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LOWER BOUNDS FOR PACKING DENSITIES

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I. Introduction

If a packing of incompressible rigid convex objects is sufficiently compressed or "compacted", one expects that the packing density will not be small. The aim of this paper is to show that certain conditions on a packing insure that there is at least a lower bound on the packing density, which generalize some previous results concerning such lower bounds.

One such condition is the notion of a compact packing of convex bodies due to L. Fejes Tóth in [6]. (Recall that a *body* in n-dimensional Euclidean space E^n is a compact set with nonempty interior, and a *packing* is a collection of sets with disjoint interiors.) We say that a body A is *enclosed* by the bodies $\{B_i\}$ if any curve, connecting a point of A with a point sufficiently far from A, intersects $\bigcup_i B_i$. If in the packing each body is enclosed by the bodies having a point in common with it, then the packing is said to be *compact.*

Two sets S_1 and S_2 in E^n are said to be *homothetic* if they are either translates or there exists a point O (as origin) and a positive real number λ such that

$$S_2 = \lambda S_1 = \{P_2 | P_2 - O = \lambda(P_1 - O),\ P_1 \in S_1\}, \tag{1.1}$$

where we always regard points as vectors. The *homogeneity* of a packing of convex bodies is the infimum of the volumes (or areas in dimension two) of the bodies divided by the supremum of the volumes. L. Fejes Tóth [6] proved that, in the Euclidean plane, the lower density of a compact packing of centrally symmetric homothetic convex sets of positive homogeneity is at least 3/4, and he conjectured that when the condition of central symmetry is dropped, then the bound 3/4 can be replaced by 1/2. This was proved by A. Bezdek, K. Bezdek, and K. Böröczky in [1]. Thus if d denotes the density of a compact packing of the Euclidean plane by homothetic convex sets such that the ratio of the areas of any two sets is bounded, then $d \geqq 1/2$. Later K. Bezdek [2] proved that in E^n $(n \geqq 3)$ the density of any compact lattice packing formed by translates of a centrally symmetric convex body is greater than $2^{1/(n-1)}/(2^{1/(n-1)}+1) > 1/2$. We shall generalize the theorems mentioned above. Namely we shall prove the following.

THEOREM 1. *If d denotes the density of a compact packing in E^n, $n \geqq 2$, consisting of homothetic centrally symmetric convex bodies with bounded volume ratios, then $d \geqq (n+1)/2n$, and for $n \leqq 3$ there is a compact lattice packing of centrally symmetric convex bodies where equality holds.*

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REMARK 1. It turns out that our lower bound $(n+1)/2n$ is never sharp for $n \geqq 4$, but we do not know of a suitable replacement. We omit the proof. See Grünbaum [8], as well as our later comments about Grünbaum's Theorem.

We say that two sets S_1 and S_2 are *homotheticly reversed* if (1.1) holds for λ negative.

THEOREM 2. *Let d denote the density of a compact packing in the Euclidean plane consisting of homothetic and homotheticly reversed convex sets with bounded area ratios. Then $d \geqq 1/2$.*

REMARK 2. When the condition of central symmetry is dropped, we present the following problems: What is the greatest lower bound of the densities of compact packings in E^n $(n \geqq 3)$ consisting of homothetic convex bodies such that the volume ratios are greater than a fixed positive number? What is the greatest lower bound if we only suppose that our convex bodies are homothetic or homotheticly reversed?

For dimensions n greater than two, the condition of being a compact packing seems to be very strong. For instance, if each of the bodies is *strictly convex*, i.e. each support plane intersects the body at a single point, then the packing cannot be compact (for $n \geqq 3$).

Thus we offer an alternative to compact packings, in dimensions greater than two, that is more general at least for centrally symmetric convex bodies. Of course, the penalty we pay is that the lower bounds are much lower than for compact packings. We say that a packing of E^n by centrally symmetric convex bodies is a *triangulated packing* if there is a triangulation of E^n such that each vertex of the triangulation is the central point of one of the packing elements, and a 1-simplex between two vertices implies that the two corresponding packing elements intersect. (Recall that a *triangulation* of a space X is a simplicial complex whose underlying space is X.) In dimension two, for packings of centrally symmetric convex sets, triangulated packings and compact packings are the same.

THEOREM 3. *Let d denote the density of a triangulated packing of homothetic centrally symmetric convex bodies in E^n, $n \geqq 2$, with bounded volume ratios. Then $d \geqq (n+1)/2^n$, and there is a triangulated lattice packing of (congruent) centrally symmetric convex bodies where equality holds.*

REMARK 3. Unfortunately for dimensions greater than two no packing of congruent spheres can be triangulated.

By using a result of Hadwiger [10] and a result of Rogers and Shephard [12] we can apply Theorem 3 to the case when the convex bodies are not necessarily centrally symmetric. It turns out that any packing $\mathcal{P}$ of translates of a convex body B has an associated packing $\hat{\mathcal{P}}$ of translates of a centrally symmetric convex body $\hat{B}$, where each B_i corresponds to a unique $\hat{B}_i \in \hat{\mathcal{P}}$ such that $B_i \cap B_j \neq \emptyset$ if and only if $\hat{B}_i \cap \hat{B}_j \neq \emptyset$. We say that $\mathcal{P}$ is a triangulated packing if and only if $\hat{\mathcal{P}}$ is a triangulated packing.

COROLLARY. *Let d denote the density of a triangulated packing of translates of a convex body in* E^n. *Then* $d \geqq (n+1)/\binom{2n}{n}$.

We thank Branko Grünbaum for (gently) pointing out that our Lemma 4 below is essentially the same as his Theorem 1 in [8]. Grünbaum's Theorem says that if there are $n+1$ symmetric homothetic convex bodies in E^n with pairwise nonempty intersections, and each body is dilated from its center by $2n/(n+1)$, then the dilated bodies have a common intersection point.

When Grünbaum's Theorem is specialized to the case when the homothetic bodies are translates (i.e. the homothetic ratios are all 1), Grünbaum points out that his Theorem can be viewed (via Helly's Theorem) as a Jung type of result. Namely, in any Minkowski space (a finite dimensional normed linear space over the reals) a ball of diameter $2n/(n+1)$ may cover, after a suitable translation, any set of diameter $\leqq 1$, which is a result of Bohnenblast [3]. See also Leichtweiss [11].

On the other hand, Grünbaum also applies his Theorem to the problem of the extensions of transformations [9].

We apply Grünbaum's Theorem (Lemma 4 below) to both our Theorem 1 and Theorem 3.

The main difficulty in Grünbaum's Theorem is handling the homothetic ratios.

We include our own version of Grünbaum's proof for two reasons. First, for the sake of completeness, it is convenient to have this important result included with the other ideas in our Theorem 1 and Theorem 3. Second, Grünbaum's version of his proof is very terse and gives no hint as to how he discovered the particular relations he used. We show how to derive the factor $2n/(n+1)$ as well as explain geometrically the two cases which Grünbaum considers in his proof.

II. Proof of Theorem 1

The following Lemma 1 is the key result needed in the proof of Theorem 1. Lemma 1 is needed for Lemma 2, and Lemma 2 and Lemma 3 are used to prove Lemma 4 which is used to find a point "close" to the packing elements that surround a "hole".

