STRUCTURES IN HYPERBOLIC SPACE

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Abstract This is an overview of some of the similarities and differences between structures such as frameworks and cabled tensegrities in the hyperbolic plane and hyperbolic space on the one hand and the Euclidean plane, the sphere and Euclidean space on the other hand.

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1. Introduction

There is a strong analogy between the unit sphere in Euclidean space and the hyperbolic plane. One can regard the hyperbolic plane $\mathbb{H}^2$ as half of the "unit sphere" in Lorenz space, namely the following set:

$$\mathbb{H}^2 = \{(x, y, t) \in E^{1,2} \mid x^2 + y^2 - t^2 = -1,\ t > 0\}.$$

Of course, there is a similar definition for higher-dimensional Euclidean, spherical and hyperbolic spaces. So one can often "change the sign" in a formula in spherical geoemtry and create a correct formula in hyperbolic geometry. For example, in spherical trigonometry many of the standard formulas for a spherical triangle involve sines and cosines. To get an appropriate correct formula in the hyperbolic plane, when these functions are applied to edge lengths, replace the sine and cosine functions by the hyperbolic sine and the hyperbolic cosines function respectively. When they are applied to angles, leave the sine and cosine functions as they were.

In the following we show some similarities and then differences between hyperbolic and non-hyperbolic situations.

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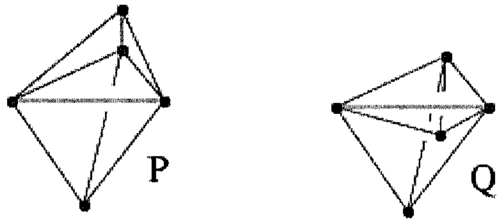

Figure 1. Non-congruent polyhedral surfaces

2. Polyhedra in dimension 3

In 1813 Cauchy [Cauchy, 1813] proved the following, which we state in modern language. A function f between sets in any space with a metric, such as Euclidean, spherical or hyperbolic space, is called a *congruence* if the distance between every pair of points p and q in one set is the same as the distance between $f(p)$ and $f(q)$ in the other set. Recall that a *face* of a convex polytope P (in Euclidean, spherical or hyperbolic space) is the intersection of a hyperplane with P that contains P on one side. (A *homeomorphism* is a function such that it and its inverse are one-to-one and continuous.)

Theorem 2.1 (Cauchy). *Suppose that $f : P \to Q$ is a homeomorphism between two convex polyhedral surfaces P and Q in $\mathbb{E}^3$ such that f restricted to each face of P is a congruence. Then f itself is a congruence.*

This is an example of a local but partly global result. It is not entirely global because of the restriction that both the source and target sets are assumed to be convex. But if the correspondence f is assumed to be close enough to the identity and the source P is convex, then the target Q will be convex as well and Cauchy's theorem applies. But if only P is convex, Figure 1 shows that Q may not be congruent to P even in the simplest case when P has just five vertices.

But what about hyperbolic space? Here it is instructive to look at Cauchy's proof. It has essentially two parts, a combinatorial topological lemma, and a local (and partly global) geometric lemma. The combinatorial lemma is the following:

Lemma 2.2. *If some subset of the edges of a convex polytope are assigned a $+$ or $-$ such that there are at least four changes in sign (or no labeled vertices) around each vertex, then no edges are labeled anywhere.*

It is instructive to try to find such a labeling for the first non-trivial case of a convex polyhedral surface, the (regular) octahedron. The proof

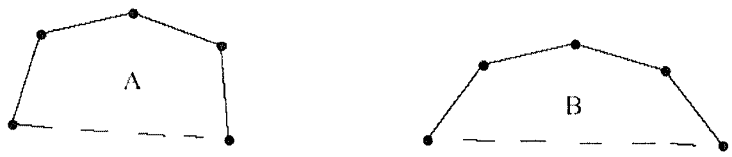

Figure 2. Polygonal arcs resembling opening arms

of this lemma is a nice application of the Euler characteristic. One can find a good description of this lemma and the next in [Lyusternik] or [Aigner-Ziegler] for example. This lemma clearly holds when the convex polyhedral surface is in $\mathbb{H}^3$, since convexity plays a minor role. It is only the topological and combinatorial structure of the graph of edges that matters.

