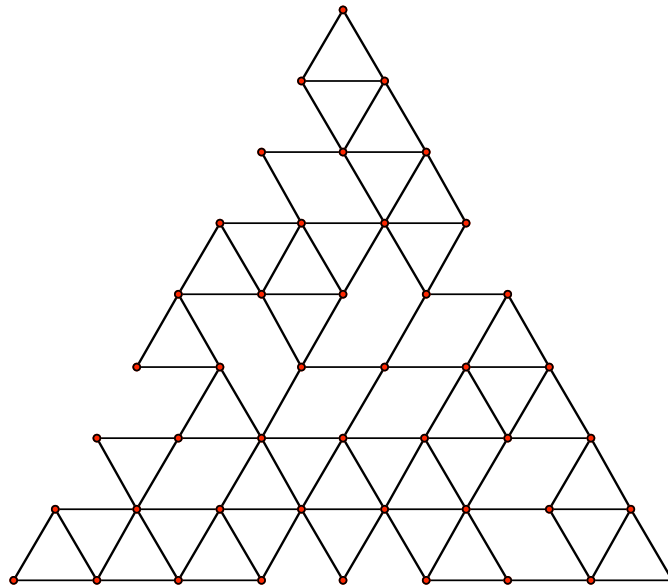
This is a suggested model for the onset of rigidity in glasses.

Start with the following bar framework, each bar is retained with a certain probability  $p$ .



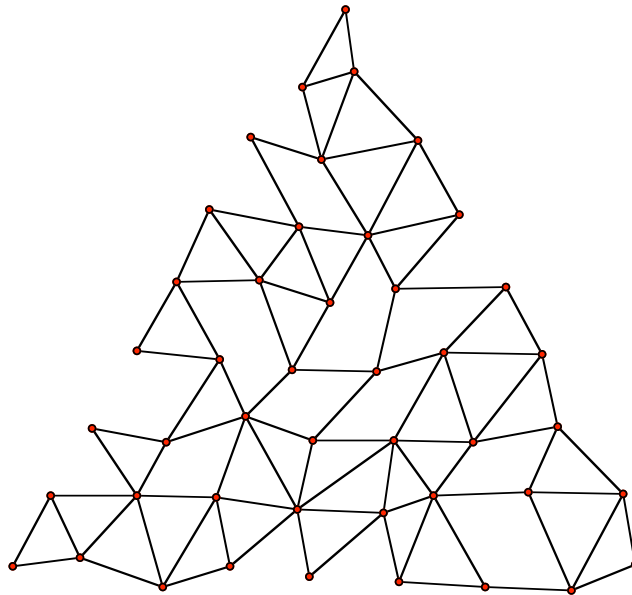
# Rigidity Percolation

Start with the following bar framework, each bar is retained with a certain probability  $p$ . Imagine perturbing the configuration so that the configuration is generic, and then apply the pebble game algorithm to compute rigidity.



# Rigidity Percolation

Start with the following bar framework, each bar is retained with a certain probability  $p$ . Imagine perturbing the configuration so that the configuration is generic, and then apply the pebble game algorithm to compute rigidity. The critical probability  $p$  where the framework becomes non-rigid is the *rigidity percolation threshold*.



