ROBERT CONNELLY* AND STEVEN J. OSTRO**

ELLIPSOIDS AND LIGHTCURVES

1. Introduction

Even the largest of the ~ 3000 catalogued asteroids are too small to be resolved by the largest optical telescopes on Earth. Therefore, an asteroid's physical properties (e.g. dimensions, spin vector, spatial orientation) must be deduced indirectly from measurements of its total (disc-integrated) brightness. In general, brightness-versus-time functions, or *lightcurves*, are periodic because asteroids rotate and are not homogeneously scattering spheres. The form of a lightcurve clearly depends on the shape and light-scattering properties of the asteroid as well as on the viewing geometry.

Although many asteroidal shapes are possible, here we consider the case when the shape is an ellipsoid, that is, the region bounded by a surface given by the equation:

$$(1) \qquad (x/a)^2 + (y/b)^2 + (z/c)^2 = 1,$$

where a, b, c are the semi-axes and satisfy $a \geqslant b \geqslant c$. For dynamical reasons we assume that this ellipsoid rotates about the c-axis. Theoretical arguments [1] and observational evidence [2] suggest that ellipsoids provide decent approximations to the gross shapes of at least the largest (maximum dimension $\gtrsim 100$ km) asteroids.

To complete this model, we must assume some light-scattering law for the asteroid surface, and then integrate over the visible, illuminated portion of the surface to determine the intensity of light reflected toward Earth. For simplicity, we merely take a plane perpendicular to the line of sight from the asteroid to the Earth, and then orthogonally project the visible, illuminated portion of the ellipsoid into this plane. The asteroid's brightness is then assumed to be proportional to this projected, visible, illuminated area (see Figure 1). For instance, the MAA logo in Figure 2 does not obey this 'geometric' scattering law. We assume that the Sun and the Earth are points infinitely far from the asteroid, so we can ignore parallax effects.

What can be said about the lightcurve of a geometrically scattering ellipsoid? In answering this question, it is helpful to have a simple, concise formula

* Partial support of this research was provided by NSF Grant MCS77-01740. The author gratefully acknowledges the support of I.H.E.S.
** Partial support of this research was provided by NASA Grant NAGW-543.

Geometriae Dedicata **17** (1984) 87–98. 0046-5755/84/0171-0087$01.80.

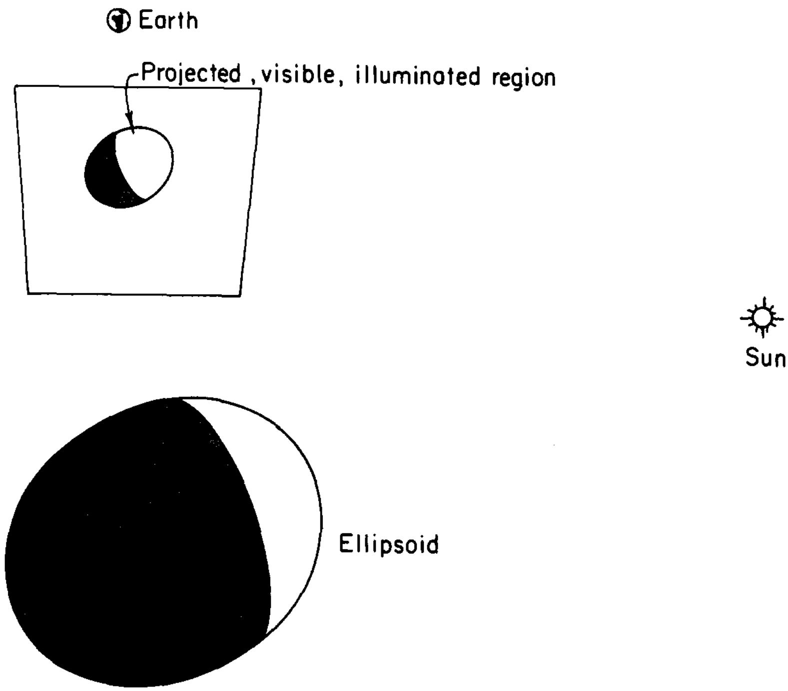

Fig. 1. The brightness of a geometrically scattering ellipsoid, as seen from Earth, is proportional to the projected, visible, illuminated portion of the ellipsoid.