Let $\langle P_1, P_2, \ldots, P_{n+1}\rangle = \sigma$ be a simplex in E^n. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ be positive real numbers, and suppose we have a point $P_{i,j}$ $(i<j)$ on each edge between P_i and P_j with the property that

$$\lambda_j(P_i - P_{i,j}) = \lambda_i(P_{i,j} - P_j), \tag{2.1}$$

where $1 \leqq i < j \leqq n+1$, and points are regarded as vectors. Define $\bar{\lambda}$ by

$$\bar{\lambda} = 1 + \frac{2(n-1)}{\left[\left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\sum_{i=1}^{n+1} \lambda_i^{-1}\right) - (n^2-1)\right]}. \tag{2.2}$$

For any set X in E^n, $P\in E^n$, α a scalar, define

$$\alpha X(P)=\{Q|Q=\alpha(P'-P)+P,\ P'\in X\}.$$

Let L_i denote the hyperplane containing $P_{1,i}, \dots, P_{i-1,i}, P_{i,i+1}, \dots, P_{i,n+1}$.

LEMMA 1. $\bigcap_{i=1}^{n+1} \bar{\lambda} L_i(P_i)\neq\emptyset$.

PROOF. The idea is to find the intersection point P as the solution to certain linear equations. This in turn will allow us to write P and $\bar{\lambda}$ explicitly in terms of matrices involving the P_i's and λ_i's.

For any column vector P in E^n let us define

$$\hat{P}=\begin{pmatrix}P\\1\end{pmatrix},$$

the vector in E^{n+1} obtained by adding a one in the $(n+1)$-st coordinate. (Regard E^n as the subset of E^{n+1} consisting of the first n coordinates. All vectors are regarded as column vectors.) Note that $\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{n+1}$ is now a basis for E^{n+1}.

We now regard the hyperplanes L_i as the solutions to certain linear equations or equivalently as null spaces of certain linear functionals. Let $f_i\colon E^{n+1}\to E^1$ be the linear functional (uniquely) defined by $f_i(\hat{P}_i)=1$, and $f_i(\hat{Q})=0$, for all Q in L_i. We will calculate the f_i's next, explicitly in terms of matrices. Rewriting (2.1) we get, for $i<j$,

$$\lambda_j(\lambda_i+\lambda_j)^{-1}\hat{P}_i+\lambda_i(\lambda_i+\lambda_j)^{-1}\hat{P}_j=\hat{P}_{i,j}.$$

Applying f_i we get, for $i\neq j$,

$$\lambda_j+\lambda_i f_i(\hat{P}_j)=0,\quad f_i(\hat{P}_j)=-\lambda_j\lambda_i^{-1}.$$

Define an $(n+1)$-by-$(n+1)$ matrix F such that

$$F\hat{P}_j=\begin{pmatrix} f_1(\hat{P}_j)\\ \vdots\\ f_j(\hat{P}_j)\\ \vdots\\ f_{n+1}(\hat{P}_j)\end{pmatrix}=\begin{pmatrix} -\lambda_j\lambda_1^{-1}\\ \vdots\\ +1\\ \vdots\\ -\lambda_j\lambda_{n+1}^{-1}\end{pmatrix}. \tag{2.3}$$

(Note that the first equality of (2.3) holds with any vector replacing $\hat{P}_j$.)

We can encode this information in a single matrix as follows: Let J be the column vector in E^{n+1} with all 1's as entries. Then JJ^t is the $(n+1)$-by-$(n+1)$ matrix with all 1's as entries, where $(\ )^t$ denotes the transpose operation. J^tJ is the one-by-one matrix with entry $n+1$. (We always regard a one-by-one matrix as a scalar.) Also note that $(-JJ^t+2I)$ is the $(n+1)$-by-$(n+1)$ matrix with $+1$'s on the diagonal and -1's elsewhere, where I denotes the $(n+1)$-by-$(n+1)$ identity matrix. Define another $(n+1)$-by-$(n+1)$ matrix A by

$$A=(\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{n+1}).$$

Let D be the $(n+1)$-by-$(n+1)$ diagonal matrix where the i-th diagonal entry is λ_i.

Then we rewrite (2.3) as

$$FA = D^{-1}(-JJ^t+2I)D.$$

Then

$$F = D^{-1}(-JJ^t+2I)DA^{-1}.$$

This is the desired explicit expression for F and thus the functionals f_i.

We next proceed to use this to find a similar expression for the intersection point. Suppose $P=\bigcap_{i=1}^{n+1} \lambda L_i(P_i)$, for some scalar λ. Then for some $Q_i\in L_i$,

$$P = \lambda(Q_i-P_i)+P_i = \lambda Q_i+(1-\lambda)P_i,$$

and thus,

$$\hat{P} = \lambda\hat{Q}_i+(1-\lambda)\hat{P}_i, \quad f_i(\hat{P}) = 1-\lambda.$$

By the definition of F,

$$F\hat{P} = -(\lambda-1)J, \quad \hat{P} = -(\lambda-1)F^{-1}J = -(\lambda-1)AD^{-1}(-JJ^t+2I)^{-1}DJ.$$

We can justify and simplify this expression for $\hat{P}$ by calculating the inverse of $(-JJ^t+2I)$, using the properties of J, for $n>1$,

$$(-JJ^t+2I)^{-1} = [-2(n-1)]^{-1}(JJ^t-(n-1)I).$$

Then

$$\hat{P} = (\lambda-1)[2(n-1)]^{-1}AD^{-1}[JJ^t-(n-1)I]DJ,$$

$$\hat{P} = (\lambda-1)[2(n-1)]^{-1}A\left[\left(\sum_{i=1}^{n+1}\lambda_i\right)D^{-1}-(n-1)I\right]J, \tag{2.4}$$

since $J^tDJ=\sum_{i=1}^{n+1}\lambda_i$. (2.4) is the desired explicit expression for $\hat{P}$.

Since the last entry of $\hat{P}$ is one, this gives us another relation to calculate λ. Let E_{n+1} be the (column) vector in E^{n+1} with the last entry 1 and all the other entries 0. Calculating the last entry of $\hat{P}$ we get

$$1 = E_{n+1}^t\hat{P} = (\lambda-1)[2(n-1)]^{-1}E_{n+1}^tA\left[\left(\sum_{i=1}^{n+1}\lambda_i\right)D^{-1}-(n-1)I\right]J.$$

But $E_{n+1}^tA=J^t$. Thus

$$1 = (\lambda-1)[2(n-1)]^{-1}\left[\left(\sum_{i=1}^{n+1}\lambda_i\right)J^tD^{-1}J-(n-1)J^tJ\right],$$

$$2(n-1)(\lambda-1)^{-1} = \left(\sum_{i=1}^{n+1}\lambda_i\right)\left(\sum_{i=1}^{n+1}\lambda_i^{-1}\right)-(n-1)(n+1).$$

From this it is easy to calculate that $\lambda=\bar{\lambda}$ in (2.2). Thus for this value of λ (only) we see that (2.4) defines $\hat{P}$ and thus P.

Remark 4. In dimension 2 it is clear that P must lie in σ. However in dimension 3 or higher, it could turn out that P lies outside σ. This can be seen by calculating the affine coordinates of P, $t_1, t_2, \ldots, t_{n+1}$ (i.e. $P=\sum_{i=1}^{n+1} t_iP_i$) by the same method as we use to find λ. Thus using (2.4)

$$t_i = (\lambda-1)[2(n-1)]^{-1}\left[\lambda_i^{-1}\left(\sum_{j=1}^{n+1}\lambda_j\right)-(n-1)\right].$$

If

$$(n-2)\lambda_i > \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j,$$

then P_i lies outside the i-th face of σ opposite P_i, because t_i is negative. Figure 1, below, shows the sets involved in Lemma 1, for $n=2$. Here it is clear geometrically that P lies in σ.