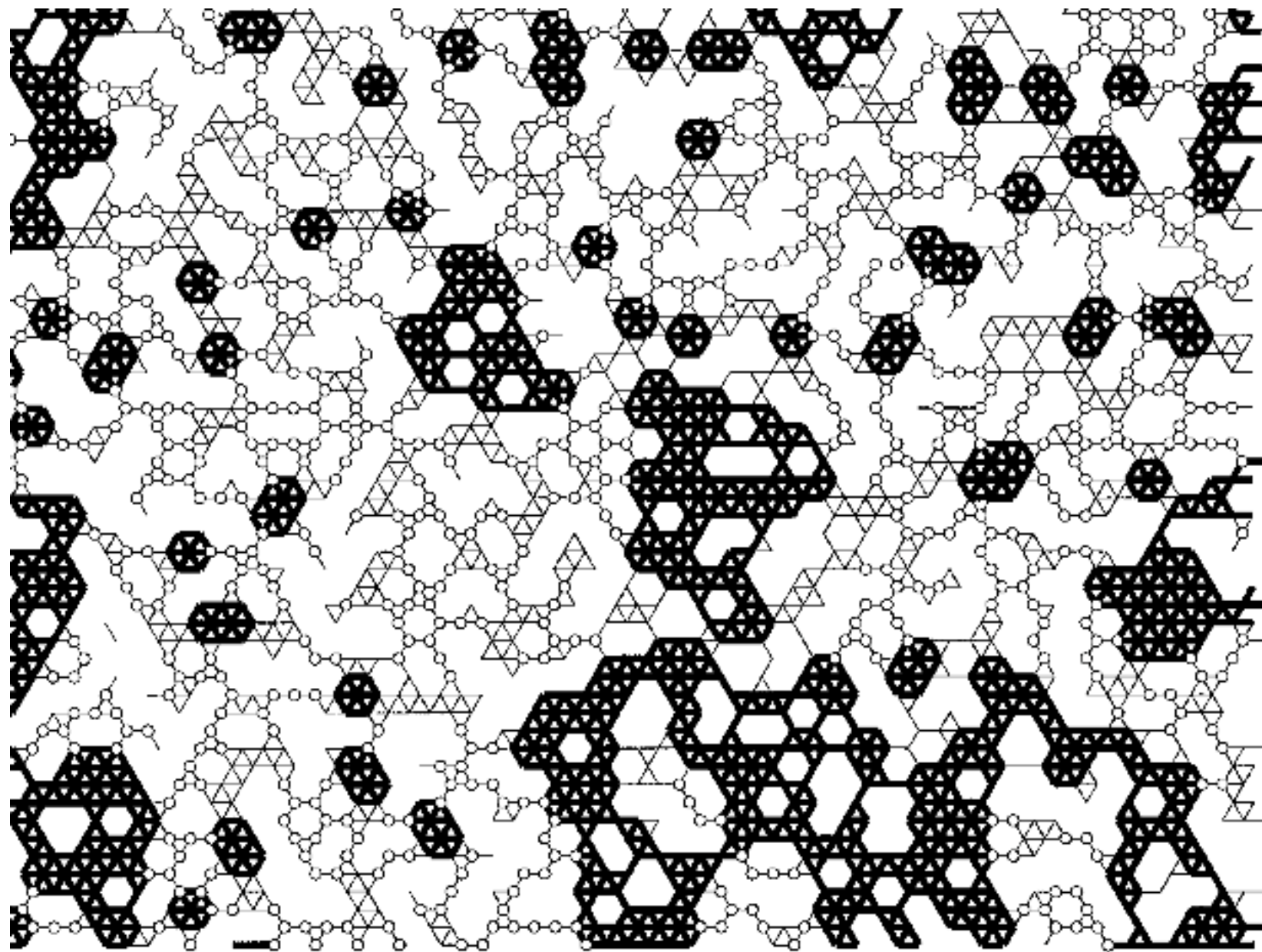
# Average degree

If a graph  $G$  has  $n$  vertices and  $e$  edges, let  $n_i$  be the number of vertices of degree  $i$ . Then

$$\sum_i i n_i = 2e,$$

define  $(\sum_i i n_i) / \sum_i n_i = (\sum_i i n_i) / n = 2e/n = Z$  as the *average degree* or *coordination number*. If  $G$  represents the bars of a generically rigid bar framework in the plane, then  $e \geq 2n-3$ , and so  $Z \geq 4 - 6/n$ .

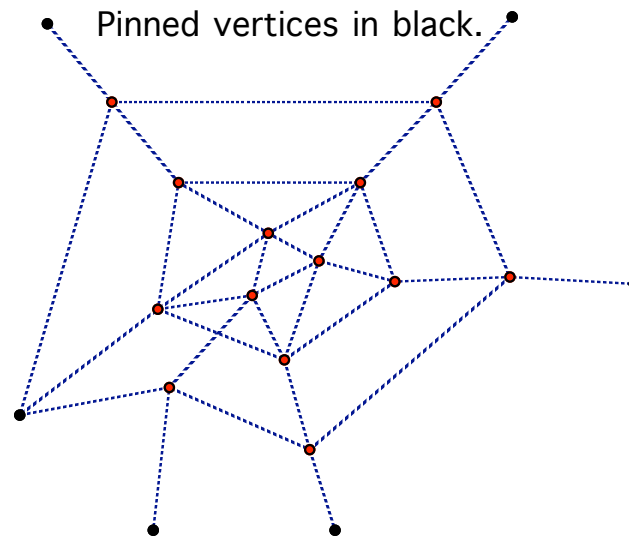
So in the percolation problem, the graph cannot be rigid until  $p \geq 2/3$ , and indeed the “floppy modes” do not decrease until a critical percolation at about  $p = 0.6603$ . (Thorpe, Duxbury, etc. 1990’s)





