Stress Matrices and M Matrices

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1. INTRODUCTION

In [2] a connection is made between what are called "M" matrices, as used in Colin de Verdière's theory of graph invariants, and stress matrices as used in rigidity theory in [1]. Following a description of stress matrices and their properties relevant to rigidity theory, it is shown how a theorem of László Lovász [2], using results about M matrices, implies a conjecture about the global rigidity of certain tensegrity frameworks by Károly Bezdek.

2. STRESS MATRICES

Given a finite graph $G = (V, E)$ without loops or multiple edges, where V is the set of n vertices labeled $1, \dots, n$ and E the edges, a stress matrix Ω is a symmetric n-by-n matrix, where the off-diagonal entries are denoted as $-\omega_{ij}$ and the following conditions hold:

(1) When $i \neq j$ and $\{ij\}$ is not in E, then $\omega_{ij} = 0$.
(2) $[1, 1, \dots, 1]\Omega = 0$.

Condition (2) defines the diagonal entries of Ω in terms of the off-diagonal entries. The i-th row and column of Ω correspond to the i-th vertex.

Consider a configuration of points $\mathbf{p} = (p_1, \dots, p_n)$, where each p_i is in Euclidian d-dimensional space $\mathbb{E}^d$. Form the d-by-n *configuration matrix* $P = [p_1, p_2, \dots, p_n]$, where each p_i is regarded a column of P. The configuration $\mathbf{p}$ is said to be in *equilibrium* with respect to the stress $\omega = (\dots, \omega_{ij}, \dots)$ if $P\Omega = 0$. This is equivalent to the vector equation for each vertex i, $\sum_j \omega_{ij}(p_j - p_i) = 0$. Some basic properties of stress matrices, which can be found in [1], are in the following proposition.

Proposition 2.1. *If the configuration* $\mathbf{p}$ *is in equilibrium with respect to the stress* ω*, then the following hold:*

(1) *The dimension of the affine span of the configuration* $\mathbf{p}$ *is at most* $n - 1 -$ *rank* Ω*.*
(2) *If the dimension of the affine span of the configuration* $\mathbf{p}$ *is exactly* $n-1-$ *rank* Ω*, and* $\mathbf{q} = (q_1, \dots, q_n)$ *is another configuration in equilibrium with respect to* ω*, then* $\mathbf{q}$ *is an affine image of* $\mathbf{p}$*.*

If Condition (2) in Proposition 2.1 holds for a configuration $\mathbf{p}$, then we say $\mathbf{p}$ is *universal* with respect to ω. It is easy to see that if a configuration $\mathbf{p}$, with a d-dimensional affine span is not universal for a given equilibrium stress ω, then there is a configuration $\mathbf{q}$, whose affine span is at least $(d + 1)$-dimensional, that projects orthogonally onto $\mathbf{p}$, and which is in equilibrium with respect to ω as well.

3. Global Rigidity

Suppose that the edges of a graph G are labeled either a cable or a strut. We say a configuration $\mathbf{q}$, corresponding to the vertices V, is *dominated by* the configuration $\mathbf{p}$ if the cables of $\mathbf{q}$ are not increased, and struts are not decreased in length. We call $G(\mathbf{p})$ a *tensegrity*, and if every configuration in $\mathbb{E}^d$ that is dominated by $\mathbf{p}$ is congruent $\mathbf{p}$, we say $G(\mathbf{p})$ is *globally rigid in* $\mathbb{E}^d$.

If $v_1, v_2, \dots$ are vectors in $\mathbb{E}^d$, we say that they lie on a *conic at infinity* if for all i, there is a non-zero d-by-d symmetric matrix C such that $v_i^T C v_i = 0$, where $()^T$ is the transpose. The following fundamental result can be found in [1].

Theorem 3.1. *If a configuration $\mathbf{p}$ in $\mathbb{E}^d$ has an equilibrium stress ω, with $\omega_{ij} > 0$ for cables, $\omega_{ij} < 0$ for struts, (called a* proper stress *for $G = (V, E)$) such that*

1. *the member directions $p_i - p_j$, for $\{ij\}$ in E, do not lie on a conic at infinity,*
2. *the matrix Ω is positive semi-definite, and*
3. *the configuration $\mathbf{p}$ is universal with respect to ω,*

then $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^N$, for all $N \geq d$.

Any configuration that satisfies the hypothesis above is called *super stable*.

4. Lovász's Result

The following result of Lovász in [2] has a situation that satisfies all the conditions of Theorem 3.1 except condition (3). Condition (1) is easy to verify.