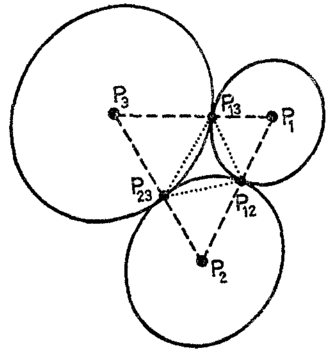

Fig. 1

There are other ways of calculating $\bar{\lambda}$, using Cramer's Rule for instance, but our method here seems as simple as any, since it does not calculate by explicitly manipulating arrays of numbers, but uses closed form matrix properties instead.

In Grünbaum's Theorem, his first case is when all the $t_i \geqq 0$. He presents $\bar{\lambda}$ (he calls it μ) as well as the affine coordinates of each point in B_i that dilates to P, and then he calculates that each point does indeed dilate to P. His second case is when some $t_i < 0$, and he handles this differently than we do below.

In what follows we reinterpret the result of Lemma 1 in terms of expanding half spaces. Using the above we define H_i as the half space containing P_i with boundary L_i (recall L_i is determined by $P_{1,i}, P_{2,i}, \ldots, P_{i-1,i}, P_{i,i+1}, P_{i,n+1}$).

Lemma 2. $\bigcap_{i=1}^{n+1} 2n(n+1)^{-1}H_i(P_i)\cap\sigma\neq\emptyset$.

Proof. Note that since the harmonic mean is less than the arithmetic mean we have

$$((n+1)^{-1}\sum_{i=1}^{n+1}\lambda_i^{-1})^{-1} \leqq (n+1)^{-1}\sum_{i=1}^{n+1}\lambda_i,$$

$$(n+1)^2 \leqq (\sum_{i=1}^{n+1}\lambda_i)(\sum_{i=1}^{n+1}\lambda_i^{-1}),$$

$$(n+1)^2-(n^2-1) = 2(n+1) \leqq (\sum_{i=1}^{n+1}\lambda_i)(\sum_{i=1}^{n+1}\lambda_i^{-1})-(n^2-1).$$

Thus

$$\bar{\lambda} = 1 + \frac{2(n-1)}{\left(\left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\sum_{i=1}^{n+1} \lambda_i^{-1}\right) - (n^2-1)\right)} \leqq 1 + \frac{2(n-1)}{2(n+1)} = 2n/(n+1).$$

Thus

$$\bar{\lambda} L_i(P_i) \subset \bar{\lambda} H_i(P_i) \subset 2n(n+1)^{-1} H_i(P_i).$$

By Lemma 1 we get

$$\bigcap_{i=1}^{n+1} 2n(n+1)^{-1} H_i(P_i) \neq \emptyset. \tag{2.5}$$

We shall prove the Lemma by induction on n. It is clearly true for $n=1$. We shall assume the result for $n-1$.

Note that $2n(n+1)^{-1}$ is a monotone increasing function for $n>0$.

Let H_i^σ denote the support half-space for σ whose boundary is the hyperplane L_i^σ spanned by the facet opposite P_i in σ. I.e. L_i^σ is spanned by $P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{n+1}$ and

$$\bigcap_{i=1}^{n+1} H_i^\sigma = \sigma. \tag{2.6}$$

We apply induction to each L_i^σ with $H_j \cap L_i^\sigma, j \neq i,$ replacing H_j, and $\sigma \cap L_i^\sigma$ replacing σ. Thus

$$\emptyset \neq (\sigma \cap L_i^\sigma) \bigcap_{\substack{j=1 \\ j\neq i}}^{n+1} 2(n-1)n^{-1}(H_j \cap L_i^\sigma)(P_j) \subset \bigcap_{i=1}^{n+1} H_i^\sigma \bigcap_{\substack{j=1 \\ j\neq i}}^{n+1} 2n(n+1)^{-1} H_j(P_j). \tag{2.7}$$

Thus by (2.5), (2.6), and (2.7) the $2(n+1)$ half-spaces

$$2n(n+1)^{-1} H_i(P_i), \quad H_i^\sigma, \quad i = 1, 2, \ldots, n+1,$$

have the property that every $n+1$ of them have a non-empty intersection. Thus by Helly's Theorem, they all must intersect, finishing the Lemma.

Let $\mathcal{P}$ be a compact packing in E^n. Let W, a *hole*, be the closure of a component of the complement of the union of the elements of $\mathcal{P}$. Let $\mathcal{P}_W \subset \mathcal{P}$ be those packing elements of $\mathcal{P}$ whose intersection with W is $(n-1)$-dimensional. W must be bounded and $\mathcal{P}_W$ is finite since $\mathcal{P}$ is a compact packing and the elements of $\mathcal{P}$ have volumes greater than a fixed positive number.

Lemma 3. *For all* $B_1, B_2 \in \mathcal{P}_W$, $B_1 \cap B_2 \neq \emptyset$.

Proof. Suppose not; suppose some $B_1, B_2 \in \mathcal{P}_W$ are such that $B_1 \cap B_2 = \emptyset$ Suppose that the volume of B_1 is not larger than the volume of B_2. Then B_2 is not a neighbor of B_1, and there is a path from B_1 through W to the center of B_2. Then the ray from the center P_2 of B_2 in the opposite direction from the center P_1 of B_1 completes a path to infinity that violates the compactness of $\mathcal{P}$.

To see this suppose not; suppose some neighbor B of B_1 intersects B_1 at Q_1 and the ray at P. Then construct Q_2 on the line segment $\langle Q_1, P \rangle$ so that the triangles $Q_1 P_1 P$ and $Q_2 P_2 P$ are similar. Since B is convex, Q_2 must be in B. But Q_2 must be in the interior of B_2 as well, since the coefficient of homogeneity for B_2

is larger than or equal to the coefficient for B_1. Packing elements cannot intersect in interior points. Thus B cannot intersect the path defined above.

Thus $B_1 \cap B_2 \neq \emptyset$ for all $B_1, B_2 \in \mathcal{P}_W$. See Figure 2.

REMARK 5. For Lemma 3 in the plane, we do not need the condition that the elements of $\mathcal{P}$ are homothetic. We can simply choose a ray going to infinity lying between the two common support lines of B_1, B_2, where $B_1 \cap B_2 = \emptyset$.

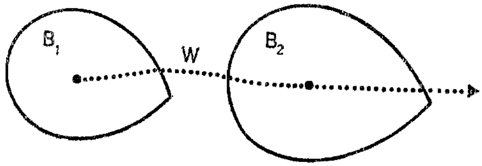

Fig. 2

LEMMA 4 (Grünbaum). $\bigcap_{B_i \in \mathcal{P}_W} 2n(n+1)^{-1} B_i(P_i) \neq \emptyset$, *where P_i is the center of B_i.*

PROOF. By Helly's Theorem we need only show the result for $n+1$ elements of $\mathcal{P}_W$, say $B_1, B_2, \ldots, B_{n+1}$. We can also assume that the centers $P_1, P_2, \ldots, P_{n+1}$ are affine independent and form a simplex σ in E^n. By Lemma 3, we know that there is a unique point $P_{i,j} = B_i \cap B_j \cap \langle P_i, P_j \rangle$, $i<j$, where $\langle P_i, P_j \rangle$ is the line segment between P_i and P_j. Let H_i be the half-space containing P_i with $P_{i,j}$, $j \neq i$, on the boundary of H_i. Clearly $H_i \cap \sigma \subset B_i$. Lemma 2 implies that

$$\emptyset \neq \bigcap_{i=1}^{n+1} 2n(n+1)^{-1} H_i(P_i) \cap \sigma \subset \bigcap_{i=1}^{n+1} 2n(n+1)^{-1} B_i(P_i),$$

finishing the Lemma.