The geometric lemma is the following. I like to call this Cauchy's "arm" lemma, because the polygon involved looks like an opening arm. (There was a problem with Cauchy's proof of this lemma. It was not corrected until some years after his 1813 publication. The message seems to be that one should be careful here. See [Connelly] for a discussion of this.)

Lemma 2.3 (Cauchy's arm). *Suppose that $f : A \to B \subset \mathbb{S}^2$ is a continuous map from a convex polygonal arc A in the unit 2-sphere $\mathbb{S}^2$ that is a congruence when restricted to each edge of A and the angle at each internal vertex of A does not decrease. Then the distance between the endpoints of A are not less than the distance between the endpoints of B.*

Figure 2 shows what this looks like in the Euclidean plane, a limiting case of the sphere.

Note that the target polygon B is not assumed to be convex. This lemma is then applied to small spheres centered at each vertex of the convex polyhedral surface P. Each edge incident to a vertex of P corresponds to a vertex on the corresponding sphere, and each dihedral angle corresponds to an internal angle of a spherical arc. The idea is that if the edges of the spherical polyhedron are labeled $+$ or $-$ depending on whether they increase or decrease, respectively, and there are only two changes in sign around a vertex, an arc with only $+$ followed by an arc with $-$ signs contradicts the Arm Lemma 2.3.

But for a hyperbolic polyhedral surface we see that the Arm Lemma 2.3 still applies since it is used for polygons in a sphere. So Cauchy's Theorem 2.1 is still true in hyperbolic space $\mathbb{H}^3$. Indeed, Cauchy's Theorem 2.1 generalizes to dimensions greater than 3, as well as to spherical and hyperbolic spaces of dimension greater than 3. See [Connelly] for more information and related results.

3. Static rigidity

A natural approach for rigidity questions is to linearize the conditions for the rigidity of constraints. In order to simplify the discussion, suppose that the polyhedral surfaces we consider all have only triangular faces. The rigidity constraints can be restricted only to pairs of vertices with an edge between them, where the edges act as bars. One version of the linear constraints can be described with forces acting on this framework of vertices and bars. For each vertex p_i in this framework there is associated a force vector F_i, the external load, and the question of static rigidity is whether each appropriate external load can be *resolved* by some internal stress in the framework. This means that there are scalars ω_{ij} associated with every edge between vertices p_i and p_j such that for each i,

$$F_i = \sum_j \omega_{ij}(p_j - p_i). \tag{1}$$

(The stress $\omega_{ij} = 0$ when there is no bar between p_i and p_j.) We say that an external force $F = (\dots, F_i, \dots)$ is an *equilibrium force* if it has 0 linear and angular momentum, which means that $\sum_i F_i = 0$, and $\sum_i F_i \wedge p_i = 0$, where $\wedge$ is the wedge product from linear algebra. There is no way that an external force $F = (\dots, F_i, \dots)$ can be resolved if it has any non-zero linear or angular momentum. So we define a bar framework to be *statically rigid* if every equilibrium force can be resolved.

Note that this definition only depends on the underlying linear algebra, and so it applies to the sphere and hyperbolic space, as well as Euclidean space, since we can regard them all as being in an appropriate ambient linear space. Note that in dimension 3 the dimension of the space of equilibrium forces is 6 (assuming the framework is not just one bar or a single vertex), and there is an equilibrium equation for each of the three coordinates. If there are n vertices in the framework, there are $3n$ equations to be satisfied. So there must be at least $3n - 6$ bars in the framework. (A similar calculation holds in other dimensions.)

We say that a framework is *rigid* if there is no continuous noncongruent motion of the points that preserves the lengths of the bars. The following is a basic result. See [Connelly] for a proof, for example.

Theorem 3.1. *If a framework is statically rigid in Euclidean, spherical or hyperbolic space, it is rigid.*

So static rigidity is a very local form of rigidity. It is easy to check that for a triangulated 2-dimensional spherical surface the number of edges in the triangulation is $3n-6$, and so there are just enough equations for static rigidity. So we have an infinitesimally local version of Cauchy's semi-global result due originally to Max Dehn. There is a clear proof in [Gluck].