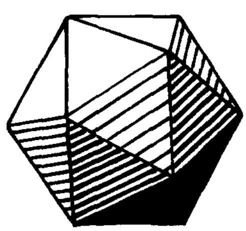

Fig. 2. A sketch of part of the logo of the Mathematical Association of America. The eight sides of the icosahedron that are both visible and illuminated have brightnesses that depend on viewing and/or illumination angle, so this figure's light-scattering law is not 'geometric'.

for the area of a projection of an ellipsoid. In this paper we derive such an expression, and use it to obtain a general formula for the projected, visible, illuminated area of a triaxial ellipsoid for arbitrary Sun–Earth–asteroid geometry. It turns out that the lightcurve of an ellipsoid has special properties that can be exploited to test the hypothesis that a given optical or radar lightcurve could be due to a geometrically scattering ellipsoid. (See [3] and [4] for related discussions.)

2. Projected areas of ellipsoids

In this section, we assume 'opposition geometry', i.e. that the source of illumination (e.g. the Sun or a radar transmitter) and the detector (e.g. an Earthbound optical telescope or radar receiver) are in the same asteroid-centric direction. Whereas most optical observations of asteroids are conducted with the asteroid-centric directions of the Sun and Earth at least several degrees apart, radar observations come very close to realizing true opposition geometry, because a single telescope serves as transmitter and receiver.

Given an ellipsoid such as (1) and a direction, what is the area of the projection or shadow (as in Figure 3) onto a plane perpendicular to this direction? The answer is surprisingly simple, and can be expressed concisely using matrix notation.

We rewrite (1) as:

$$\mathbf{x}^t\mathbf{Q}\mathbf{x} = 1, \tag{2}$$

where

$$\mathbf{Q} = \begin{pmatrix} (1/a)^2 & 0 & 0 \\ 0 & (1/b)^2 & 0 \\ 0 & 0 & (1/c)^2 \end{pmatrix}$$

$\mathbf{x}^t = (x\,y\,z)$, $\quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $(\ \ )^t$ denotes transpose.

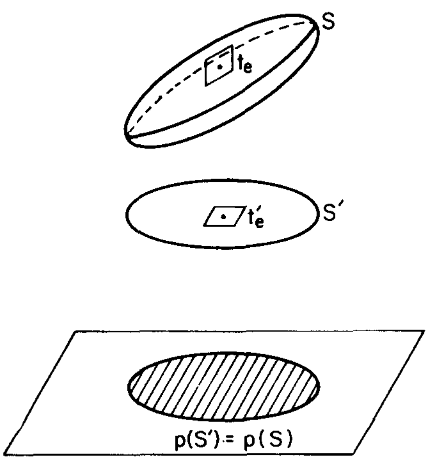

Fig. 3. Projection of an ellipsoid onto the plane perpendicular to a specified direction. The cross-hatched area A is given by Theorem 1.

Note that if (1) had xy, xz, or yz terms, (2) would be the same but $\mathbf{Q}$ would have some nonzero off-diagonal elements. We choose a coordinate system with origin at the center of the ellipsoid. Suppose that

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

is a unit vector in the direction of the Earth (so $e_1^2 + e_2^2 + e_3^2 = 1$). Note that the determinant of $\mathbf{Q}$, $\det \mathbf{Q}$, is $(1/abc)^2$. In the following, we regard one-by-one matrices as scalars, and we regard $\mathbf{Q}$ as a symmetric matrix, not necessarily diagonal.

LEMMA. *With the ellipsoid defined by* (2), *the following statements about a particular vector* $\mathbf{x}$ *are equivalent*:

(i) $\mathbf{x}$ *is an eigenvector for* $\mathbf{Q}$ (*i.e.* $\mathbf{Qx} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$);

(ii) $\mathbf{x}$ *lies on one of the axes of the ellipsoid defined by* (2), *or, if two axes are the same length,* $\mathbf{x}$ *lies in the plane containing them, or* (2) *is a sphere*;

(iii) *the tangent plane to* (2) *in the direction* $\mathbf{x}$ (*i.e. at the point* $(1/\mathbf{x}^t\mathbf{Qx})\mathbf{x}$) *is perpendicular to* $\mathbf{x}$.

Proof. There is a basis for 3-space consisting of vectors that are mutually perpendicular and all eigenvectors for $\mathbf{Q}$. This is the standard polar decomposition theorem. With respect to this basis formula, (2) takes the form (1). Thus (i) and (ii) are equivalent if the axes are of different lengths. The other cases follow easily.

