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On Hyperplanes and Polytopes

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Abstract. We call a convex subset N of a convex d-polytope $P \subset E^d$ a k-nucleus of P if N meets every k-face of P, where $0 < k < d$. We note that P has disjoint k-nuclei if and only if there exists a hyperplane in E^d which bisects the (relative) interior of every k-face of P, and that this is possible only if $\left[\frac{d+2}{2}\right] \leqslant k \leqslant d-1$. Our main results are that any convex d-polytope with at most $2d-1$ vertices ($d \geqslant 3$) possesses disjoint $(d-1)$-nuclei and that $2d-1$ is the largest possible number with this property. Furthermore, every convex d-polytope with at most $2d$ facets ($d \geqslant 3$) possesses disjoint $(d-1)$-nuclei, $2d$ cannot be replaced by $2d+2$, and for $d = 3$, six cannot be replaced by seven.

§ 0. Introduction. Let $P \subset E^d$ be a convex d-polytope with $f_j(P)$ j-dimensional faces, $0 \leqslant j \leqslant d-1$ and $d \geqslant 3$. A convex subset of P is called a *k-nucleus* of P if it meets every k-face of P, $0 < k < d$. (For the terminology, the reader should consult [3].) It is easy to check (Lemma 2) that P possesses disjoint k-nuclei if and only if there exists a hyperplane which bisects the (relative) interior of every k-face of P. One might expect that there exists a P with a "large" number of j-dimensional faces such that there are no disjoint k-nuclei of P. Nevertheless, we show (Lemma 2) that there exists a convex d-polytope in E^d having disjoint k-nuclei if and only if $\left[\frac{d+2}{2}\right] \leqslant k \leqslant d-1$, $d \geqslant 3$ ($[x]$ means the largest possible integer which is not greater than $x \in \mathbb{R}$.) Thus the following question arises naturally: What is the largest possible integer $F_j(d,k)$ ($d \geqslant 3$,

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$0 \leqslant j \leqslant d-1, \left\lceil \frac{d+2}{2} \right\rceil \leqslant k \leqslant d-1$) such that $f_j(P) \leqslant F_j(d,k)$ implies that $P \subset E^d$ possesses disjoint k-nuclei? Lemma 2 shows that $d+1 \leqslant F_j\left(d, \left\lceil \frac{d+2}{2} \right\rceil\right) \leqslant F_j\left(d, \left\lceil \frac{d+2}{2} \right\rceil + 1\right) \leqslant \ldots \leqslant F_j(d, d-1)$ for any $0 \leqslant j \leqslant d-1$. We also show that $F_0(d, d-1) = 2d-1$, $d \geqslant 3$ (Theorem 1, Remark 1), $2d \leqslant F_{d-1}(d, d-1) \leqslant 2d+1$, $d \geqslant 4$ (Theorem 3, Remark 3) and that $F_2(3,2) = 6$ (Theorem 3, Remark 2). As an open problem, we ask whether there exists a convex d-polytope in E^d, $d \geqslant 4$, with $2d+1$ facets such that it possesses no disjoint $(d-1)$-nuclei? If such a polytope exists, then it does not have a simple vertex (Theorem 4). Finally, it would be very interesting to determine $F_j(d,k)$ in some other cases as well. Since our problem is connected with sections of polytopes, it is worth mentioning that some other problems about convex polytope cross-sections can be found in [1], [2], [4], [5], [6] and [8].

We wish to remark that work on this article was started at the Department of Geometry of Eötvös Lòrànd University (Budapest) in 1986.

§ **1.** We start with the following important observation.

Lemma 1. *If P is a convex d-polytope in E^d and H is a hyperplane of E^d $(d \geqslant 1)$ which does not contain any vertex of P, then on at least one side of H there exists a face of P of dimension at least* $\left\lceil \frac{d}{2} \right\rceil$.

Proof. Since the statement has already stimulated many nice generalizations, we omit its simple proof. (See [2] and also [1], [4], and [5].)

Definition. Let $H \subset E^d$ be a hyperplane, $d \geqslant 1$. Then H *bisects* a subset S of E^d if there are points p_1 and p_2 in $S \setminus H$ which are on opposite sides of H.

As an immediate consequence, we get the following.

