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Higher-Order Rigidity—What Is the Proper Definition?*

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Abstract. We show that there is a bar-and-joint framework $G(p)$ which has a configuration p in the plane such that the component of p in the space of all planar configurations of G has a cusp at p. At the cusp point, the mechanism $G(p)$ turns out to be third-order rigid in the sense that every third-order flex must have a trivial first-order component. The existence of a third-order rigid framework that is not rigid calls into question the whole notion of higher-order rigidity.

1. Introduction

Suppose a finite configuration of labeled points $p = (p_1, \dots, p_n)$, where each p_i is in Euclidean space $\mathbb{R}^d$, is given. Let G be a graph whose vertices correspond to the labels $\{1, \dots, n\}$. An edge of G, denoted by $\{i, j\}$, is called a *bar*. The configuration p together with the graph G is called a *bar-and-joint framework*, and is denoted by $G(p)$. If $\{i, j\}$ is a bar of G, then during any continuous one-parameter motion $p(t) = (p_1(t), \dots, p_n(t))$ with $p(0) = p$, we insist that the distance from $p_i(t)$ to $p_j(t)$ be kept fixed. When is this bar-and-joint framework $G(p)$ *rigid*? That is to say, when does every such continuous motion of the points of the framework, preserving the bar lengths, arise as a restriction of a one-parameter family of

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congruences of $\mathbb{R}^d$? Equivalently, when does every continuous motion preserve the distance between any two points, whether they are connected by a bar or not?

Considering each of the points p_i of the configuration as a vector in $\mathbb{R}^d$, the distance between p_i and p_j can be written in terms of the usual dot product, and the distance constraints are expressed by the system of quadratic equations

$$(p_i - p_j)\cdot(p_i - p_j) = c_{ij} \qquad \text{for all } \{i,j\} \text{ bars of } G, \tag{1}$$

where each c_{ij}, a constant, is the squared length of bar $\{i,j\}$. If there is an analytic motion of the points p_i satisfying (1), then all the derivatives of the left-hand side must be identically zero. The resulting homogeneous differential equations provide the motivation behind the concept of first-order and second-order rigidity.

In the case of the first derivative, the definition of first-order rigidity is natural, and things work out well. Again, let $p = (p_1, \ldots, p_n)$ denote a fixed configuration corresponding to the graph G. A *first-order flex* of $G(p)$ is a configuration of vectors $p' = (p'_1, \ldots, p'_n)$ such that the equations

$$(p_i - p_j)\cdot(p'_i - p'_j) = 0 \qquad \text{for all } \{i,j\} \text{ bars of } G \tag{2}$$

are satisfied. Equation (2) comes from the formal derivative of (1). We say that p' is *trivial* if $p'_i = (d/dt)\Phi_t(p_i)|_0$ for a one-parameter family Φ_t, $0 \le t \le 1$, of congruences of $\mathbb{R}^d$, where Φ_0 is the identity.

If $d = 2$, then we may, without loss of generality, take any bar and require that bar to be fixed by any motion, essentially pinning its endpoints. Since the only one-parameter family of congruences of the plane which fixes some edge is the identity, the trivial first-order flex is easily identified by the equation $p' = 0$.

We say that $G(p)$ is *first-order rigid* if it has only trivial first-order flexes. With this definition of first-order rigidity, it is easy and natural to show that if $G(p)$ is first-order rigid, then it is rigid. See [1] and [4] for a proof and a discussion of this concept.

In the case of the second derivative, the definition of second-order rigidity is still reasonable and things still work out well. A second-order flex of $G(p)$ is a pair of configurations of vectors (p', p''), where p' is a first-order flex of $G(p)$ and $p'' = (p''_1, \ldots, p''_n)$, with $p''_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, is such that the equations

$$(p_i - p_j)\cdot(p''_i - p''_j) + (p'_i - p'_j)\cdot(p'_i - p'_j) = 0 \qquad \text{for all } \{i,j\} \text{ bars of } G \tag{3}$$

are satisfied. We also insist that $p''_i = 0$ for all pinned vertices i. Equation (3) comes from the formal derivative of (2). Physically we can regard p' as formal velocities and p'' as formal accelerations permitted by the distance constraints of the framework $G(p)$.

We say that a bar framework $G(p)$ is *second-order rigid* if every second-order flex (p', p'') has p' trivial as a first-order flex. In the case when G has pinned vertices, we say that $G(p)$ is *second-order rigid* if every second-order flex (p', p'') has $p' = 0$.