For what follows, we need to compare volumes, and it helps to consider a slight generalization of the notion of the volume bounded by a surface. Let P be a point in E^n, and let S be an oriented surface possibly with boundary. For instance, S could be a polyhedral surface with an orientation, or the boundary of a component of the intersection of the complements of a finite number of convex bodies. In the case of a polyhedral surface we define the *signed volume* from a point P to S by

$$\mathrm{Vol}[S, P] = (n!)^{-1} \sum_{\sigma \in S} \det (P_1 - P, P_2 - P, \ldots, P_n - P),$$

for $\sigma = \langle P_1, \ldots, P_n \rangle$, an oriented simplex of S. "det" denotes the determinant, and vectors are n-by-one columns, as usual. If S is a closed surface enclosing a bounded region in E^n, then $\mathrm{Vol}[S, P]$ is the volume enclosed by S. By taking limits of polyhedral surfaces, we can extend this definition to the case of more general surfaces, such as the piecewise convex surfaces mentioned above.

We say $C \subset E^n$ is a *cone from* $P \in E^n$ if $tC(P) = C$, for all $0 < t$. For any set X let $\mathrm{bdy}(X)$ denote the topological boundary of X. For a convex body B we choose an orientation on $\mathrm{bdy}(B)$ such that $\mathrm{Vol}[\mathrm{bdy}(B), P]$ is positive and thus equal to $\mathrm{Vol}(B)$, the usual Euclidean volume.

LEMMA 5. *Let C be a cone from P in E^n, and B a convex body containing P in its interior. Let $P_0 \in E^n$ and $\lambda > 1$ be such that $P_0 \in \lambda B(P)$. Then*

$$(2.8)\qquad (\lambda-1)\,\mathrm{Vol}(B\cap C) = (\lambda-1)\,\mathrm{Vol}[\mathrm{bdy}(B)\cap C, P] \geqq \mathrm{Vol}[\mathrm{bdy}(B)^- \cap C, P_0],$$

where $(\cdot)^-$ indicates the opposite orientation.

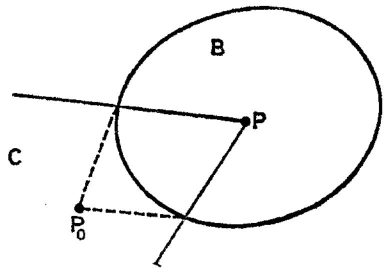

Fig. 3

PROOF. We will show the lemma first in the case when C and B are both polyhedral. The more general case follow by approximating the surfaces with polyhedral sets. Furthermore, by subdividing the boundary of B into simplices, we can further reduce our considerations to the cases when $B\cap C$ is a simplex σ. We simply sum over all simplices on the boundary of B, where each term is the case when C is the cone from P over each simplex in $B\cap C$. Let H denote the half-space containing σ with boundary L containing the face opposite P. Let d denote the distance of P from L, and let d_0 denote the signed distance of P_0 from L, where d_0 is negative if P_0 is in H. Then

$$n\,\mathrm{Vol}(H\cap C) = d\,\mathrm{Vol}_{n-1}(L\cap C),\quad n\,\mathrm{Vol}[L^-\cap C, P_0] = d_0\,\mathrm{Vol}_{n-1}(L\cap C),$$

where Vol_{n-1} is the $(n-1)$-dimensional volume in L.

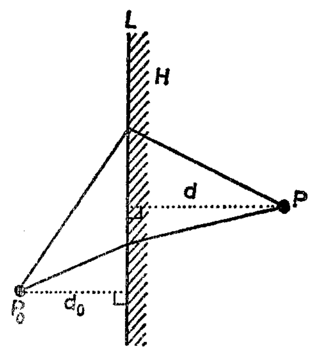

Fig. 4

Then,

$$d+d_0 \leqq \lambda d,\quad d_0 \leqq (\lambda-1)d,$$

since $P_0 \in \lambda B(P)$. See Figure 4. Thus

$$n \operatorname{Vol}[L^- \cap C, P_0] = d_0 \operatorname{Vol}_{n-1}(L \cap C) \leqq$$

$$\leqq (\lambda - 1) d \operatorname{Vol}_{n-1}(L \cap C) = (\lambda - 1) n \operatorname{Vol}(H \cap C) = (\lambda - 1) n \operatorname{Vol}(B \cap C).$$

(2.8) then follows, finishing the Lemma.

PROOF OF THEOREM 1. The idea is to compare the volume of the holes of the packing to the volume of the packing elements using Lemma 5. We get our estimate to be the sharpest when there is a point sufficiently near to all of the packing elements next to the hole. Lemma 4 guarantees that there is such a point.

Let W be a hole for the compact packing $\mathcal{P}$. Recall that $\mathcal{P}_W$ is the collection of those elements of $\mathcal{P}$ whose boundary and W intersect in an $(n-1)$-dimensional set. Let P_i be the center of B_i, as in our previous notation. Let V_i denote the cone over $B_i \cap W$ from P_i, namely

$$V_i = \{\langle Q, P_i \rangle | Q \in B_i \cap W\},$$

where $\langle Q, P_i \rangle$ is the line segment between Q and P_i.

By Lemma 4 there is a point

$$P_0 \in \bigcap_{B_i \in \mathcal{P}_W} 2n(n+1)^{-1} B_i(P_i).$$

By Lemma 5 for $\lambda = 2n(n+1)^{-1}$, and $B_i \in \mathcal{P}_W$,

$$\operatorname{Vol}[\operatorname{bdy}(B_i)^- \cap W, P_0] \leqq (n-1)(n+1)^{-1} \operatorname{Vol}(V_i).$$

But

$$\sum_{B_i \in \mathcal{P}_W} \operatorname{Vol}[\operatorname{bdy}(B_i)^- \cap W, P_0] = \operatorname{Vol}(W).$$

Thus

$$\operatorname{Vol}(W) \leqq (n-1)(n+1)^{-1} \sum_{B_i \in W} \operatorname{Vol}(V_i).$$

Let $V = \bigcup_{B_i \in \mathcal{P}_W} V_i$. Then in $W \cup V$

$$\frac{n+1}{2n} = \frac{\operatorname{Vol}(V)}{(n-1)(n+1)^{-1}\operatorname{Vol}(V) + \operatorname{Vol}(V)} \leqq$$

$$\leqq \frac{\operatorname{Vol}(V)}{\operatorname{Vol}(W) + \operatorname{Vol}(V)} = \frac{\operatorname{Vol}(V)}{\operatorname{Vol}(W \cup V)}.$$

Since the sets $\{W \cup V\}$ have disjoint interiors and cover the complement of the packing elements of $\mathcal{P}$ and since the volume ratios of the packing elements are bounded, we have that the lower packing density of $\mathcal{P}$ is $\geqq \frac{n+1}{2n}$.

To complete the proof of Theorem 1, we need the following:

CONSTRUCTION 1. Let

$$B = \{(x_1, \ldots, x_n) \big| |x_1 + \ldots + x_n| \leqq 1, |x_i| \leqq 1, \ i = 1, \ldots, n\},$$

for $n=2$ or 3. Let $\mathscr{P}$ be the packing defined by taking translates of B by the lattice

$$\Lambda = \{(2k_1, \ldots, 2k_n) \mid k_1, \ldots, k_n \text{ are integers}\}.$$

Figure 5 shows B for $n=2$ and $n=3$.