Theorem 4.1. *If a tensegrity framework $G(\mathbf{p})$ is defined by putting cables for the edges of a convex 3-dimensional polytope P and struts from any interior vertex to each of the vertices of P, then the configuration $\mathbf{p}$ has a proper equilibrium stress ω, and any such non-zero stress has a stress matrix Ω with exactly one negative eigenvalue and 4 zero eigenvalues.*

If one takes the stress matrix Ω from Theorem 4.1 and removes the row and column corresponding to the central vertex, then one gets an M matrix as used in the definition of Colin de Verdière's number defined as a graph invariant. The problem is to get rid of the offending negative eigenvalue.

5. The Conjecture

Suppose P is a convex polytope in $\mathbb{E}^3$ and one creates a tensegrity $G(\mathbf{p})$ by assigning the vertices of P as the vertices of a configuration for G, the edges of P as the cables for $G(\mathbf{p})$, and assigning struts as some of the internal diagonals such that $G(\mathbf{p})$ has a proper equilibrium stress. Then it appears that the resulting stress matrix Ω satisfies the conditions for being super stable, but no proof is known in general. Károly Bezdek specialized that conjecture to the case when the polytope P is centrally symmetric. With the help of Theorem 4.1 by Lovász, we can prove that conjecture, which is the following.

Theorem 5.1. *For any* 3*-dimensional centrally symmetric convex polytope* P*, the associated tensegrity* $G_P(\mathbf{p})$*, with struts between all antipodal vertices, has a stress* ω *such that* $G_P(\mathbf{p})$ *is super stable. Furthermore any such proper equilibrium stress for* $G_P(\mathbf{p})$ *is such that it serves to make* $G_P(\mathbf{p})$ *super stable.*

Proof. Let ω' denote any non-zero proper stress determined by the conclusion of Theorem 4.1 for the centrally symmetric polytope P with the central vertex as the interior point. Let $\hat{\omega}'$ denote the stress on $G_P(\mathbf{p})$ obtained by replacing each cable and strut stress with the stress on its antipode. Then $\hat{\omega}'$ is an equilibrium stress for $G_P(\mathbf{p})$ as well. Hence $\omega' + \hat{\omega}'$ is a proper equilibrium stress for $G_P(\mathbf{p})$, where stresses on antipodal cables and struts are equal. So we assume without loss of generality that the stresses in ω' are symmetric, and Theorem 4.1 assures us that the associated stress matrix has only one negative eigenvalue.

Suppose that i and j correspond to antipodal vertices of P. Let $\omega'_{i0} = \omega'_{j0} < 0$ be the stresses from the central vertex to the i and j vertices coming from the stress ω'. Form a small tensegrity $G_{ij}(p_i, p_j, 0)$ with just three vertices i, j, and the central vertex 0, where $\{i, j\}$ is a strut, while $\{0, i\}$ and $\{0, j\}$ are cables. Let $\omega_{ij} = 2\omega'_{i0} = 2\omega'_{j0} < 0$, and replace ω'_{i0} and ω'_{j0} with $-\omega'_{i0}$. It is easy to check that this is an equilibrium stress for $G_{ij}(p_i, p_j, 0)$ whose associated stress matrix is positive semi-definite. Extend this to all the vertices of G by having all other stresses 0. The associated stress matrix Ω'_{ij} defined on all the vertices of G is still positive semi-definite. But now $\Omega' + \Omega'_{ij}$ has its $(0, i)$ and $(0, j)$ entry 0. Let $\Omega' + \sum_{ij} \Omega'_{ij} = \Omega$, where the sum is over all antipodal vertices $\{i, j\}$. We obtain a stress matrix Ω corresponding to a stress ω, where all the $\omega_{0i} = 0$ and $\omega_{ij} < 0$ for pairs $\{i, j\}$ of antipodal vertices. Otherwise $\omega_{ij} = \omega'_{ij}$. Since Ω is obtained by adding a positive semi-definite matrix $\sum_{ij} \Omega'_{ij}$ to Ω', none of the eigenvalues of Ω' decrease. It is clear that the stress ω is an equilibrium stress for a configuration whose affine span is 4-dimensional, since the central vertex can be displaced into $\mathbb{E}^4$, where it has essentially been disconnected from all the other vertices. So the 0 eigenvalues of Ω must stay at 0, while the negative eigenvalue must increase to provide the extra 0. If we remove the central vertex, then the resulting framework with antipodal vertices connected by struts is super stable, as desired.

It clear that the above process can be reversed, starting with an arbitrary equilibrium stress ω for the centrally symmetric polytope P to obtain a proper stress for a tensegrity as in Theorem 4.1. So any such proper equilibrium stress ω for struts connecting antipodal vertices of P will be super stable. □

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References

[1] R. Connelly, *Rigidity and Energy*, Invent. Math. **66** (1982), no. 1, 11–33.

[2] László Lovász, *Steinitz Representations of Polyhedra and the Colin de Verdière Number*, Journal of Combinatorial Theory, Series B **82** (2001), 223–236.