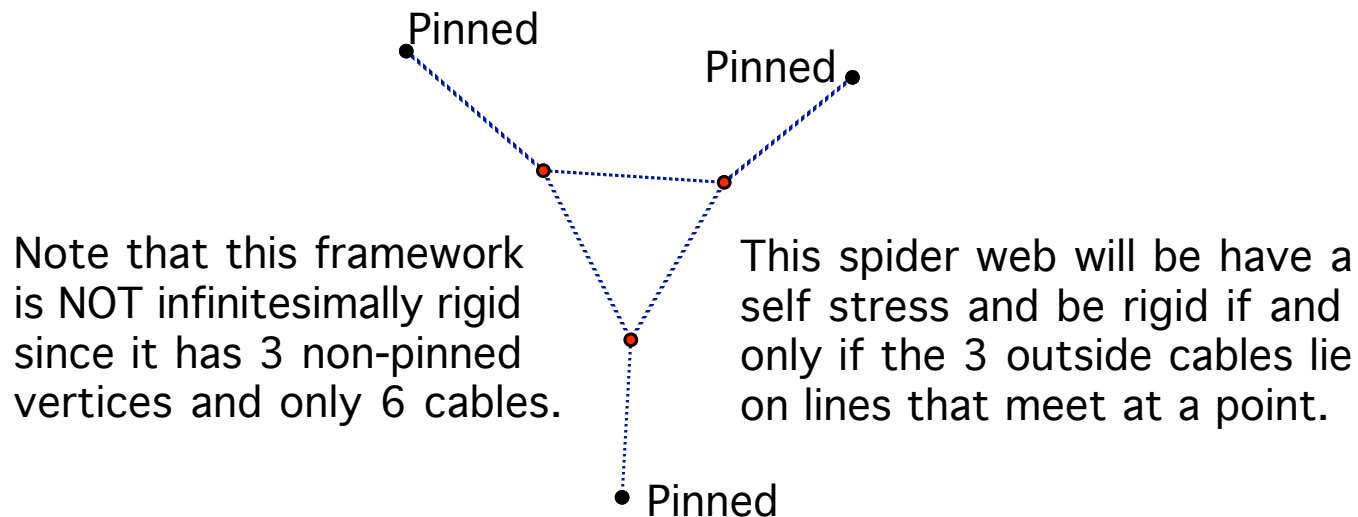
# Tension Percolation

Spider Webs: Consider any tensegrity framework  $G(\mathbf{p})$  with all member cables and some pinned vertices, called a *spider web*. When is it rigid?



# Tension Percolation

**Theorem:** If a spider web has a proper self stress, strictly positive on all cables all connected to a pinned vertex, then it is rigid (in any dimension).