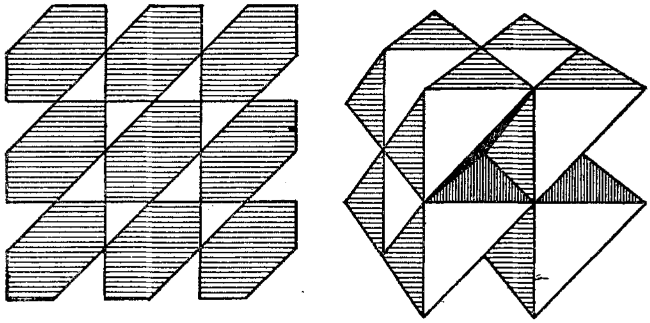

Fig. 5

We claim that $\mathscr{P}$ is a compact packing of convex symmetric bodies with density $(n+1)/(2n)$ for $n=2$ and $n=3$. It is clear that $\mathscr{P}$ is a packing of convex symmetric bodies. It is easy to check that by translating each facet F of the square or cube to the opposite facet $\bar{F}$ that the relative interior of $B \cap F$ is translated into the relative complement of $B \cap \bar{F}$ and the two sets cover the facet $\bar{F}$. Thus the two simplices that make up the complement of B in the cube or square are holes in the packing $\mathscr{P}$. Thus $\mathscr{P}$ is a compact packing. The density is easily calculated to be $(n+1)/(2n)$ for $n=2$ and $n=3$. This finishes the proof of Theorem 1.

III. Proof of Theorem 2

Let $\mathscr{P}$ be a compact packing of the Euclidean plane by homothetic and homotheticly reversed compact convex sets such that the area of all the packing elements have a positive lower bound.

Recall that a hole W is a connected component of the complement of the union of the elements of $\mathscr{P}$.

By Remark 5 after Lemma 3 each W is the connected component of the complement of a finite number $S_1, \ldots, S_n \in \mathscr{P}$, where $S_i \cap S_j \neq \emptyset$, for $i \neq j$. Since each S_i is a convex set with non-empty interior in the plane, each set of 3 of the S_i's, say S_1, S_2, S_3 must bound a connected region in the plane. If S_4 is in this bounded region then W is not connected. If S_4 is outside this region it is not part of the boundary of W. Thus $n=3$.

Let C_i be the centroid of S_i, $i=1, 2, 3$. Let $i, j=1, 2, 3$, $i \neq j$. If S_i and S_j are homotheticly reversed we define $P_{i,j}=P_{j,i}$ to be the unique point on the line segment from C_i to C_j in $S_i \cap S_j$. Note in this case $P_{i,j}$ is the center of dilation which moves S_i to S_j. If S_i and S_j are not homotheticly reversed, then we choose $P_{i,j}$ to be any point in $S_i \cap S_j$. However, if S_i and S_j correspond to another hole we must be careful to choose the same $P_{i,j}$.

Let $H(W)$ be the hexagon whose boundary consists of the union of the line segments $[C_i, P_{i,j}]$, $i \neq j$, $i, j=1, 2, 3$. See Figure 6.

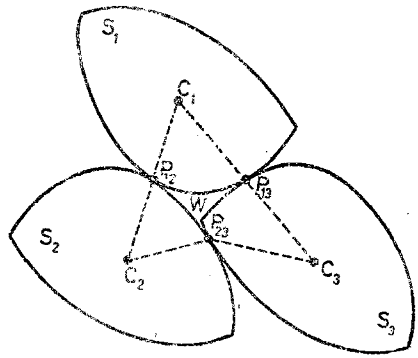

Fig. 6

Note that if at least one of the sets is homotheticly reversed and at least one is not homotheticly reversed, then two pairs of adjacent sides of the hexagon are colinear, and we can think of our hexagon as a quadrilateral.

LEMMA 6. *The collection of hexagons* $\{H(W)|W$ *is a hole of* $\mathcal{P}\}$ *have disjoint interiors and the union covers the complement of the union of the elements of* $\mathcal{P}$.

PROOF. $E^2 \setminus [S_1 \cup S_2 \cup S_3 \cup H(W)]$ is connected and unbounded, thus $H(W)$ must contain the bounded component of $E^2 \setminus [S_1 \cup S_2 \cup S_3]$. I.e., $W \subset H(W)$. Since there are no unbounded components of the complement of the union of the elements of $\mathcal{P}$, the union of the hexagons must cover the complement of the union of the elements of $\mathcal{P}$.

By the construction of $S_1, S_2, S_3 \in \mathcal{P}$ for each hole, we see that no $H(W)$ contains an element of $\mathcal{P}$. Thus $H(W)$ contains no centroid of an element of $\mathcal{P}$ and no $P_{i,j}$. Since no two of the segments that define the boundaries of the hexagons can intersect except at their endpoints, any two hexagons must have disjoint interiors. This finishes the proof of the Lemma.

If we know that the density of the packing $\mathcal{P}$, when each element is intersected with one of the hexagons, is not smaller than 1/2, then the overall packing density of $\mathcal{P}$ is not smaller than 1/2. Thus the following Lemma finishes Theorem 2.

LEMMA 7. *For each hole* W *of* $\mathcal{P}$

$$2(\text{area } W) \leqq \text{area } H(W).$$

PROOF. If all three of the packing elements corresponding to W are homothetic or all three are homotheticly reversed, then the methods of A. Bezdek, K. Bezdek, and K. Böröczky [1] imply the result.

Thus we are left with the case when one of the packing elements is different (homotheticly) from the other two. We assume that S_1 is homotheticly reversed from S_2 and S_3 (and thus S_2 and S_3 are homothetic). Since affine linear transformations take centroids to centroids, preserve area ratios, and homothetic pairs of sets, we may also assume that the triangle $\langle P_{1,2}, P_{2,3}, P_{3,1}\rangle$ is an equilateral triangle of side length 1. We must carefully estimate the area of W. Let C_i', for $i=1, 2, 3$, be the point in S_i furtherest from the line through $P_{i,i+1}, P_{i,i-1}$ (indices mod 3) on the same side as W. The quadrilateral $\langle C_i, P_{i,i+1}, C_i', P_{i,i-1}\rangle$ is contained in S_i and roughly speaking we will use its area as a lower bound for the area of $H(W)\cap \cap S_i$. See Figure 7. Note that it might happen that C_i and C_i' are on the same side of $P_{i,i+1}, P_{i,i-1}$.

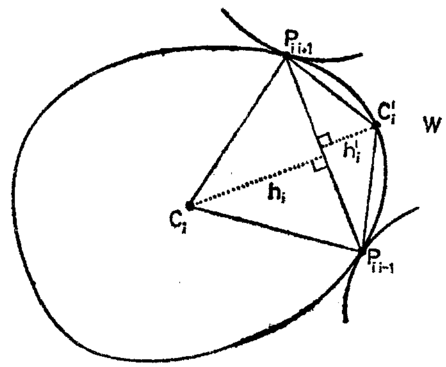

Fig. 7

Let absolute value of h_i be the altitude of $\langle C_i, P_{i,i+1}, P_{i,i-1}\rangle$ from the vertex C_i. We define h_i to be positive if C_i and C_i' are on opposite sides of $\langle P_{i,i+1}, P_{i,i-1}\rangle$, otherwise h_i is negative (or zero if C_i is on the line through $P_{i,i+1}, P_{i,i-1}$).

Let $h_i'>0$ be the altitude of $\langle C_i', P_{i,i-1}, P_{i,i+1}\rangle$ from the vertex C_i'. We claim:

$$\text{area}\,[H(W)\cap S_i] \geqq \frac{1}{2}(h_i+h_i').$$

In case C_i and C_i' are on opposite sides of the line through $P_{i,i-1}$ and $P_{i,i+1}$, or C_i lies inside the triangle $\langle C_i', P_{i,i-1}, P_{i,i+1}\rangle$, then $\frac{1}{2}(h_i+h_i')$ is the area of the quadrilateral $\langle C_i, P_{i,i-1}, C_i', P_{i,i+1}\rangle$ which is contained in $H(W)\cap S_i$. On the other hand, it is easy to see that when $h_i<0$, since $|h_i|\leqq h_i'$, the claim still holds, since the triangle $\langle C_i, P_{i,i+1}, C_i'\rangle$ has area $\geqq\frac{1}{2}(h_i+h_i')$ and is contained in $H(W)\cap S_i$, assuming (without loss of generality) that the segments $\langle C_i, P_{i,i+1}\rangle$ and $\langle C_i', P_{i,i-1}\rangle$ cross. See Figure 8.