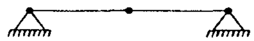

Fig. 1. A second-order rigid framework which is not first-order rigid.

Clearly, if $G(p)$ is first-order rigid, then it is second-order rigid. In Fig. 1 we see a framework which is second-order rigid but not first-order rigid. The symbol at the endpoints represents a pinned vertex. In [2] it is shown that if $G(p)$ is second-order rigid, then it is rigid.

Just considering the pinned case, at first sight it might seem more natural to define a framework to be second-order rigid if every second-order flex (p', p'') satisfies $p' = p'' = 0$. The difficulty with this definition is that it leads to a notion of second-order rigidity which is equivalent to first-order rigidity.

Proposition 1. *A pinned framework $G(p)$, in the plane, is first-order rigid, if and only if every second-order flex $(p', p'') = (0, 0)$.*

Proof. Suppose that $G(p)$ is not first-order rigid. Let p' be a nonzero first-order flex of $G(p)$ that satisfies (2). Then $(0, p')$, where p' takes the place of a formal acceleration, is a nonzero second-order flex satisfying both (2) and (3).

Suppose that $G(p)$ is first-order rigid. Let (p', p'') be any second-order flex. Since p' is a first-order flex, $p' = 0$. Then (3) reduces to (2) with p'' playing the role of a first-order flex. So $p'' = 0$ as well. □

A similar result holds in $\mathbb{R}^d$.

Thus it is clear that we must control only the formal first derivative in the definition of second-order rigidity. In what follows we discuss the difficulties of extending this definition to the case of higher-order rigidity. Some authors have attempted to make such definitions, but we believe that there are some inherent difficulties with any reasonable definition. See [9, pp. 60–61] or [10], for example, for some attempts to make such a definition of higher-order rigidity. We discuss one very natural possibility here and show the difficulties. This example is also a counterexample to the claim of [9] on pp. 60–61 that such examples must be "geometrically invariant."

2. Higher-Order Rigidity

The question is how to extend these definitions of first- and second-order rigidity to higher-order rigidity. The formal derivatives satisfy

$$\sum_{a=0}^{k} \binom{k}{a} (p_i^{(a)} - p_j^{(a)}) \cdot (p_i^{(k-a)} - p_j^{(k-a)}) = 0 \qquad \text{for all } \{i, j\} \text{ bars of } G, \tag{4}$$

where $p^{(a)} = (p_1^{(a)}, \ldots, p_n^{(a)})$ represents the formal nth derivative, $p^{(0)} = p$, and $\binom{k}{a}$ is the binomial coefficient. We say that $(p^{(1)}, \ldots, p^{(N)})$ is an Nth-*order flex* if all the

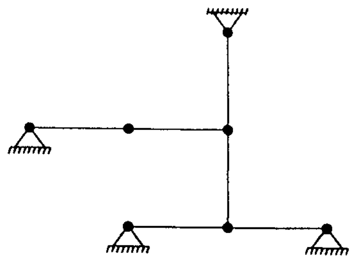

Fig. 2. An eighth-order rigid framework which is not fourth-order rigid.

equations in (4) are satisfied for all $k = 1, 2, \ldots, N$, and for all $\{i, j\}$ bars of G. In [2] the following was suggested as a definition of higher-order rigidity.

Definition 1. A bar-and-joint framework $G(p)$ is Nth-order rigid if every Nth-order flex $(p^{(1)}, \ldots, p^{(N)})$ has $p^{(1)}$ trivial as a first-order flex.

Note that this definition agrees with the definitions above for second-order rigidity and first-order rigidity, and it is clear that nth-order rigidity implies mth-order rigidity for $1 \leq n < m$. The hierarchy of rigidity is also nontrivial, as seen in Fig. 2, which indicates how to create a sequence of frameworks of higher-order rigidity out of pieces like Fig. 1. For details see [10].