Corollary. *Let P be a convex d-polytope in E^d $(d \geqslant 1)$ and let k be an integer such that $0 \leqslant k \leqslant \left\lceil \frac{d}{2} \right\rceil$. Then there is no hyperplane of E^d which bisects the (relative) interior of every k-face of P.*

Lemma 2. *For the integers $0 < k < d$, there exists a convex d-polytope P in E^d and a hyperplane (in E^d) which bisects the (relative) interior of every k-face of P if and only if* $\left[\frac{d+2}{2}\right] \leqslant k \leqslant d-1$. *$P$ possesses disjoint k-nuclei if and only if there is a hyperplane in E^d which bisects the (relative) interior of every k-face of P.*

Proof. If P is a convex d-polytope in E^d and H is a hyperplane of $E^d (d \geqslant 1)$ which bisects the (relative) interior of every k-face of P, then we may suppose that H does not contain any vertex of P. Thus $k \geqslant \left[\frac{d}{2}\right] + 1 = \left[\frac{d+2}{2}\right]$ by Lemma 1. On the other hand, if $\left[\frac{d+2}{2}\right] \leqslant k \leqslant d-1$, then the case of a regular d-simplex of E^d shows the existence of a convex d-polytope P in E^d and of a hyperplane H (in E^d) which bisects the (relative) interior of every k-face of P. (We split the set of the vertices of a regular d-simplex by a hyperplane of E^d into two sets of cardinality $\left[\frac{d+1}{2}\right]$ and $\left[\frac{d+2}{2}\right]$. The sets determine simplices of dimension $\left[\frac{d-1}{2}\right]$ and $\left[\frac{d}{2}\right]$. Consequently, any k-face of our regular d-simplex with $\left[\frac{d}{2}\right] + 1 = \left[\frac{d+2}{2}\right] \leqslant$ $\leqslant k \leqslant d-1$ will contain some vertices from both sets; that is, a hyperplane bisects the (relative) interior of every k-face.)

Finally, any k-nucleus of P contains a compact convex subset of P which is also a k-nucleus. Hence if P possesses two disjoint k-nuclei, then we may suppose that there is a hyperplane H which is disjoint from the k-nuclei and separates them. Since each k-face F of P meets both components of $P \setminus H$, H bisects the (relative) interior of F. Conversely, let H be a hyperplane which bisects the (relative) interiors of the k-faces of P. Let H' be parallel and close to H. Then $H \cap P$ and $H' \cap P$ are disjoint k-nuclei of P. □

Theorem 1. *Let P be a convex d-polytope such that $f_0(P) \leqslant 2d-1$, $d \geqslant 3$. Then P possesses disjoint $(d-1)$-nuclei.*

Proof. We fix a facet of P, say, F^*. Since F^* has at least d vertices, there exist $k \geqslant 0$ and $m \geqslant 1$ such $k + m \leqslant d-1$,

$P = \operatorname{conv}\{p_1, \ldots, p_{d+k}, q_1, \ldots, q_m\}$ and $F^* = \operatorname{conv}\{p_1, \ldots, p_{d+k}\}$. Since $m \leqslant d-1$, every facet of P contains a vertex of F^*. (But no facet of P contains all the vertices of F^*, except F^*.) Consequently, since $d \geqslant 3$, there is a vertex, say, p_1 of F^* such $N_1 = \operatorname{conv}\{q_1, \ldots, q_m, p_1\}$ is not a facet of P. (This is obvious for $m < d-1$, and for $m = d-1$, it is enough to observe that if N_1 is a facet then $\operatorname{conv}\{q_1, \ldots, q_{d-1}, p_1\}$ is a $(d-1)$-simplex. So $\operatorname{conv}(q_1, \ldots, q_{d-1}\}$ is a $(d-2)$-face of P belonging to two facets of P.) We now set $N_2 = \operatorname{conv}\{p_2, p_3, \ldots, p_{d+k}\}$ and observe that $N_1 \cap N_2 = \emptyset$.

Let F be a facet of P. If $F = F^*$, then $p_1 \in N_1 \cap F$ and $p_{d+k} \in N_2 \cap F$. If $F \neq F^*$, then of course some $q_i \in N_1 \cap F$. The preceding argument shows that $N_1 \neq F$ and thus, some $p_j \in N_2 \cap F$ where $2 \leqslant j \leqslant d+k$. Hence N_1 and N_2 are disjoint $(d-1)$-nuclei of P. □

Remark 1. For any $d \geqslant 3$, there is a convex d-polytope P with $2d$ vertices such that no hyperplane meets the (relative) interior of every facet of P.