We now must estimate h_i+h_i', for $i=1, 2, 3$.

We observe that if L is a support line for S_i, $i=1, 2, 3$, and b_i is the breadth of S_i in the direction perpendicular to L, then $d(C_i, L)\geqq b_i/3$, where $d(C_i, L)$ is the distance from the centroid C_i to L. (See Bonnesen and Fenchel [4], page 52.) In particular when L_i is the support line through C_i' parallel to $P_{i,i-1}, P_{i,i+1}$, then $d(C_i, L_i)=h_i+h_i'$. Thus we now look for lower bounds for the breadth in the direction perpendicular to the line through $P_{i,i-1}, P_{i,i+1}$.

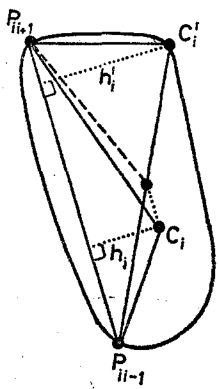

Fig. 8

We use the homotheties to find points far away from C_i' in the given direction. Let λ_i, $i=1, 2, 3$ be the absolute scalar constants of homothety for S_i. That is, the ratio of the lengths of correspondig line segments from S_i to S_j is λ_i/λ_j. Recall S_1 is homotheticly reversed from S_2 and S_3. Let $h_{k,l}$: $S_k \to S_l$ be the homothetic dilation that takes the set S_k onto S_l, where $k \neq l$, $k, l=1, 2, 3$. Define $P_{i,j}^{k,l} = h_{k,l}(P_{i,j})$. Note that $P_{i,j}^{k,l}$ is defined only when $P_{i,j} \in S_k$, i.e. $i=k$ or $j=k$.

We now compute the distance of $P_{2,3}^{3,2}$ from the line through $P_{2,3}$, $P_{1,2}$. See Figure 9.

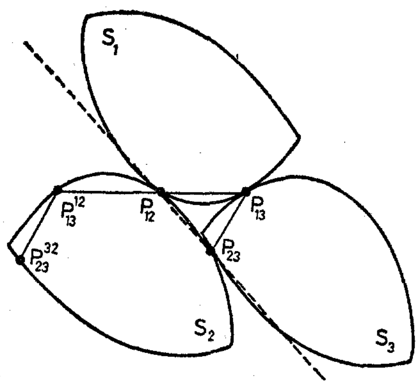

Fig. 9

Recall $h_{1,2}(P_{1,2})=P_{1,2}$ and $h_{1,3}(P_{1,3})=P_{1,3}$

$$|P_{1,2}^{1,2}-P_{1,2}| = |h_{1,2}(P_{1,3})-h_{1,2}(P_{1,2})| = \frac{\lambda_2}{\lambda_1}|P_{1,3}-P_{1,2}| = \frac{\lambda_2}{\lambda_1}. \tag{3.1}$$

But

$$P_{1,3}^{1,2} = h_{1,2}(P_{1,3}) = h_{1,2}h_{3,1}(P_{1,3}) = h_{3,2}(P_{1,3}) = P_{1,3}^{3,2},$$

since a composition of homothetic dilations is a homothetic dilation. Then

$$(3.2)\qquad |P_{1,3}^{1,2}-P_{2,3}^{3,2}| = |P_{1,3}^{3,2}-P_{2,3}^{3,2}| =$$

$$= |h_{3,2}(P_{1,3})-h_{3,2}(P_{2,3})| = \frac{\lambda_2}{\lambda_3}|P_{1,3}-P_{2,3}| = \frac{\lambda_2}{\lambda_3}.$$

Each of the above line segments makes a 60° angle with the line through $P_{1,2}$ and $P_{2,3}$. So by (3.1) and (3.2),

$$\frac{\sqrt{3}}{2}(|P_{1,3}^{1,2}-P_{1,2}|+|P_{1,3}^{1,3}-P_{2,3}^{3,2}|)+h_2' \leqq b_2,$$

$$(3.3)\qquad \frac{\sqrt{3}}{2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_2}{\lambda_3}\right)+h_2' \leqq b_2 \leqq 3(h_2+h_2').$$

Similarly

$$(3.4)\qquad \frac{\sqrt{3}}{2}\left(\frac{\lambda_3}{\lambda_1}+\frac{\lambda_3}{\lambda_2}\right)+h_3' \leqq b_3 \leqq 3(h_3+h_3').$$

We get an estimate for h_1+h_1' by calculating $P_{2,3}^{3,1}$.

$$|P_{2,3}^{3,1}-P_{1,3}| = |h_{3,1}(P_{2,3})-h_{3,1}(P_{1,3})| = \frac{\lambda_1}{\lambda_3}|P_{2,3}-P_{1,3}| = \frac{\lambda_1}{\lambda_3}.$$

Thus

$$\frac{\sqrt{3}}{2}|P_{2,3}^{3,1}-P_{1,3}|+h_1' \leqq b_1,$$

$$(3.5)\qquad \frac{\sqrt{3}}{2}\frac{\lambda_1}{\lambda_3}+h_1' \leqq b_1 \leqq 3(h_1+h_1').$$

Adding (3.3), (3.4), and (3.5) we get

$$\frac{\sqrt{3}}{2}\left(\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_1}+\frac{\lambda_2}{\lambda_3}+\frac{\lambda_3}{\lambda_1}+\frac{\lambda_3}{\lambda_2}\right)+h_1'+h_2'+h_3' \leqq 3(h_1+h_2+h_3+h_1'+h_2'+h_3').$$

Since $\frac{\lambda_1}{\lambda_3}+\frac{\lambda_3}{\lambda_1}\geqq 2$ and $\frac{\lambda_2}{\lambda_3}+\frac{\lambda_3}{\lambda_2}\geqq 2$, we get

$$\frac{\sqrt{3}}{2}\cdot 4+\sum_{i=1}^{3} h_i' \leqq 3\left(\sum_{i=1}^{3} h_i+h_i'\right), \quad 2\sqrt{3} \leqq 3\sum_{i=1}^{3} h_i+2\sum_{i=1}^{3} h_i'.$$

But $h_i'\geqq 0$, for all $i=1,2,3$, so

$$2\sqrt{3} \leqq 3\sum_{i=1}^{3}(h_i+h_i').$$

Thus

$$\frac{\sqrt{3}}{3} \leqq \sum_{i=1}^{3}\frac{(h_i+h_i')}{2} \leqq \sum_{i=1}^{3}\text{area}\,[H(W)\cap S_i] = \text{area}\left[H(W)\cap\left(\bigcup_{i=1}^{3} S_i\right)\right].$$

But $W \subset \langle P_{1,2}, P_{2,3}, P_{3,1} \rangle$ and

$$\text{area}\,\langle P_{1,2}, P_{2,3}, P_{3,1} \rangle = \frac{\sqrt{3}}{4}.$$

Thus area $W \leqq \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{3} \leqq \text{area}\,[H(W) \cap (\bigcup_{i=1}^{3} S_i)]$. Thus

$$2\,\text{area}\,W \leqq \text{area}\,[(H(W) \cap (\bigcup_{i=1}^{3} S_i)) \cup W] = \text{area}\,H(W)$$

finishing the Lemma and the Theorem.