For (iii) note that the gradient of the function $\mathbf{x}^t\mathbf{Qx}$ is the vector $2\mathbf{Qx}$. Thus $\mathbf{Qx}$ is perpendicular to the tangent plane of (2) at $(1/\mathbf{x}^t\mathbf{Qx})\mathbf{x}$, and $\mathbf{x}$ is perpendicular to this tangent plane if and only if $\mathbf{x}$ is parallel to $\mathbf{Qx}$, i.e. if and only if $\mathbf{x}$ is an eigenvector for $\mathbf{Q}$. Therefore (i) is equivalent to (iii).

THEOREM 1. *The area* A *of the projection of the ellipsoid* (1) *or* (2) *onto the plane perpendicular to* $\mathbf{e}$ *is given by*:

$$\text{(3)} \qquad \begin{aligned} A &= \pi abc[(e_1/a)^2 + (e_2/b)^2 + (e_3/c)^2]^{1/2} \\ &= \pi(\mathbf{e}^t\mathbf{Qe}/\det \mathbf{Q})^{1/2}. \end{aligned}$$

Proof. By the remarks above, if $\mathbf{T}$ is an orthogonal matrix, then in a coordinate system defined by $\mathbf{T}$, the area of the projection remains the same;

defining $\mathbf{Q}' = \mathbf{T}^t\mathbf{Q}\mathbf{T}$ and $\mathbf{e}' = \mathbf{Te}$, then $\det \mathbf{Q} = \det \mathbf{Q}'$, $|\mathbf{e}'| = |\mathbf{e}| = 1$. Thus we know by Lemma 1 that (3) holds (in the form of the last right-hand expression) for any vector $\mathbf{e}$ such that the following hold:

$$|\mathbf{e}| = 1$$

$$\mathbf{e} \text{ is an eigenvector for } \mathbf{Q},$$

where A in (3) is the projection of an ellipsoid onto a plane perpendicular to $\mathbf{e}$.

Now let us remove the condition that $\mathbf{e}$ be an eigenvector for $\mathbf{Q}$. Let $p(\mathbf{x}) = \mathbf{x} - (\mathbf{e}\cdot\mathbf{x})\mathbf{e}$ be the orthogonal projection of a vector $\mathbf{x}$ onto the plane perpendicular to $\mathbf{e}$. Then we find a linear function $\mathbf{T}\colon R^3 \to R^3$ such that

(a) $\mathbf{T}(\mathbf{e}) = \mathbf{e}$;
(b) $p(\mathbf{x}) = p[T(\mathbf{x})]$, for all $\mathbf{x}$ in R^3;
(c) $\mathbf{T}(e^\perp) = t_e$, where $e^\perp$ is the plane perpendicular to $\mathbf{e}$ in R^3, and t_e is the tangent plane to (2) at $(1/\mathbf{e}^t\mathbf{Q}\mathbf{e})\mathbf{e}$.

We see that there is one and only one $\mathbf{T}$ as follows. Complete $\mathbf{e}$ to an orthogonal basis of R^3. (a) defines $\mathbf{T}(\mathbf{e})$. (b) and (c) define the images of the other basis vectors. Clearly (b) and (c) hold if they hold on a basis.

Call the new matrix $\mathbf{Q}' = \mathbf{T}^t\mathbf{Q}\mathbf{T}$. Let S be the surface of the ellipsoid defined by (2), and S' be the corresponding surface for $\mathbf{Q}'$. Let A' be the area of the projection, defined by p, of S'.

(b) implies that $p(S) = p(S')$, and thus $A = A'$.

Let t'_e be the tangent plane of S' at the point on the ray through $\mathbf{e}$, $(1/\mathbf{e}^t\mathbf{Q}'\mathbf{e})\mathbf{e}$. (c) implies that t'_e is perpendicular to $\mathbf{e}$. Thus (by the Lemma), $\mathbf{e}$ is an eigenvector for $\mathbf{Q}'$, so formula (3) applies to $\mathbf{e}$ and $\mathbf{Q}'$.

(a), (b), and (c) together imply that $\det \mathbf{T} = 1$, and thus $\det \mathbf{Q} = \det \mathbf{Q}'$. (Geometrically $\mathbf{T}$ is a 'shear' parallel to $\mathbf{e}$.)