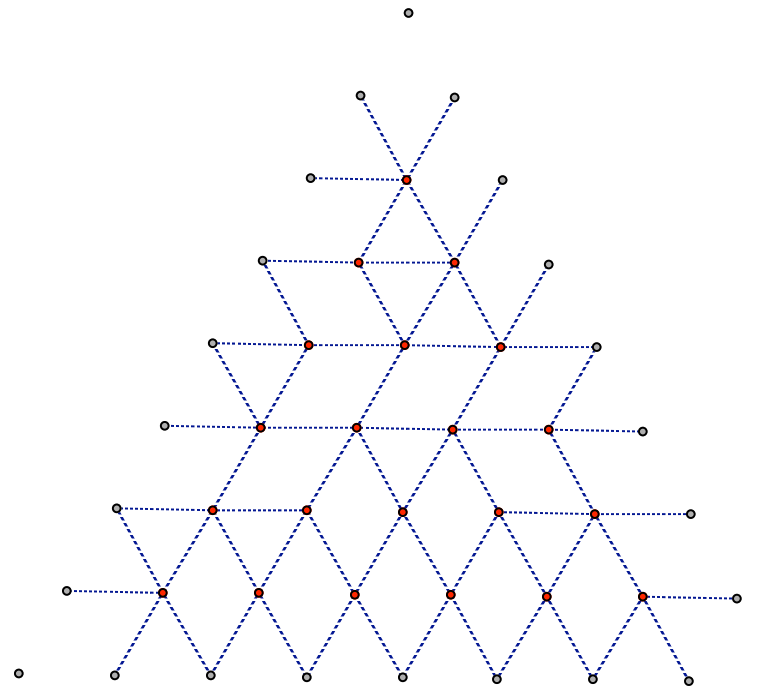
# Spider proof

Let  $\sigma = (\dots, \sigma_{ij}, \dots)$  be the proper self stress, non-zero on all the cables at the configuration  $\mathbf{p}$ . For each configuration  $\mathbf{q}$ , define the energy function:

$$E(\mathbf{q}) = \sum_{i < j} \sigma_{ij} |q_i - q_j|^2.$$

Note that  $E$  is a quadratic function, and that since  $E(\mathbf{q}) = \frac{1}{2} \mathbf{R}(\mathbf{q})\mathbf{q}$ , we have that  $\mathbf{p}$  is a critical point for  $E$ , since  $\mathbf{R}(\mathbf{p}) = 0$ . Since each  $\sigma_{ij} > 0$ ,  $\mathbf{p}$  is clearly a minimum. If  $\mathbf{q}$  is another minimum point for  $E$ , then  $E((1-t)\mathbf{q} + t\mathbf{p})$ , for  $t$  real, would also be a minimum, which is not possible, since cables would be stretched to infinity. //

# Tension Percolation



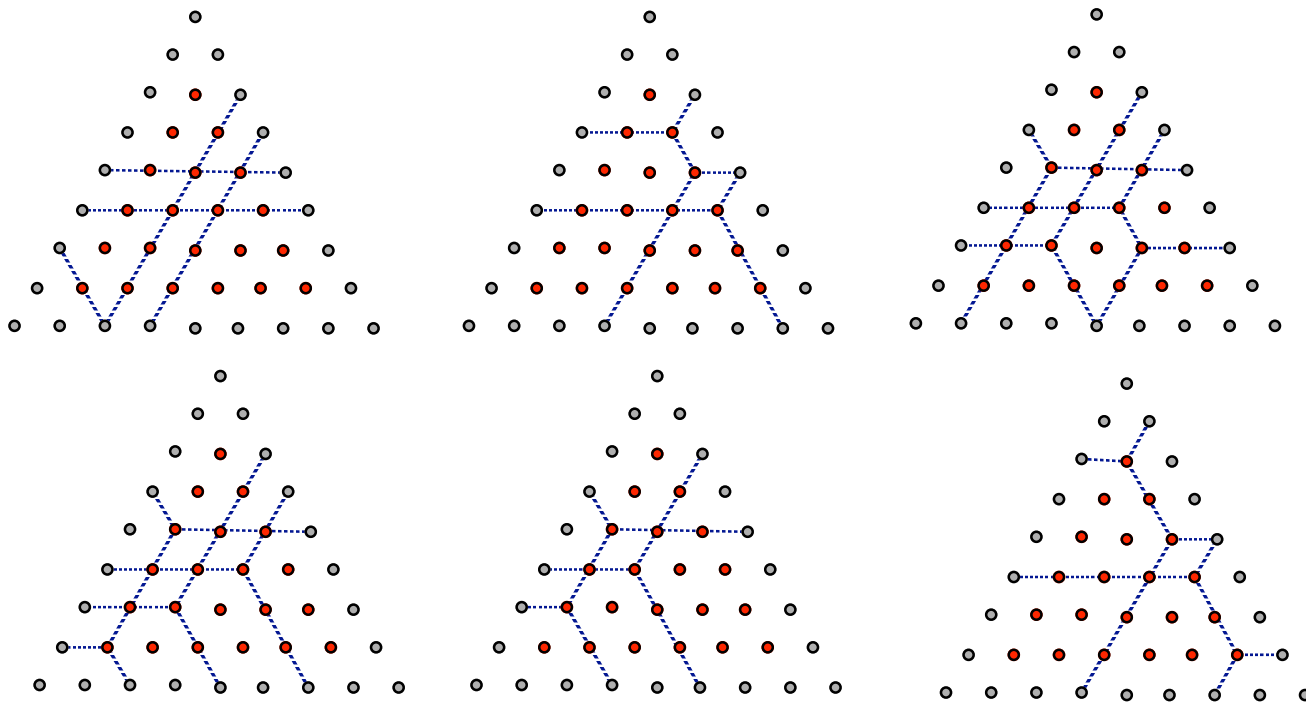
A cable framework in the plane with the boundary vertices pinned. Internal bars are deleted with a certain probability  $p$ .