In E^d, let p_i be the point with 1 in its i'th coordinate and zero elsewhere and $q_i = -p_i$; $i = 1, \ldots, d$ and $d \geqslant 3$. Then the set $P^d = \operatorname{conv}\{p_1, \ldots, p_d, q_1, \ldots, q_d\}$ is a *d-crosspolytope* (cf. [3], p. 55). Suppose that there is a hyperplane H in E^d which meets the (relative) interior of every facet of P^d. Then we may assume that H contains no vertex of P^d. Let H^+ and H^- be two open half-spaces bounded by H. We claim that if $\{p_i, q_i\} \subset H^+$ for some $i = 1, \ldots, d$, then $\{p_j, q_j\} \cap H^+ \neq \emptyset$ for all $j = 1, \ldots, d$. If not, then for some $j \neq i$, $\{p_j, q_j\} \subset H^-, \dfrac{p_j + q_j}{2} = 0 \in H^-$ and $\dfrac{p_i + q_i}{2} = 0 \in H^+$; a contradiction. Thus by relabelling, we may assume that $\{p_1, \ldots, p_d\} \in H^+$. Recall that P^d has 2^d facets, and each facet is a $(d-1)$-simplex which contains either p_i or q_i, $i = 1, \ldots, d$. Therefore $\operatorname{conv}\{p_1, \ldots, p_d\}$ is a facet contained in H^+. Thus we have shown that H contains a facet of P^d on one side, which is a contradiction. Hence P^d does not have disjoint $(d-1)$-nuclei (see Lemma 2). We remark that every $(d-1)$-nucleus of P^d contains the origin of E^d. □

§2. We now introduce the results which we use for our examination of d-polytopes with a "large" number of facets.

A graph G on a set of vertices V is a subset of unordered pairs (x, y), $x, y \in V$. The elements of G are called edges. G is *simple* if it has no loops or parallel edges. A *1-factor* of G is a system of independent edges (no two of them have an endpoint in common) covering all the vertices of V. We shall use the following well known fact.

Theorem 2. *Let G be a simple graph on $2n$ vertices with all degrees at least n. Then G has a 1-factor.*

Proof. ([7], p. 51) We suppose that G has no 1-factor. Then there exists a matching F in G, with a maximum number of edges, and two vertices u, v not in F. (A matching in G is a collection of edges of G such that each vertex belongs to at most one of them.) Let the edge $(x, y) \in F$. If there are at least three edges joining $\{x, y\}$ to $\{u, v\}$ then there are two of them which are independent, say, (x, u) and (y, v). But then $(F \setminus \{(x, y)\}) \cup \{(x, u), (y, v)\}$ is a bigger matching then F; a contradiction. So each edge of F is joined to $\{u, v\}$ by at most two edges. Since u and v are not joined to each other or to any point not on the edges of F (otherwise, F would not be maximal), this implies that $\deg(u) + \deg(v) \leqslant 2 \cdot |F| \leqslant 2(n-1)$. This, however, contradicts the hypothesis that $\deg(u)$ and $\deg(v)$ are at least n. □

Lemma 3. *Let $A_1, A_2, \ldots, A_d$ be d convex sets in E^d of dimension at least $d-2$, $d \geqslant 3$. Then*

(2.1) *there is a $(d-2)$-flat which contains two of the A_i,*

or

(2.2) *there is a hyperplane which contains all the A_i,*

or

(2.3) *there is a hyperplane which bisects each A_i.*

Proof. Assume that (2.1) and (2.2) are not true for $A_1, A_2, \ldots, A_d$ $(d \geqslant 3)$. Let H be a hyperplane containing the points $p_i \in \operatorname{relint} A_i$, $1 \leqslant i \leqslant d-1$, and $p_d^1 \in \operatorname{relint} A_d$. Since (2.2) is not true, at least one of our sets, say, A_d contains a point $p_d^2 \in \operatorname{relint} A_d$ outside H. Since (2.1) is not true, the $(d-2)$-flat $\bar{H} \subset H$, which passes through the points $p_1, p_2, \ldots, p_{d-1}$, may be chosen so that the sets $A_1, A_2, \ldots, A_{d-1}$ have points outside $\bar{H}$ with the possible exception of, say, A_1. In that case, the dimension of A_1 is $d-2$. Then there exists a point $p_d \in \operatorname{conv}\{p_d^1, p_d^2\} \subset \operatorname{relint} A_d$ such that the hyperplane spanned by p_d and $\bar{H}$ bisects all the sets $A_1, A_2, \ldots, A_d$, except possibly A_1 and in that case, A_1 is a $(d-2)$-dimensional subset of the hyperplane in question. But then, a suitable rotation of the hyperplane yields the hyperplane satisfying (2.3). □