Collecting these results, we obtain:

$$\begin{aligned} A &= A' \\ &= \pi(\mathbf{e}^t\mathbf{Q}'\mathbf{e}/\det \mathbf{Q}')^{1/2} \\ &= \pi(\mathbf{e}^t\mathbf{Q}\mathbf{e}/\det \mathbf{Q}')^{1/2} \\ &= \pi(\mathbf{e}^t\mathbf{Q}\mathbf{e}/\det \mathbf{Q})^{1/2}, \end{aligned}$$

which is the desired formula. This completes the proof.

Suppose that we have an ellipsoid whose equation is given by the matrix

$\mathbf{T}^t\mathbf{Q}\mathbf{T}$, where $\mathbf{Q}$ is diagonal as in (2) and

$$\mathbf{T} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\alpha & 0 & -\cos\alpha \\ 0 & 1 & 0 \\ \cos\alpha & 0 & \sin\alpha \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix}.$$

Here θ is the angle of rotation of the asteroid, α is the angle from the asteroid's spin vector to the direction of Earth, and ω (defined in Section 3) is zero at opposition. By (3), any lightcurve taken at opposition must have the form:

$$A = \pi abc[(\sin\alpha \sin\theta/a)^2 + (\sin\alpha\cos\theta/b)^2 + (\cos\alpha/c)^2]^{1/2}. \tag{4}$$

In the language of Fourier analysis, this says that the square of the lightcurve has only the zeroth and second harmonics. The presence of any other harmonics in an opposition lightcurve indicates that one of our assumptions is invalid, i.e. either the asteroid is not an ellipsoid or the back-scattering law is not geometric, or both.

3. Nonopposition geometry

We now generalize Theorem 1 to the situation when the asteroid-centric directions $\mathbf{e}$ and $\mathbf{s}$ of the Earth and Sun are not equal. In this case, the solar phase angle φ between $\mathbf{e}$ and $\mathbf{s}$ is nonzero, and it is convenient to define the obliquity angle ω as the dihedral angle between the plane containing $\mathbf{e}$ and $\mathbf{e} \times \mathbf{s}$ and the plane containing $\mathbf{e}$ and the spin axis.

In the following, we assume that the ellipsoid has Equation (2), with $\mathbf{Q}$ any (not necessarily diagonal) symmetric matrix.

THEOREM 2. *The area A of the projection onto the plane perpendicular to* $\mathbf{e}$ *of the visible portion of the ellipsoid* (2), *illuminated in the given direction* $\mathbf{s}$, *is given by*:

$$A = \pi abc[(\mathbf{e}^t\mathbf{Q}\mathbf{e})^{1/2}(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2} + \mathbf{e}^t\mathbf{Q}\mathbf{s}]/2(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2}. \tag{5}$$

Proof. The projection of the whole ellipsoid onto the plane P perpendicular to $\mathbf{e}$ has as its boundary an ellipse E_1, and Theorem 1 implies that the area bounded by E_1 is given by (3).

The *terminator*, or boundary between the illuminated and unilluminated portions of the ellipsoid, is an ellipse that is the intersection of a plane P_t

through the center of the ellipsoid with the ellipsoid itself. The projection of this terminator into the plane perpendicular to $\mathbf{e}$ is yet another ellipse E_2.

It is clear that E_1 and E_2 are tangent and share the same center. Thus the projected, visible, illuminated region is bounded by arcs of E_1 and E_2. (See Figure 4.)

If A_1 and A_2 are the areas of E_1 and E_2, respectively, then the projected, visible, illuminated area A is either $(A_1 + A_2)/2$ or $(A_1 - A_2)/2$, depending on whether the Earth and the Sun are on the same (+) or opposite (−) side(s), respectively, of the plane P_t of the terminator. (This result follows from the symmetry of both E_1 and E_2.) Since $A_1 = \pi abc[\mathbf{e}^t\mathbf{Q}\mathbf{e}]^{1/2}$ by (3), we need only find A_2. By Theorem 1, the projection of the illuminated portion of the ellipsoid onto the plane perpendicular to $\mathbf{s}$ is given by

$$A_s = \pi abc[\mathbf{s}^t\mathbf{Q}\mathbf{s}]^{1/2}.$$

By the discussion preceding Theorem 1, $\mathbf{Qs}$ is perpendicular to the terminator plane P_t. Let β be the dihedral angle between P_t and the plane perpendicular to s. Then

$$A_s/A_t = \cos\beta = \mathbf{s}\cdot\mathbf{Qs}/|\mathbf{Qs}| = \mathbf{s}^t\mathbf{Qs}/|\mathbf{Qs}|,$$

where A_t is the area of P_t contained by the ellipsoid. Applying the same reasoning to A_t and A_2 yields