**How can you tell easily when it has a positive self stress?**

# Remark about packings

Recall that by the Roth-Whiteley criterion, this same stress condition, but with the opposite sign, relates to packings of circles in a triangular lattice. Can you randomly delete  $(1-p)$  of the contacts and hope to have a self stress, all negative on all the struts?

# The final stress



With the sum of these stresses, each of 1 unit in each cable, every vertex has an incident cable with a positive self stress and so this is rigid.

# The critical probability

Take the triangular lattice, and for each edge independently decide to keep each edge with probability  $p$ . Is there a critical probability  $p_c < 1$  that determines a transition from having a stress to not having a stress?

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Take the triangular lattice, and for each edge independently decide to keep each edge with probability  $p$ . Is there a critical probability  $p_c < 1$  that determines a transition from having a stress to not having a stress?

Answer: No. For a large enough lattice, and for any  $p < 1$ , the network will lose tension.

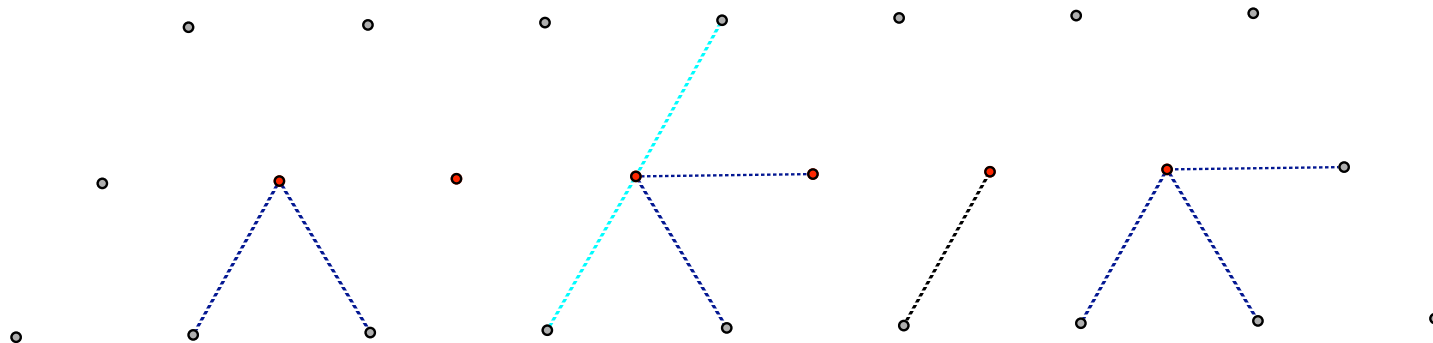
Connelly, Robert(1-CRNL); Rybnikov, Konstantin(1-CRNL); Volkov, Stanislav(4-BRST)  
Percolation of the loss of tension in an infinite triangular lattice. (English. English summary)  
J. Statist. Phys. 105 (2001), no. 1-2, 143--171.