Lemma 4. *Let P be a convex d-polytope which possesses d $(d-2)$-faces $A_1, A_2, \ldots, A_d$ such that any facet of P contains at least one A_i, $d \geqslant 3$. Then there is a hyperplane of E^d which bisects the (relative) interior of every facet of P.*

Proof. Let $1 \leqslant i < j \leqslant d$. Since A_i and A_j are distinct $(d-2)$-faces of P, it follows that the affine hull of $A_i \cup A_j$ is either E^d or a hyperplane of E^d. Thus (2.1) is not true: By (2.2), we may assume that there is a hyperplane H such that $\bigcup_{i=1}^{d} A_i \subset H$. Since any facet of P contains at least one A_i, $H \cap P$ is determined by d supporting hyperplanes of P. Thus $H \cap P$ is a $(d-1)$-simplex in H. If H is a supporting hyperplane of P, then P is a d-simplex and the assertion follows by Theorem 1. If H does not support P, then P is a bipyramid with base $H \cap P$ and P possesses $d+2$ vertices. Since $d \geqslant 3$ implies that $d+2 \leqslant 2d-1$, we again apply Theorem 1. Finally, (2.3) immediately yields the Lemma. □

§3. **Theorem 3.** *Let P be a convex d-polytope such that $f_{d-1}(P) \leqslant 2d$, $d \geqslant 3$. Then there is a hyperplane in E^d which bisects the (relative) interior of every facet of P.*

Proof. We note that it is sufficient to consider the case $f_{d-1}(P) = 2d$. Let $F_1, F_2, \ldots, F_{2d}$ be the $2d$ facets of P. We observe that each facet contains at least d $(d-2)$-faces and of course, each $(d-2)$-faces is uniquely determined by two facets.

Let G be a graph on $2d$ vertices, labelled $1, 2, \ldots, 2d$, such that there is an edge between distinct i and j if and only if $F_i \cap F_j$ is a $(d-2)$-face of P. Thus each vertex of G has degree at least d and G is simple. By Theorem 2, G has a 1-factor. Thus P possesses d $(d-2)$-faces such that any facet of P contains one of them. Hence the assertion follows from Lemma 4. □

Remark 2. In E^3, there is a convex 3-polytope with seven facets such that no plane meets that (relative) interior of every facet. As depicted in Fig. 1, let $P = \text{conv}\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (0,1,1)\}$. Suppose that there is a plane H which meets the (relative) interior of every facet of P. As in Remark 1, we assume that H contains no vertex of P and introduce H^+ and H^-.

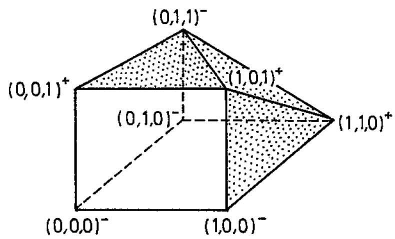

Fig. 1

Since $\operatorname{conv}\{(1,1,0), (1,0,1), (0,1,1)\}$ is a facet of P, we may assume that, say, $\{(1,1,0), (1,0,1)\} \subset H^+$ and $(0,1,1) \in H^-$. Thus $(1,0,0) \in H^-$ as well, and $(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1,0,0) + \frac{1}{2}(0,1,1) \in H^-$. Then $(1,0,1) \in H^+$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1,0,1) + \frac{1}{2}(0,1,0) \notin H^+$ imply that $(0,1,0) \in H^-$. A similar argument yields that $(0,0,0) \in H^-$. As $\operatorname{conv}\{(0,0,0), (0,1,0), (0,1,1), (0,0,1)\}$ is a facet of P, the preceding implies that $(0,0,1) \in H^+$. But then $(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(0,0,1) + \frac{1}{2}(1,1,0) = \frac{1}{2}(1,0,0) + \frac{1}{2}(0,1,1) \in H^+ \cap H^-$; a contradiction. Thus there is no hyperplane H which bisects every facet. □

Lemma 5. *Let $P \subset E^d$, $d \geqslant 3$, be a convex d-polytope with $f_{d-1}(P) = 2d+1$. Then for any set of $d+1$ facets $F_1, F_2, \ldots, F_{d+1}$ of P, there is a hyperplane H in E^d such that H bisects the (relative) interior of each F_i, $i = 1, 2, \ldots, d+1$.*

We introduce the following notion.