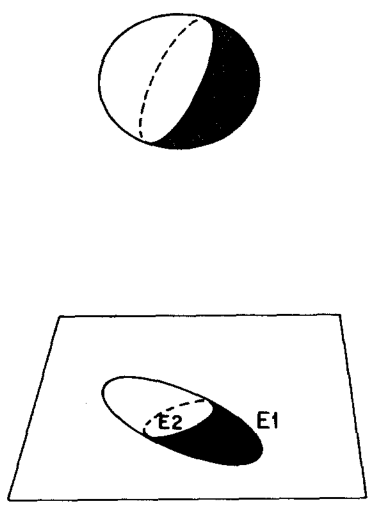

Fig. 4. Projected, visible, illuminated area A of the geometrically scattering ellipsoid (top) is the average of the areas A_1 and A_2 of the ellipses E_1 and E_2. These ellipses are the projections (in the plane perpendicular to the ellipsoid–Earth line) of the ellipsoid and the terminator.

$$A_2/A_t = |\mathbf{e}^t\mathbf{Q}\mathbf{s}|/|\mathbf{Q}\mathbf{s}|.$$

Combining these formulas gives

$$A_2 = \pi abc|\mathbf{e}^t\mathbf{Q}\mathbf{s}|/[\mathbf{s}^t\mathbf{Q}\mathbf{s}]^{1/2}. \tag{6}$$

Since $\mathbf{Qs}$ is perpendicular to the terminator plane P_t, $\mathbf{e}$ is on the same side of P_t as $\mathbf{s}$ if and only if $\mathbf{e}^t\mathbf{Qs}$ is positive. Thus the formula above for A_2, with absolute value signs removed, gives the desired signed area. Combining this result with the formula for A_1 gives (5) and finishes the proof.

4. THE OPPOSITION LIGHTCURVE AS AN APPROXIMATION TO A NONOPPOSITION LIGHTCURVE

The general formula (5) reduces to the simpler formula (3) for the special case of opposition ($\mathbf{e} = \mathbf{s}$, $\varphi = 0$). In the previous section's notation, the extent to which A_1 approximates A increases as $|\varphi|$ decreases. Furthermore, whereas opposition lightcurves can have only zeroth and second harmonics, non-opposition lightcurves can also have higher even harmonics.

In this section, we use the Fourier properties of ellipsoid lightcurves to derive a constraint on how well A_1 approximates A. This constraint depends on the ellipsoid's maximum axis ratio (a/c) as well as on the solar phase angle (φ), and provides a simple test of the hypothesis that a given lightcurve could actually be due to a geometrically scattering ellipsoid with a/c no larger than an *a priori* upper bound, m.

Let ε be the upper bound on the fractional error incurred by using A_1 as an approximation to A:

$$(A_1 - A)/A \leqslant \varepsilon$$

or, since $A = (A_1 + A_2)/2$,

$$(A_1 - A_2)/(A_1 + A_2) \leqslant \varepsilon, \tag{7}$$

where A_2 can be negative.

We defer evaluation of ε to Section 5, digressing here to derive a constraint on the Fourier properties of A^2. Manipulating (7),

$$A_1 - A \leqslant \varepsilon A$$

$$A_1 \leqslant (1 + \varepsilon)A$$

$$A_1^2 \leqslant (1 + 2\varepsilon + \varepsilon^2)A^2$$

$$A_1^2 - A^2 \leqslant (2\varepsilon + \varepsilon^2)A^2$$

$$(A_1^2 - A^2)^2 \leqslant \tau A^4$$

where $\tau = (2\varepsilon + \varepsilon^2)^2$. Recalling that A_1 and A are functions of the rotational phase angle, θ, we integrate over a full cycle:

$$(1/\pi)\int_0^{2\pi} (A_1^2 - A^2)^2\, d\theta \leqslant \tau(1/\pi)\int_0^{2\pi} A^4\, d\theta. \tag{8}$$