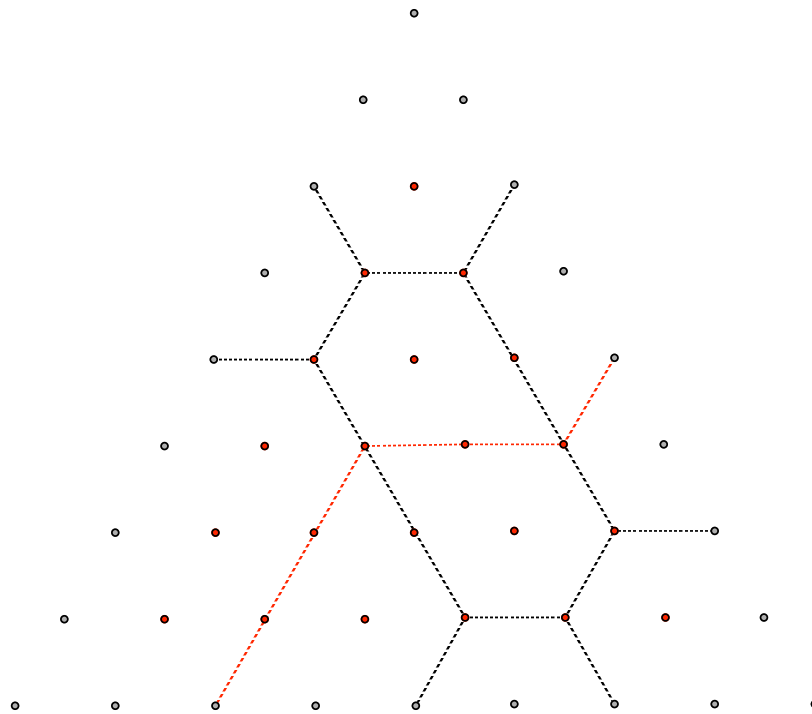
# Eliminating edges

There are instances when one can show that the stresses in many members is 0.

After the cables have been chosen, we can eliminate more as in the dark edges below.

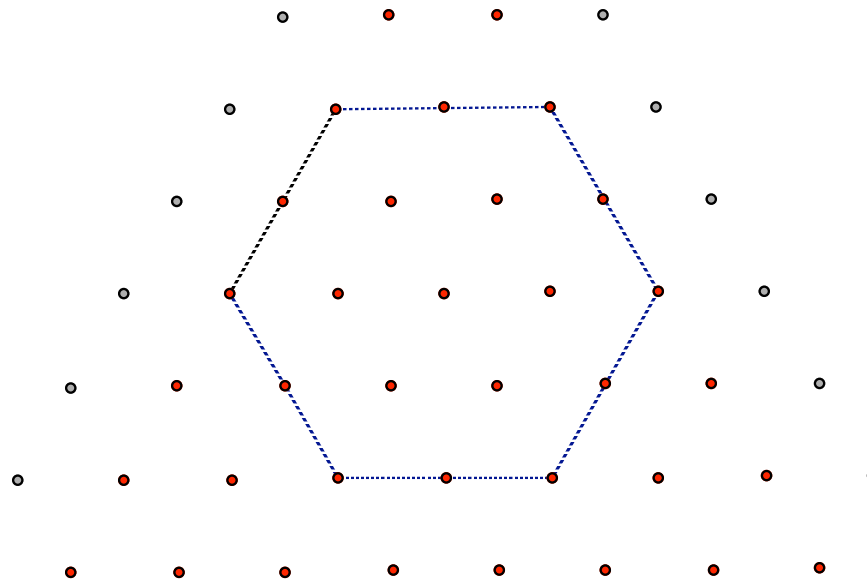


# Nuances



Some configurations are subtle. This is an example, where each vertex can be locally in equilibrium with positive stresses, but the stress on the red cables must be 0.

# The idea of the proof



Look inside ever larger hexagons, and argue inductively that by using the unnuanced process for eliminating edges that the interior edges have 0 stress. Then with high probability, each of the edges of the hexagon have an edge that is missing. This eliminates the next layer.

# Calculating the probabilities

Suppose that a hexagon with  $m$  edges on a side has stresses all 0 inside. Then the probability that at least one edge from each of the 6 sides will have been deleted, and hence creating an empty hexagon with  $m+1$  sides, is  $(1-p^m)^6$ . So the probability that this will continue to infinity is  $\prod_m (1-p^n)^6$ . For fixed  $p < 1$  and  $m \rightarrow \infty$  this tends to 1.//

# A continuous model

Imagine the plane as a piece of paper clamped outside a large convex polygon. Cut out holes in the paper.

- Can the paper be folded in such a way that it can be flexed up into space?
- Is there a critical probability such that when the area of the holes have a small density, then it cannot be folded?