IV. Proof of Theorem 3

We repeat here the same notation of Section II, where $\langle P_1, \ldots, P_{n+1} \rangle = \sigma$ is a simplex in E^n, and $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are positive real numbers. $P_{i,j}$ $(i<j)$ is a point on each edge $\langle P_i, P_j \rangle$ between P_i and P_j with the property that

$$\lambda_j(P_i - P_{i,j}) = \lambda_i(P_{i,j} - P_j),$$

where $1 \leqq i < j \leqq n+1$. We regard $P_{i,j} = P_{j,i}$.

LEMMA 8. $\text{Vol}\,[\text{Conv}\,\{P_{i,j}\}_{i \neq j}] \leqq \left(1 - \frac{n+1}{2^n}\right) \text{Vol}\,\sigma.$

PROOF. Without loss of generality, by applying an affine linear transformation we may assume that each edge has length 1. We will proceed by induction on n. For $n=1$ and $n=2$ the statement is trivial and follows from the analysis in Section II respectively.

Call $W = \text{conv}\,\{P_{i,j}\}_{i \neq j}$. Let $\tau_i = \text{conv}\,[\bigcup_{j=1, j \neq 1}^{n+1} \{P_{i,j}\}]$. Then

$$\sigma = W \cup \bigcup_{i=1}^{n+1} P_i * \tau_i, \tag{4.1}$$

where $P_i * \tau_i$ is the convex hull of P_i and τ_i, and each of the sets in the union has disjoint interiors. $P_i * \tau_i$ is the i-th "corner" of σ outside W. See Figure 10.

We now apply Lemma 2 to find a point $\hat{\sigma} \in \sigma$ such that for all $i = 1, 2, \ldots, n+1$ the half-space with τ_i in its boundary containing P_i when dilated (from P_i) by $2n/(n+1)$ contains $\hat{\sigma}$. Fix $i = 1, 2, \ldots, n+1$. Let l_i be the altitude from P_i in $P_i * \tau_i$. Let $\hat{l}_i$ be the altitude from $\hat{\sigma}$ in $\hat{\sigma} * \tau_i$ (the convex hull of $\hat{\sigma}$ and τ_i), where, if $\hat{\sigma}$ is on the same side of τ_i as P_i, then $\hat{l}_i < 0$, and $\hat{\sigma} * \tau_i$ is regarded as having negative volume. Then

$$l_i + \hat{l}_i \leqq \frac{2n}{n+1} l_i, \quad \hat{l}_i \leqq \frac{n-1}{n+1} l_i,$$

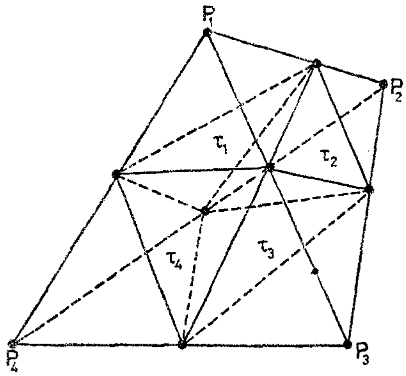

Fig. 10

and

$$\text{(4.2)} \qquad \mathrm{Vol}\, \hat{\sigma} * \tau_i \leqq \frac{n-1}{n+1} \mathrm{Vol}\, P_i * \tau_i.$$

Let $\sigma_i = \langle P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{n+1} \rangle$ be the face opposite P_i. Let

$$W_i = \mathrm{conv}\, \{P_{i,j}\}_{i \neq j \neq k \neq i} \subset \sigma_i.$$

Then

$$W = \bigcup_{i=1}^{n+1} \hat{\sigma} * W_i \cup \bigcup_{i=1}^{n+1} \hat{\sigma} * \tau_i.$$

Let h_i be the perpendicular distance of $\hat{\sigma}$ from the plane of σ_i. Since each pair of sets in the above union have disjoint interiors, we get

$$\mathrm{Vol}\, W = \sum_{i=1}^{n=1} \frac{1}{n} h_i \mathrm{Vol}_{n-1} W_i + \sum_{i=1}^{n+1} \mathrm{Vol}\, \hat{\sigma} * \tau_i,$$

where Vol_{n-1} denotes $(n-1)$-dimensional volume. By induction

$$\mathrm{Vol}_{n-1} W_i \leqq \left(1 - \frac{n}{2^{n-1}}\right) \mathrm{Vol}_{n-1} \sigma_i.$$

Thus using (4.2)

$$\mathrm{Vol}\, W \leqq \sum_{i=1}^{n+1} \frac{h_i}{n} \left(1 - \frac{n}{2^{n-1}}\right) \mathrm{Vol}_{n-1} \sigma_i + \sum_{i=1}^{n+1} \frac{n-1}{n+1} \mathrm{Vol}\, (P_i * \tau_i) \leqq$$

$$\leqq \left(1 - \frac{n}{2^{n-1}}\right) \frac{\mathrm{Vol}_{n-1} \sigma_1}{n} \sum_{i=1}^{n+1} h_i + \frac{n-1}{n+1} \sum_{i=1}^{n+1} \mathrm{Vol}\, (P_i * \tau_i),$$

since $\mathrm{Vol}_{n-1} \sigma_i = \mathrm{Vol}_{n-1} \sigma_1$, for all i. Since σ is regular $\sum_{i=1}^{n+1} h_i$ = any altitude of σ. Thus

$$\frac{\mathrm{Vol}_{n-1} \sigma_1}{n} \sum_{i=1}^{n+1} h_i = \mathrm{Vol}\, \sigma.$$

By (4.1)

$$\sum_{i=1}^{n+1} \mathrm{Vol}\,(P_i * \tau_i) = \mathrm{Vol}\,\sigma - \mathrm{Vol}\,W.$$

Putting this together we get

$$\mathrm{Vol}\,(W) \leqq \left(1-\frac{n}{2^{n-1}}\right)\mathrm{Vol}\,\sigma + \frac{n-1}{n+1}(\mathrm{Vol}\,\sigma - \mathrm{Vol}\,W),$$

$$\frac{2n}{n+1}\mathrm{Vol}\,(W) \leqq \left(\frac{2n}{n+1}-\frac{n}{2^{n-1}}\right)\mathrm{Vol}\,\sigma, \quad \mathrm{Vol}\,(W) \leqq \left(1-\frac{n+1}{2^n}\right)\mathrm{Vol}\,\sigma,$$

finishing the Lemma.

We now can show the density estimate in Theorem 3. If σ is an n-simplex in the triangulation for the triangulated packing $\mathscr{P}$, then each vertex P_i of σ is the center of some body $B_i \in \mathscr{P}$. If λ_i is the constant of homothety for B_i, then points $P_{i,j} \in \langle P_i, P_j \rangle \cap B_i \cap B_j$ are as in Lemma 8. Thus $P_i * \tau_i \subset B_i$ and $W \cup \bigcup_{i=1}^{n+1} B_i \supset \sigma$, and $\bigcup_{i=1}^{n+1} (B_i \cap \sigma) \supset \sigma \setminus W$. So

$$\mathrm{Vol}\,(\sigma \setminus W) \leqq \mathrm{Vol}\,(\bigcup_{i=1}^{n+1} B_i \cap \sigma), \qquad \mathrm{Vol}\,\sigma - \mathrm{Vol}\,W \leqq \mathrm{Vol}\,(\bigcup_{i=1}^{n+1} B_i \cap \sigma),$$

$$\mathrm{Vol}\,\sigma - \left(1-\frac{n+1}{2^n}\right)\mathrm{Vol}\,\sigma \leqq \mathrm{Vol}\,[\bigcup_{i=1}^{n+1} B_i \cap \sigma], \qquad \frac{n+1}{2^n} \leqq \frac{\mathrm{Vol}\,[\bigcup_{i=1}^{n+1} B_i \cap \sigma]}{\mathrm{Vol}\,\sigma}.$$

Thus $\frac{n+1}{2^n} \leqq d$, where d=density of $\mathscr{P}$.