We expand the pertinent functions as Fourier series:

$$A^2 = \sum_{n=0}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$A_1^2 = \hat{a}_0 + \hat{a}_2 \cos 2\theta + \hat{b}_2 \sin 2\theta$$

$$(1/\pi)\int_{n=0}^{2\pi} A^4\, d\theta = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$(1/\pi)\int_0^{2\pi} (A_1^2 - A^2)^2\, d\theta$$
$$= 2(a_0 - \hat{a}_0)^2 + (a_2 - \hat{a}_2)^2 + (b_2 - \hat{b}_2)^2 + \sum_{\substack{n \neq 0,2 \\ n > 0}} (a_n^2 + b_n^2).$$

Combining this result with (8) we get

$$\sum_{n \neq 0,2} (a_n^2 + b_n^2) \leqslant (1/\pi)\int_0^{2\pi} (A_1^2 - A^2)^2\, d\theta \leqslant \tau(1/\pi)\int_0^{2\pi} A^4\, d\theta$$
$$= \tau\left(2a_0^2 + \sum_{n \geqslant 1} (a_n^2 + b_n^2)\right).$$

Thus we finally obtain our test

$$\hat{\tau} = \sum_{n \neq 0,2} (a_n^2 + b_n^2) \Bigg/ \left(2a_0^2 + \sum_{n \geqslant 1} (a_n^2 + b_n^2)\right) \leqslant \tau = (2\varepsilon + \varepsilon^2)^2, \tag{9}$$

where the test statistic $\hat{\tau}$ can be calculated from the actual lightcurve, A. If (9) is violated, then the asteroid is not an ellipsoid or the scattering law is not geometric, or both.

5. Evaluation of ε

The following theorem indicates that the upper bound ε in (7) should be equated to $(a/c)^2 \tan^2(\varphi/2)$, where a/c is the ellipsoid's maximum axis ratio (Section 1). This completes the definition of τ in (9).

THEOREM 3. *The maximum value of* $(A_1 - A_2)/(A_1 + A_2)$ *is*

$$(a/c)^2 \tan^2(\varphi/2).$$

Proof. We apply (5) and (6) to the above expression:

$$\frac{A_1 - A_2}{A_1 + A_2} = \frac{(\mathbf{e}^t\mathbf{Q}\mathbf{e})^{1/2} - \mathbf{e}^t\mathbf{Q}\mathbf{s}/(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2}}{(\mathbf{e}^t\mathbf{Q}\mathbf{e})^{1/2} + \mathbf{e}^t\mathbf{Q}\mathbf{s}/(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2}} = \frac{(\mathbf{e}^t\mathbf{Q}\mathbf{e})^{1/2}(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2} - \mathbf{e}^t\mathbf{Q}\mathbf{s}}{(\mathbf{e}^t\mathbf{Q}\mathbf{e})^{1/2}(\mathbf{s}^t\mathbf{Q}\mathbf{s})^{1/2} + \mathbf{e}^t\mathbf{Q}\mathbf{s}}. \tag{10}$$

Write $\mathbf{Q} = \mathbf{B}^t\mathbf{B}$ for some matrix $\mathbf{B}$. This is always possible since (2) defines an ellipsoid (i.e. is positive definite). Write $\mathbf{Be} = \bar{\mathbf{e}}$, $\mathbf{Bs} = \bar{\mathbf{s}}$. So using $\cdot$ to denote the inner product, we have

$$\bar{\mathbf{e}}\cdot\bar{\mathbf{s}} = \mathbf{Be}\cdot\mathbf{Bs} = \mathbf{e}^t\mathbf{B}^t\mathbf{Bs} = \mathbf{e}^t\mathbf{Q}\mathbf{s}.$$

Thus we rewrite (10) as

$$\frac{A_1 - A_2}{A_1 + A_2} = \frac{|\bar{\mathbf{e}}||\bar{\mathbf{s}}| - \bar{\mathbf{e}}\cdot\bar{\mathbf{s}}}{|\bar{\mathbf{e}}||\bar{\mathbf{s}}| + \bar{\mathbf{e}}\cdot\bar{\mathbf{s}}} = \frac{1 - \cos\bar{\varphi}}{1 + \cos\bar{\varphi}} = \tan^2(\bar{\varphi}/2)$$

where $\bar{\mathbf{e}}\cdot\bar{\mathbf{s}} = |\bar{\mathbf{e}}||\bar{\mathbf{s}}|\cos\bar{\varphi}$, and $\bar{\varphi}$ is the angle between $\bar{\mathbf{e}}$ and $\bar{\mathbf{s}}$. Thus (10) is maximized when $\bar{\varphi}$ is maximized, since the tangent function is monotone increasing for $0 \leqslant \bar{\varphi}/2 \leqslant \pi/2$.