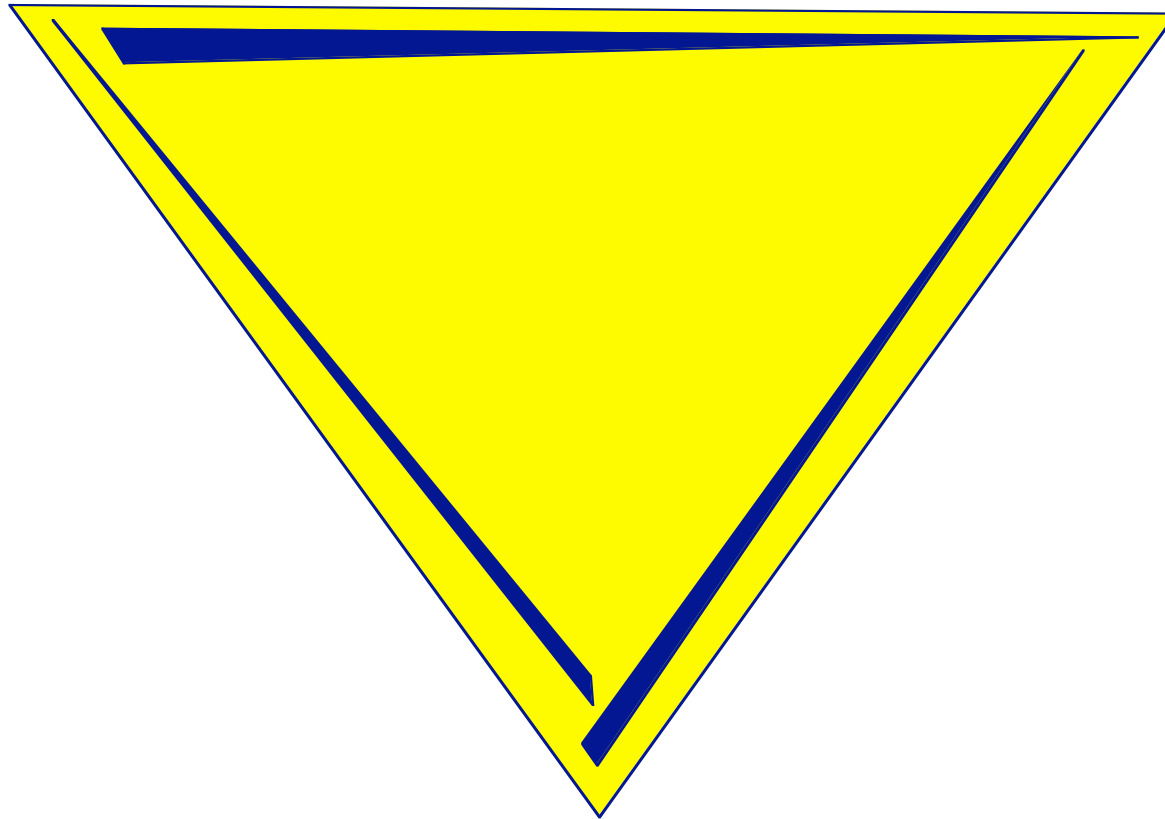
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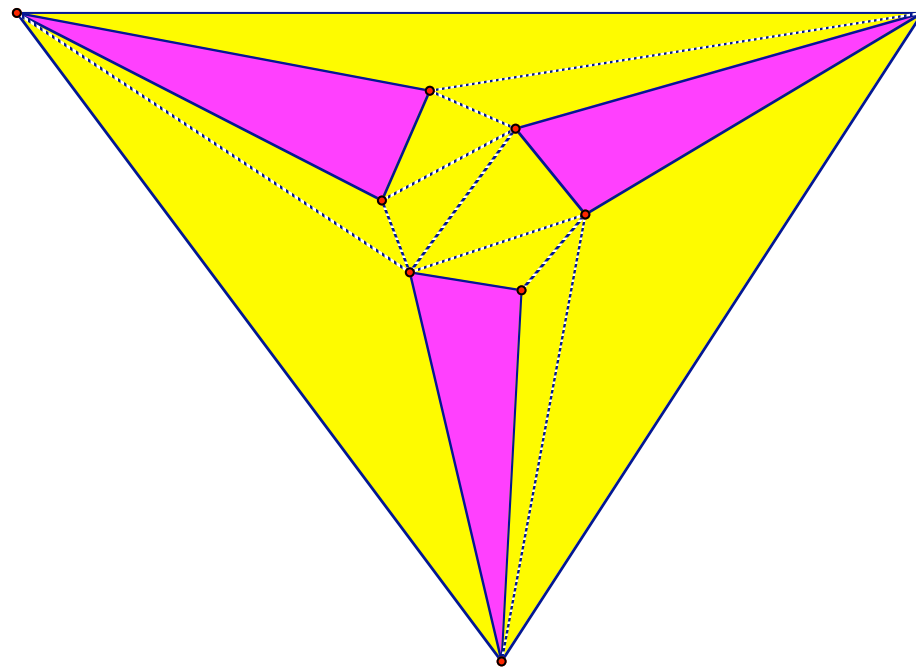
- Can the paper be folded in such a way that it can be flexed up into space?
- Is there a critical probability such that when the area of the holes have a small density, then it cannot be folded?