Theorem 3.2 (Dehn). *If P is a convex polyhedral surface in Euclidean, spherical or hyperbolic space, where all the 2-dimensional faces of P are triangles, then P is statically rigid.*

Note that since there are just enough equations for static rigidity, it is enough to show that the homogenized version of the linear equilibrium equations (1) only have the 0 solution. In other words, in this case, it is enough to show that the 0 external force has only the 0 stress as a resolution. (A stress that resolves the 0 external force is called a *self stress.*) It is easy to check that there cannot be just two sign changes in the stresses in the edges adjacent to a vertex, and so Cauchy's combinatorial Lemma applies us to show that a self stress for P is 0. (This is the idea in [Gluck] following some ideas of A. D. Alexsandrov, but Dehn's proof is somewhat different.) Incidentally, W. Whiteley in [Connelly-Whiteley] showed that it is possible to get Dehn's Theorem 3.2 formally from Cauchy's Theorem 2.1 by a technique related to Pogorelov's correspondence mentioned below.

4. The Pogorelov correspondence

There is a very interesting formal corresondence from pairs of configurations of points in Euclidean space $\mathbb{E}^d$ to pairs of configurations of points in hyperbolic space $\mathbb{H}^d$ that has very useful properties with respect to distances. This is described in [Pogorelov].

Theorem 4.1 (Pogorelov). *There is a rational function $f : \mathbb{E}^d \times \mathbb{E}^d \to \mathbb{H}^d \times \mathbb{H}^d$ such that for all (p_1, q_1) and (p_2, q_2) in $\mathbb{E}^d \times \mathbb{E}^d$ we have*

$$\begin{aligned} |p_1 - p_2| &\le |q_1 - q_2| \quad \textit{if and only if} \\ |f_1(p_1, q_1) - f_1(p_2, q_2)| &\le |f_2(p_1, q_1) - f_2(p_2, q_2)|, \quad \textit{and} \\ |p_1 - p_2| &\ge |q_1 - q_2| \quad \textit{if and only if} \\ |f_1(p_1, q_1) - f_1(p_2, q_2)| &\ge |f_2(p_1, q_1) - f_2(p_2, q_2)|, \end{aligned}$$

where $f = (f_1, f_2)$, and $|\dots|$ is the usual Euclidean norm for $\mathbb{E}^d$ and the norm using the Lorentz inner product when the points are in $\mathbb{H}^d$.

This map can be regarded as taking the configurations $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ to the configurations

$$\begin{aligned} f_1(\mathbf{p}, \mathbf{q}) &= (f_1(p_1, q_1), \ldots, f_1(p_1, q_1)) \quad \text{and} \\ f_2(\mathbf{p}, \mathbf{q}) &= (f_2(p_1, q_1), \ldots, f_2(p_1, q_1)), \end{aligned}$$

where, in particular if some pair of vertices in the $\mathbf{p}$ configuration is the same distance apart as in the $\mathbf{q}$ configuration in Euclidean space, then the corresponding pair of vertices are also the same distance apart in hyperbolic space. But an important consideration is that the image pair of configurations in hyperbolic space each depend on both configurations $\mathbf{p}$ and $\mathbf{q}$. Nevertheless, the image of a point $f_1(p_i, q_i)$, say, only depends on p_i and q_i, and it turns out that it is possible to extend correspondences from $\mathbf{p}$ to $\mathbf{q}$ so the conditions of Theorem 4.1 hold. Furthermore, if both $\mathbf{p}$ and $\mathbf{q}$ are convex polyhdra in $\mathbb{E}^d$, so are $f_1(\mathbf{p})$ and $f_2(\mathbf{p})$ in $\mathbb{H}^d$. So this provides another proof of Cauchy's Theorem in hyperbolic space. But this still provides no direct way of transfering global results in Euclidean space to hyperbolic space.

5. Global motions

There are applications, where it is desirable to have a truly global result, with no restriction on the target configuration. But there can be problems when it comes to the case of hyperbolic geometry. For example, consider the following Lemma, which we called a "Leapfrog Lemma" and which was an essential component in the paper [Bezdek-Connelly].