Notice that the right-hand side of (10) is defined for any positive definite symmetric matrix $\mathbf{Q}$ of any size. In particular, we will first consider the case when $\mathbf{Q}$ is 2×2. We can think of this as restricting the given $\mathbf{Q}$ to the plane of $\mathbf{e}$ and $\mathbf{s}$. Replacing $\mathbf{Q}$ by a scalar times $\mathbf{Q}$ (or similarly for $\mathbf{B}$) does not affect the value of (10), and we can choose an orthogonal basis so that $\mathbf{Q}$ is diagonal. All these considerations imply that we choose the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix},$$

where $m > 1$ is the maximum axis ratio for the ellipse defined by (2). In this coordinate system

$$\mathbf{e} = \begin{pmatrix} \cos\theta_e \\ \sin\theta_e \end{pmatrix}, \qquad \mathbf{s} = \begin{pmatrix} \cos(\theta_e + \varphi) \\ \sin(\theta_e + \varphi) \end{pmatrix}$$

The transformation defined by the matrix $\mathbf{B}$ takes lines that make an angle θ with the x-axis into lines that make an angle $\bar{\theta}$ with the x-axis. Define

$$f(\theta) = \mathbf{B}\mathbf{x}/|\mathbf{B}\mathbf{x}| = (\cos^2\theta + m^2\sin^2\theta)^{-1/2}\begin{pmatrix}\cos\theta \\ m\sin\theta\end{pmatrix} = \begin{pmatrix}\cos\bar{\theta} \\ \sin\bar{\theta}\end{pmatrix},$$

where $\mathbf{x} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$. Note that $\tan\bar{\theta} = m\tan\theta$. Thus, if θ_i corresponds to $\bar{\theta}_i$ for $i = 0, 1$, then

$$\bar{\theta}_1 - \bar{\theta}_0 = \int_{\theta_0}^{\theta_1} |f'(\theta)|\,\mathrm{d}\theta.$$

For $\theta_1 = \theta_e + \varphi, \theta_0 = \theta_e$,

$$\bar{\varphi} = \int_{\theta_e}^{\theta_e+\varphi} |f'(\theta)|\,\mathrm{d}\theta. \tag{11}$$

The problem is to find θ_e such that (11) is maximized for fixed φ. However, $|f'(\theta)|$ has a relative maximum only for multiples of π, and is symmetric about 0. Explicitly,

$$|f'(\theta)| = \mathrm{d}\bar{\theta}/\mathrm{d}\theta = m/(\cos^2\theta + m^2\sin^2\theta).$$

Thus the maximum of (11) occurs when the interval $\theta_e \leqslant \theta \leqslant \theta_e + \varphi$ is symmetrically placed about 0 or π, i.e. $\theta_e = -\varphi/2$. In either case, the right-hand side of (10), using the special form for $\mathbf{Q}$, simplifies to

$$m^2\tan^2(\varphi/2). \tag{12}$$

The intersection of any plane through the center of the ellipsoid (2) is an ellipse and (12) serves as a maximum for (10) for $\mathbf{e}$·s on that ellipse. The only difference from one ellipse to another in (12) is m, the maximum axis ratio, since φ is assumed to be constant. But for the ellipsoid (1) the major and minor semiaxes (a and c, respectively) correspond to the directions of the points on the surface having the maximum and minimum distances from the center of the ellipsoid. Thus a/c represents the maximum axis ratio for any ellipse through the center. Thus, for $m = a/c$, $\varepsilon = m^2\tan^2(\varphi/2)$ is the upper bound sought for $(A - A_1)/A$. See [3] for applications of Theorem 3 to interpretation of actual asteroid lightcurves.

ACKNOWLEDGEMENTS

The authors thank O. Gabber, N. H. Kuiper and L. O'Raifeartaigh for helpful conversations. The authors also thank C. Ikuta for secretarial assistance and B. Boettcher for drafting the figures.

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(Received May 4, 1984)