As before, no, by a similar argument!

An example that can be folded



# An example that cannot be folded: an underlying spider web

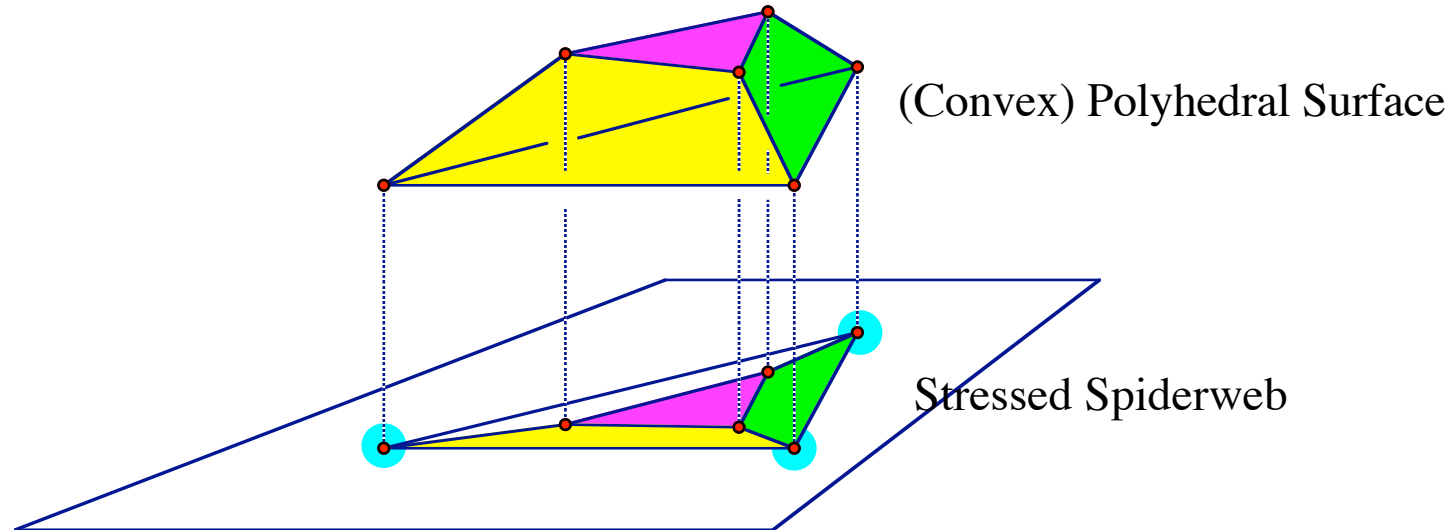


If there is a triangulation of the complement of the holes (in purple here) such that there is a positive equilibrium stress on each edge, then the paper cannot be folded to flex.



# Graphical Statics

In the 19th century a method of computing forces graphically, as above, was common. And James Clerk Maxwell and Luigi Cremona showed that there is a visual correspondence between polyhedral surfaces and the equilibrium forces in a tensegrity as below.



# Lifting holes

- If a system of holes can be lifted to a convex surface with each hole as a convex face, the Maxwell-Cremona correspondence provides a stress from the convex hull of the lifted holes to a spider web in the plane.
- The existence of the lift can be determined by a linear programming feasibility algorithm.