To finish the Theorem we rely on the following:

CONSTRUCTION 2. Let $\Gamma = \{(z_1, \ldots, z_n) | z_i$ is an integer $i=1, 2, \ldots, n\} \subset E^n$ be the usual integral lattice. Let $C \subset \Gamma$ be vertices of the unit cube, i.e., $C = \{(z_1, \ldots, z_n) |$for each i, $z_i = 0$ or $1\}$. We define a relation on Γ by saying for $P, Q \in \Gamma$, $P \prec Q$ if $Q - P \in C$. Note the $\prec$ is a partial ordering on C, but not all of Γ.

We now define the triangulation $\mathscr{T}$ of E^n. A simplex $\sigma = \langle P_1, \ldots, P_m \rangle$ of $\mathscr{T}$ consists of $P_i \in \Gamma$ such that $P_1 \prec P_2 \prec \ldots P_m$ and $P_i \prec P_j$ if $1 \leqq i < j \leqq m \leqq n+1$. I.e., well-ordered subsets of Γ are the vertices of simplices of $\mathscr{T}$. This defines a well-known triangulation of E^n apparantly originally due to Freudenthal [7]. See Todd [13], for example, for a proof that we have indeed defined a triangulation.

For P a vertex of $\mathscr{T}$, we claim that the *star* of P in $\mathscr{T} = \mathrm{st}\,(P, \mathscr{T}) = \bigcup \{\sigma | \sigma \in \mathscr{T}, P$ is a vertex of $\sigma\}$ is the convex body B_P defined by $B_P = P + \{(x_1, \ldots, x_n) | |x_i| \leqq 1, |x_i - x_j| \leqq 1, i \neq j, i, j = 1, \ldots, n\}$.

To show this claim we assume, without loss of generality, that $P = O$, the origin. If a simplex of $\mathscr{T}$, $\sigma \subset \mathrm{st}\,(P, \mathscr{T})$, then we can order the vertices of σ so that $P_1 \prec \ldots \prec 0 = P_i \prec \ldots \prec P_{n+1}$ (possibly with repeated P_j's). All the coordinates of

$P_1, \ldots, P_{i-1}$ are 0 or -1, and all the coordinates of $P_{i+1}, \ldots, P_{n+1}$ are 0 or 1. In order for $P_1 \prec P_{n+1}$ the non-zero coordinates in the two collections must be disjoint. Thus the vertices of σ are in B_0 and hence $\sigma \subset B_0$.

Suppose $Q=(x_1, \ldots, x_n) \in B_0$. By reordering the coordinates, we may suppose that $x_1 \leqq \ldots \leqq x_{i-1} \leqq 0 = x_i \leqq x_{i+1} \leqq \ldots \leqq x_n$. Let

$$P_j = (-1, \ldots, -1, 0, \ldots, 0), \quad j = 1, \ldots, i-1,$$

where the first j coordinates are -1, and the rest are 0. Let $P_i = O$.

$$P_j = (0, \ldots, 1, \ldots, 1), \quad j = i+1, \ldots, n+1,$$

where the last j coordinates are 1. Then $P_1 \prec \ldots \prec P_{n+1}$ defines a simplex $\sigma = \langle P_1, , \ldots, P_{n+1} \rangle$ in $\mathscr{T}$, and σ is in $\operatorname{st}(O, \mathscr{T})$ since $P_i = O$.

To show that Q is in $\operatorname{st}(O, \mathscr{T})$ define t_i, $i=1, \ldots, n+1$, by

$$\begin{aligned} t_1 &= x_2 - x_1 \\ &\vdots \\ t_{i-2} &= x_{i-1} - x_{i-2} \\ t_{i-1} &= -x_{i-1} \\ t_i &= 1 - x_n + x_1 \\ t_{i+1} &= x_n - x_{n-1} \\ &\vdots \\ t_n &= x_{i+1} - x_i \\ t_{n+1} &= x_i. \end{aligned}$$

It is easy to check that $t_i \geqq 0$, for all $i=1, \ldots, n+1$, $\sum_{i=1}^{n+1} t_i = 1$, and $\sum_{i=1}^{n+1} t_i P_i = Q$.

It is clear that $\mathscr{T}$ is symmetric about any vertex of $\mathscr{T}$. Then we define a symmetric convex set $\hat{B}_P = P + \frac{1}{2} B_O$ for each $P \in \Gamma$. This is the convex hull of the midpoints of all of the 1-simplices with P as one vertex. $\hat{B}_P$ is symmetric since $\mathscr{T}$ is, and each $\hat{B}_P$ is congruent to $\hat{B}_Q$, $P, Q \in \Gamma$ by a translation. Each $\hat{B}_P \cap \sigma$ where $\sigma \in \mathscr{T}$, $P \in \sigma$, is precisely the corner defined by $\lambda_1 = \lambda_2 = \ldots = \lambda_{n+1}$ in Lemma 8. Thus the density is exactly $(n+1)/2^n$. See Figure 11.

This finishes the proof of Theorem 3.

REMARK 6. It turns out in Construction 2 that the sets B_P are all (congruent) zonotopes. They are the convex hull of a cube of side length 1/2 and the reflection of that cube about one of its vertices.

Note also that although no triangulated packing of equal spheres exists for dimensions greater than 2, in dimension 3 we can find a triangulated packing with just two sizes of spheres. Take the usual close lattice packing of equal spheres with radius 1 say. Joining the centers of adjacent spheres we get a 1-dimensional complex that can be regarded as the edges of a tiling of E^3 by regular octahedra and

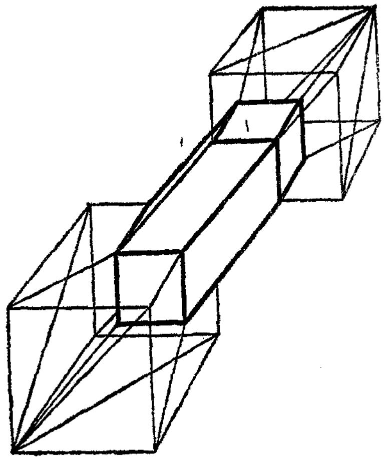

Fig. 11

tetrahedra of side length 2. In the center of each octahedron place a smaller sphere of radius $\sqrt{2}-1$, which touches all 8 of the surrounding spheres of the regular octahedron. This then yields a triangulated packing of E^3 by spheres of just two sizes.

PROOF OF THE COROLLARY. Let O be the origin in the interior of a convex body B. Then the Minkowski average of B and $-B$,

$$\tilde{B} = \frac{1}{2}B - \frac{1}{2}B = \left\{\frac{1}{2}x - \frac{1}{2}y \,\middle|\, x, y \in B\right\}$$

is a centrally symmetric convex body. If $B_i = p_i + B$ and $B_j = p_j + B$ are translates of B with disjoint interiors, then by a Theorem of Hadwiger [10] $\tilde{B}_i = p_i + \tilde{B}$, $\tilde{B}_j = p_j + \tilde{B}$ have disjoint interiors and intersect if and only if B_i and B_j intersect, as mentioned in the introduction.

A theorem of Rogers and Shephard [12] says that

$$\operatorname{Vol} \tilde{B} \leqq \frac{\binom{2n}{n}}{2} \operatorname{Vol} B.$$

This then yields the Corollary via Theorem 3.

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