The difficulty arises in trying to show that Nth-order rigidity implies rigidity. If a framework has a nontrivial motion, which we may assume, without loss of generality, to be analytic (see [3] for a proof of this analytic parametrization), then it would at first appear that we can take as many derivatives as needed to distinguish the motion from a trivial one. Those derivatives will be a nontrivial flex of some order. So the following question arises. Are there *mechanisms* (with pinned vertices), that is, nonrigid frameworks, all of whose nontrivial flexes have a trivial first-order component? The mechanism must at least have this property to be a candidate for being an example of being Nth-order rigid, but not rigid. Regard each configuration p as a single vector in $\mathbb{R}^{nd}$ and consider the configuration space X,

$$X = \{p \mid p \text{ is a configuration satisfying (1)}\} \subset \mathbb{R}^{nd},$$

where pinned vertices are fixed in the definition of X. A motion of a mechanism corresponds to a path in X, and so a curve in $\mathbb{R}^{nd}$. In this general context, if a curve has a cusp (such as the curve given by $y^2 = x^3$ at the point $(0, 0)$ in the plane), then the velocity of any analytic parametrization at the cusp point must be 0. If the only third-order flexes are those coming from parametrizations of the curve in configuration space, then we will have an example of a mechanism that is third-order rigid. These considerations motivate the following example.

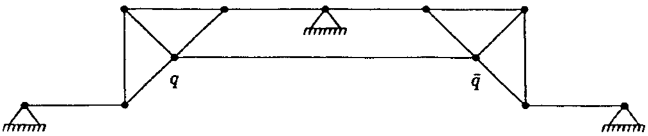

Fig. 3. A third-order rigid framework that is a mechanism.

3. A Cusp Mechanism

We note first that it is not enough to require that some point on the candidate mechanism traces out a cusp, since this corresponds at most to a projection of the configuration space, which may be perfectly regular.

In Fig. 3 we have an example of a mechanism whose configuration space X has a cusp at this position. We see that it is in fact third-order rigid by Definition 1. The framework basically is made up of two mirror image Watt mechanisms joined by a horizontal rod. For a discussion of the Watt mechanism see [8]. A Watt mechanism is pictured in Fig. 4, where the path of the midpoint q of the middle bar under the motion of the framework is indicated. Note that, although there is a self-intersection of this path, from the configuration pictured, the motion of q must follow first along the arc having the vertical tangent. With some additional bars added to allow the attachment of an edge at this midpoint, it follows that the central horizontal bar of Fig. 3 is confined to the well of Fig. 5(a), from which position it cannot ascend, and may descend in two alternative ways shown in Fig. 5(b) and Fig. 5(c), depending on which endpoint falls faster. It is clear that infinitesimally the two paths away from this configuration coincide, hence the configuration space has a cusp at the position of Fig. 3.

Proposition 2. *The framework in Fig. 3 is third-order rigid but not rigid.*

Proof. The point p_1 in Fig. 4 is constrained by the bar $\{0, 1\}$ to move in a circle about p_0, so the most general third-order flex at p_1 satisfies

$$p_1' = (0, a_1), \qquad p_1'' = (-a_1^2, b_1), \quad \text{and} \quad p_1''' = (-3a_1 b_1, c_1).$$

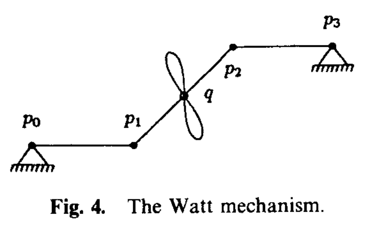

Fig. 4. The Watt mechanism.

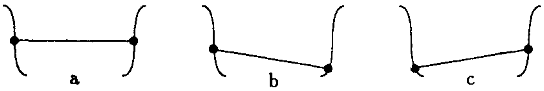

Fig. 5. Possible configurations near the cusp.

Similarly, the most general third-order flex at p_2 satisfies

$$p_2' = (0, a_2), \qquad p_2'' = (a_2^2, b_2), \quad \text{and} \quad p_2''' = (3a_2 b_2, c_2),$$

and the bar $\{1, 2\}$, with $p_2 - p_1 = (1, 1)$, gives three equations:

$$(1, 1) \cdot (0, a_2 - a_1) = 0,$$
$$(1, 1) \cdot (2a^2, b_2 - b_1) = 0,$$
$$(1, 1) \cdot (3a(b_2 + b_1), c_2 - c_1) = 0,$$

where $a_1 = a_2 = a$ from the first equation. Thus

$$b_2 - b_1 = -2a^2, \qquad c_2 - c_1 = -3a(b_2 + b_1).$$

It follows that all the nonpinned vertices on the left augmented Watt mechanism of Fig. 3 will have $(0, a)$ as their first-order flex. Moreover, since all these vertices are part of an infinitesimally rigid framework, all the higher-order $p^{(a)}$'s are determined by $p_1^{(a)}$ and $p_2^{(a)}$. See [2] for a discussion of this principle. In particular, the point q in Fig. 4 will have its most general third-order flex (q', q'', q''') expressible as the arithmetic mean of (p_1', p_1'', p_1''') and (p_2', p_2'', p_2'''). So we may write

$$q' = (0, a), \qquad q'' = (0, (b_2 + b_1)/2), \quad \text{and} \quad q''' = (-6a^3, (c_2 + c_1)/2).$$