Lemma 5.1 (Leapfrog).
Suppose that $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *are two configurations in* $\mathbb{E}^d$*. Then there is a motion* $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$ *in* $\mathbb{E}^{2d}$*, which is analytic in* t*, such that* $\mathbf{p}(0) = \mathbf{p}$*,* $\mathbf{p}(1) = \mathbf{q}$ *and for* $0 \le t \le 1$*,* $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ *is monotone.*

The method in [Bezdek-Connelly] uses a specific formula, but we will discuss here another method. Both methods have difficulties in extending the results to hyperbolic space. For any configuration of points $\mathbf{p} = (p_1, \ldots, p_n)$ in $\mathbb{E}^d$ define the *Gramm matrix* Gr as an $(n-1)$-by-$(n-1)$ symmetric matrix, whose ij entry is the inner product $p_i \cdot p_j$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-1$, where we assume $p_n = 0$. Define the *configuration matrix* as

$$P = \begin{bmatrix} p_1 & p_2 & \ldots & p_{n-1} \end{bmatrix},$$

where each p_i for $i = 1, \ldots, n-1$ is regarded as a column vector. Then the Gramm matrix $Gr = P^T P$, where P^T is the transpose of P.

So any Gramm matrix coming from a configuration of points in $\mathbb{E}^d$ is positive semi-definite with a kernel of dimension at most d. Conversely, any positive semi-definite matrix Gr with a kernel of dimension at most d can be written as $Gr = P^T P$, where P is a d-by-n matrix and the columns of P serve as the coordinates of a configuration in $\mathbb{E}^d$ with $p_n = 0$. So the Leapfrog Lemma 5.1 can be viewed as coming from the convexity of positive semi-definite matrices.

But what about hyperbolic space? One can still define a Gramm matrix using the indefinite inner product, where

$$< p_i, p_j >= x_i x_j + y_i y_j + z_i z_j + \cdots - t_i t_j$$

and for each i

$$p_i = (x_i, y_i, z_i, \ldots, t_i),$$

and hyperbolic space itself is defined as in the first section:

$$\mathbb{H}^d = \{(x, y, z, \ldots, t) = p \in E^{1,d} \,|< p, p >= -1,\ t > 0\}.$$

We can assume that 0 is in the configuration as before. But now the Gramm matrix for this inner product is $Gr = P^T DP$, where P is the configuration matrix, as before, and D is a diagonal matrix with all 1's on the diagonal, except for the last which is -1. So Gr is indefinite with one negative eigenvalue while all the rest are positive. Convex combinations of these matrices could have more than one negative eigenvalue, and we have a quandry. Do we have a leapfrog lemma in hyperbolic space? We are left with the following rather unsettling questions and conjecture.

Conjecture 5.2 (Hyperbolic-Leapfrog).
Suppose that $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *are two configurations in* $\mathbb{H}^d$. *Then there is a motion* $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$ *in* $\mathbb{H}^{2d}$, *which is analytic in* t, *such that* $\mathbf{p}(0) = \mathbf{p}$, $\mathbf{p}(1) = \mathbf{q}$ *and for* $0 \le t \le 1$, $< \mathbf{p}_i(t) - \mathbf{p}_j(t), \mathbf{p}_i(t) - \mathbf{p}_j(t) >$ *is monotone.*

It is not clear if there is even a continuous monotone motion in any $\mathbb{H}^N$, for N sufficiently large, although this seems more likely to be true. On the positive side, we do have the following analogous result for the spherical case.

Lemma 5.3 (Spherical-Leapfrog). *Suppose that* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *are two configurations in* $\mathbb{S}^d$. *Then there is a motion* $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$ *in* $\mathbb{S}^{2d+1}$, *which is analytic in* t, *such that* $\mathbf{p}(0) = \mathbf{p}$, $\mathbf{p}(1) = \mathbf{q}$ *and for* $0 \le t \le 1$, $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ *is monotone.*

This follows easily from the Euclidean case Lemma 5.1, but notice that the ambient space has dimension $2d + 1$ instead of $2d$. If $2d + 1$

could be replaced with $2d$ even for $d = 2$, it would yield an extension of the main result (about unions or intersections of spherical disks) in [Bezdek-Connelly] to $\mathbb{S}^2$ following a method of Csikos. Similarly, there would be another extension to $\mathbb{H}^2$ if Conjecture 5.2 were true. Even if these conjectures are true, it is clear that there seems to be some particular asymmetry between spherical space and hyperbolic space.

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