Definition. Two facets of a convex d-polytope are *adjacent* if their intersection is a $(d-2)$-face of the polytope.

Proof of Lemma 5. We consider first the case when no two of the facets $F_1, F_2, \ldots, F_{d+1}$ are adjacent. Then $F_1, F_2, \ldots, F_{d+1}$ are $(d-1)$-simplices and all of them are adjacent to any of the other d facets of P. Say, F is a facet of P which is adjacent to the facets $F_1, F_2, \ldots, F_{d+1}$. Let H be the supporting hyperplane spanned by F. Then a hyperplane H', close and parallel to H, bisects the relative interior of each facet F_i, $i = 1, 2, \ldots, d+1$. Next we suppose that, say, F_1 and F_2 are adjacent. Then we choose points $p_1 \in \operatorname{relint}(F_1 \cap F_2)$ $p_2 \in \operatorname{relint} F_3, \ldots, p_d \in \operatorname{relint} F_{d+1}$. Thus there is a hyperplane H going through the points $p_1, p_2, \ldots, p_d$. In addition, we choose the points $p_1, p_2, \ldots, p_d$ such that $p_1 \in F_1 \cap F_2 \not\subset H$ (since $d \geqslant 3$). Consequently, H bisects the (relative) interior of each F_i, $i = 1, 2, \ldots, d+1$. □

Lemma 6. *Let $P \subset E^d$, $d \geqslant 3$, be a convex d-polytope with a vertex w which lies on exactly d facets $F_1, F_2, \ldots, F_d$ of P. Let H be a hyperplane such that $w \in H$ and $H \cap \operatorname{int} P \neq \emptyset$. Then there is a hyperplane H', close to H and not containing w, which bisects the (relative) interior of each F_i, $i = 1, \ldots, d$.*

Proof. Since w lies on exactly d-facets of P, there is a simplex $Q \subset P$ such that w is a vertex of Q, $H \cap \operatorname{int} Q \neq \emptyset$ and $F_i \cap Q$ are facets of Q, $i = 1, \ldots, d$. Let F_w be the remaining facet of Q. Then $w \notin F_w$, $H \cap \operatorname{relint} F_w \neq \emptyset$ and H separates in some manner the vertices of F_w. If both components of $Q \setminus H$ contain at least two vertices of F_w, then any H', close to H and not containing w, bisects each relint F_i. If one component of $Q \setminus H$ contains exactly one vertex of F_w, then we choose H' so that each component of $Q \setminus H'$ contains at least two vertices of Q. □

Now we are in a position to prove our last theorem.

Theorem 4. *Let $P \subset E^d$ be a convex d-polytope such that it has a simple vertex (a vertex which belongs to exactly d facets of P) and $f_{d-1}(P) = 2d + 1$, $d \geqslant 4$. Then there is a hyperplane in E^d which bisects the (relative) interior of every facet of P.*

Proof. Let w be a simple vertex of P. Let $\mathscr{F} = \{F_1, \ldots, F_d\}$ be the set of facets which contain w and $\mathscr{F}^* = \{F_1^*, \ldots, F_d^*, F_{d+1}^*\}$ be the set of facets which do not contain w. We claim that there exist adjacent facets in $\mathscr{F}^*$.

Suppose that $F_i^* \in \mathscr{F}^*$ is not adjacent to any $F_j^* \in \mathscr{F}^*$ for $i \neq j$. Then F_i^* is necessarily adjacent to each $F_k \in \mathscr{F}$ and F_i^* is a $(d-1)$-simplex. Since $w \notin F_i^*$, $Q = \operatorname{conv}\{F_i^*, w\} \subset P$ is a d-simplex. Let $\bar{H}_k$ $(\bar{H}_i^*)$ denote the closed halfspace bounded by the affine hull aff F_k (aff F_i^*) and containing P, $F_k \in \mathscr{F}$ $(F_i^* \in \mathscr{F}^*)$. Since Q is a d-simplex, it follows that $P \subset \left(\bigcap_{k=1}^{d} \bar{H}_k\right) \cap \bar{H}_i^* = Q$ and P is a d-simplex; a contradiction.