By symmetry,

$$\bar{q}' = (0, \bar{a}), \qquad \bar{q}'' = (0, (\bar{b}_2 + \bar{b}_1)/2), \quad \text{and} \quad \bar{q}''' = (6\bar{a}^3, (\bar{c}_2 + \bar{c}_1)/2).$$

The central horizontal bar, with, say, $\bar{q} - q = (2, 0)$ yields three equations. The first-order equation is already satisfied. The second-order equation is

$$(2, 0) \cdot (0, (\bar{b}_2 + \bar{b}_1 - b_2 - b_1)/2) + (0, \bar{a} - a) \cdot (0, \bar{a} - a) = 0$$

giving $a = \bar{a}$, and hence $(\bar{q}' - q') = 0$. Thus the third-order equation gives

$$(2, 0) \cdot (12\text{a}^3, (\bar{c}_2 + \bar{c}_1 - c_2 - c_1)/2) = 0,$$

so $a = 0$, and the first-order flex $p' = 0$. □

We believe that Fig. 3 is the first known example of a cusp mechanism. A well-known result of Kempe [7] says that to any compact subset of a plane curve defined by polynomial inequalities, there corresponds a mechanism, one vertex of which traces out that subset. See [6] for a careful proof. We note, however, that this is insufficient to guarantee the existence of a cusp mechanism, since we are considering the curve comprising the configuration space, which takes into account the motion of all the vertices simultaneously. More to the point, N. Mnev has recently claimed in a private communication that any compact algebraic set is equivalent to a finite number of components of the configuration space of some bar mechanism in the plane by a nonsingular diffeomorphism with rational coordinate functions. His result is constructive, and so should also yield a variety of cusp mechanisms, though none perhaps so simple and direct as that of Fig. 3. In any case, his result is not quite enough to ensure that such a mechanism is also Nth-order rigid, since that would require, in addition, that any Nth-order flex correspond only to the first N derivatives of some parametrization of the configuration space. Thus it seems that, even with Mnev's result, the calculations of this section still need to be done for some particular mechanism for a complete proof of Proposition 2.

4. Redefining Higher-Order Rigidity

The problem still remains as to what kind of definition might be used for "higher-order rigidity." The following properties seem to us to be necessary for any reasonable definition:

1. First- and second-order rigidity agree with Definition 1.
2. For $N = 1, 2, \ldots$, if a framework is Nth-order rigid, then it is $(N + 1)$-order rigid (but not conversely).
3. For $N = 1, 2, \ldots$, if a framework is Nth-order rigid, then it is rigid.

We just saw that Definition 1 does not satisfy Property 3. The definition proposed in [9] on pp. 60–61 does not satisfy Property 3 either. However, Definition 1 does satisfy the other two properties and the following additional desirable properties as well:

4. If a given framework is rigid, then it is Nth-order rigid for some $N = 1, 2, \ldots$.
5. For $N = 1, 2, \ldots$, for a given framework, Nth-order rigidity can be computed in a finite time.

One idea to strengthen the previous definitions of higher-order rigidity is to use Proposition 1 as the model. It may be required, say, that some fixed fraction of any flex be trivial. In fact, second-order rigidity is equivalent to the condition that every flex of even order $(p^{(1)}, \ldots, p^{(2N)})$ has $(p^{(1)}, \ldots, p^{(N)})$ trivial, as was shown in [2] in the course of proving that second-order rigidity implies rigidity. Proceeding in this manner, a hierarchy of weaker definitions is obtained, and it can be shown that the first four conditions above are all satisfied. Unfortunately, we do not yet know if the necessary finiteness requirement of Property 5 can be achieved for $N > 3$.

Lastly, we remark that much of what has been discussed here can be put in a more general context. For example, the usual definition of Nth-order rigidity may be carried over to arbitrary systems of algebraic equations, $f(X) = 0$, by saying that f is nth-order rigid at the solution X if every solution of the system $f^{(1)}(X, X^{(1)}), \ldots, f^{(n)}(X, \ldots, X^{(n)})$, where $f^{(j)}$ is the formal jth derivative of f, requires that $X^{(1)} = 0$. It can be shown that first- and second-order rigidity implies rigidity.

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