As an immediate consequence of the preceding, we observe that

(i) there exist four facets of $\mathscr{F}^*$, say, F_1^*, F_2^*, F_3^* and F_4^* such that F_1^* and F_2^* (F_3^* and F_4^*) are adjacent or

(ii) there is an $F_i^* \in \mathscr{F}^*$ which is adjacent to each $F_j^* \in \mathscr{F}^* \setminus \{F_i^*\}$.

If (i), then we choose points $p \in \operatorname{relint}(F_1^* \cap F_2^*)$, $q \in \operatorname{relint}(F_3^* \cap F_4^*)$ and $r_j \in \operatorname{relint} F_j^*$, $j = 5, \ldots, d+1$. Then there is a hyperplane H containing w, p, q and the $(d-3)$ points $r_5, \ldots, r_{d+1}$. Since $d \geqslant 4$ we may choose the r_j's so that H contains neither $F_1^* \cap F_2^*$ nor $F_3^* \cap F_4^*$. Thus a hyperplane H', close to H, also bisects the (relative) interior of each $F_k^* \in \mathscr{F}^*$. Finally Lemma 6 implies that there is an H' which bisects the (relative) interior of every facet of P.

We assume (ii). Let $\bar{H}$ denote supporting hyperplane spanned by F_i^*. In $\bar{H}$, F_i^* is a convex $(d-1)$-polytope whose facets include $F_i^* \cap F_j^*$ for $i \neq j$, $1 \leqslant j \leqslant d+1$. Obviously, the number of facets of F_i^* in $\bar{H}$ is at most $2d$. If this number of facets is at most $2(d-1)$ then by Theorem 3, there exists a hyperplane H^* in $\bar{H}$ which bisects the (relative) interior of every facet of F_i^* in $\bar{H}$. If the number of facets of F_i^* in $\bar{H}$ is $2(d-1)+1 = 2d-1$ then by Lemma 5, there exists a hyperplane H^* in $\bar{H}$ which bisects the (relative) interiors of the facets $F_i^* \cap F_j^*$, $j \neq i$, $1 \leqslant j \leqslant d+1$. In both cases, Lemma 6 yields a hyperplane in E^d, close to $\operatorname{aff}\{H^*, w\}$, which bisects the relative interior of every facet of P. Finally, if F_i^* is adjacent to any other facet of P, then a hyperplane H in E^d, close and parallel to $\bar{H}$, bisects the (relative) interiors of the facets of P (except possibly that of F_i^*). But then, after a little rotation of H, the resulting hyperplane bisects the relative interior of every facet of P. □

Remark 3. Let $d \geqslant 4$. Then there is a convex polytope $P \subset E^d$ with $f_{d-1}(P) = 2d+2$ such that no hyperplane in E^d meets the (relative) interior of every facet of P.

Let $Q = \operatorname{conv}\{q_1, \ldots, q_d, q_{d+1}\}$ be a d-simplex in E^d. As the vertices of Q are in general position, it follows that $Q^* = \operatorname{conv}\{q_1^*, \ldots, q_d^*, q_{d+1}^*\}$ is a d-simplex whenever q_i^* is sufficiently close to q_i, $i = 1, \ldots, d+1$. Now let P be a convex d-polytope obtained by slicing off each q_i of Q by a hyperplane arbitrarily close to q_i, $i = 1, \ldots, d+1$. Then each q_i is replaced by a facet of P, the points of which are arbitrarily close to q_i. Thus P has $2d+2$ facets and any nucleus of P contains $d+1$ points q_i^* as described above. □

References

[1] BEZDEK, A., BEZDEK, K., ODOR, T.: On a Caratheodory-type theorem. Preprint (1988).

[2] GOODMAN, J. E., PACH, J.: Cell decomposition of polytopes by bending. Israel J. Math. **64**, 129—138 (1988).

[3] GRÜNBAUM, B.: Convex Polytopes. New York: Interscience. 1967.

[4] HOVANSKI, A. G.: Hyperplane sections of polytopes, toric varieties and discrete groups in Lobachevsky space. Functional Anal. Appl. **20**, 50—61 (1986).

[5] KINCSES, J.: Convex hull representation of cut polytopes. Preprint (1988).

[6] KLEINSCHMIDT, P., PACHNER, U.: Shadow-boundaries and cuts of convex polytopes. Mathematika **27**, 58—63 (1980).

[7] LOVÁSZ, L.: Combinatorial Problems and Exercises. Amsterdam–New York–Oxford: North-Holland. p. 51, problem 7.22.

[8] SHEPHARD, G. C.: Sections and projections of convex polytopes. Mathematika **19**, 144—162 (1972).

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