Discrete Comput Geom 13:313–319 (1995)

Discrete & Computational
Geometry

Finite and Uniform Stability of Sphere Coverings*

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Abstract. By attaching cables to the centers of the balls and certain intersections of the boundaries of the balls of a ball covering of E^d with unit balls, we can associate to any ball covering a collection of cabled frameworks. It turns out that a finite subset of balls can be moved, maintaining the covering property, if and only if the corresponding finite subframework in one of the cabled frameworks is not rigid. As an application of this cabling technique we show that the thinnest cubic lattice sphere covering of E^d is not finitely stable.

1. Definitions and Terminologies

Definitions Concerning Sphere Coverings and Packings. Most of the notions we use here were first introduced by Fejes Tóth and were discussed in [F1] and [F2], which seem to be the most often used references in the field of discrete geometry. In this paper we discuss the arrangements of d-dimensional balls in the Euclidean d-dimen-

*The first two authors were partially supported by the Hungarian National Science Foundation under Grant No. 326-0413.

sional space E^d. A *ball covering* (also called a *sphere covering*) in E^d is an arrangement of d-dimensional closed balls whose union is E^d. If the centers of the balls of a sphere covering form a discrete point set, then we speak about *discrete sphere covering*. The discreteness of a sphere covering with unit balls (balls with radii equal to 1) means that each point of E^d belongs to at least one but only finitely many balls of the covering.

In order to describe how firm or steady covering as a structure is, different notions were introduced.

A sphere covering C is said to be:

(i) *Stable* (*also* 1-*stable*) if each ball is fixed by its neighbors, so that no ball can be moved separately and maintain the covering property.
(ii) n-*Stable* if each set of n balls is fixed by its neighbors, so that no subset of n balls can be moved and maintain the covering property.
(iii) *Finitely stable* if it is n-stable for every $n \geq 1$.
(iv) *Uniformly stable* if there is an $\varepsilon > 0$ such that no finite subset of the balls of C can be rearranged such that each ball is moved a distance less than ε and the rearranged balls together with the rest of balls form a covering different from C.

There are dual counterparts of the above definitions. A *ball packing* (also called a *sphere packing*) in E^d is an arrangement of d-dimensional closed balls which have a disjoint interior. If the centers of thc balls of a sphere packing form a discrete point set, then we speak about *discrete sphere packing*. A sphere packing with unit balls is automatically discrete.

A sphere packing P is said to be:

(i) *Stable* (*also* 1-*stable*) if each ball is fixed by its neighbors.
(ii) n-*Stable* if each set of n balls is fixed by its neighbors.
(iii) *Finitely stable* if it is n-stable for every $n \geq 1$.
(iv) *Uniformly stable* if there is an $\varepsilon > 0$ such that no finite subset of the balls of P can be rearranged such that each ball is moved a distance less than ε and the rearranged balls together with the rest of balls form a packing different from P.

Delone Triangulation. Let C be a discrete sphere covering of E^d with unit balls. Let S_0 be the family of center points of the balls of C. Consider a d-dimensional ball whose interior does not contain any center point from S_0 and whose boundary intersects S_0 in a set spanning E^d. The convex hull of the intersection of the d-dimensional ball with S_0 is a *Delone cell*. Every Delone cell can be obtained this way. It turns out that the Delone cells from a tiling D or D^d, the *Delone tiling*. We illustrate the planar case in Fig. 1, which shows the Delone tiling determined by the centers of a discrete circle covering of the plane with unit circles. The circle on Fig. 1 is to show how an empty circle determines a Delone cell. Note that, since C is a covering, each Delone cell of D is inscribed in a sphere of radius ≤ 1 and if it is inscribed in a sphere of radius 1, then its interior contains the center of that sphere. Moreover, a slight perturbation of the covering C yields a covering whose Delone

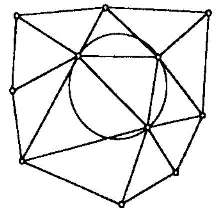

Fig. 1. Delone tiling.

tiling consists only of simplices. Thus, having obtained C as a limit of sphere coverings of E^d with unit balls whose center points lie in general position, we can triangulate the Delone tiling of C into simplices called *Delone simplices*, without introducing new vertices. The triangulation obtained in this way is called a *Delone triangulation*, associated with the covering C. The Delone triangulation is not necessarily unique.

Frameworks. The following definitions are taken from [AR]. Let G be a finite or infinite graph with no loops or multiple edges. We refer to the vertices of G by $1, 2, \ldots$. Each edge is labeled either a *bar*, *cable*, or a *strut*. Each vertex is labeled either *fixed* or *variable*. A realization of G in E^d is an assignment of a point p_i in E^d, for the ith vertex of G, $i = 1, 2, \ldots$. There is no restriction about edges crossing each other. A realization of G is denoted by $G(p)$ where $p = (p_1, p_2, \ldots)$. $G(p)$ is called the *(tensegrity) framework*.

Let $p(t) = (p_1(t), p_2(t), \ldots)$, where $p_i(t)$ is continuous for $0 \le t \le 1$, $p(0) = p$, and $p_i(t) = p_i$, for the fixed vertices i of G, for all t. We say $p(t)$ is a *continuous motion or a flex* of $G(p)$ if the cables are not increased, the bars are not changed, and the struts are not decreased in length. If $G(p)$ admits only the identity flex, then we say $G(p)$ is *rigid*.

A framework $G(p)$ with infinitely many vertices is said to be *finitely rigid* if labeling all but finitely many vertices fixed always result in a rigid framework. This is the same as saying that the framework does not allow any motion once all but finitely many vertices are pinned down. Pinned vertices are not allowed to move.

A *stress* $\omega = (\ldots, \omega_{ij}, \ldots)$ of the finite framework $G(p)$ is an assignment of a scalar $\omega_{ij} = \omega_{ji}$ for each edge $\{i, j\}$ of G such that the following equilibrium equation holds for each variable vertex i of G:

$$\sum_j \omega_{ij}(p_j - p_i) = 0.$$

This vector sum is taken over all vertices j adjacent to i. A *proper stress* is a stress ω such that $\omega_{ij} = \omega_{ij} \ge 0$ of $\{i, j\}$ is a cable, and $\omega_{ij} = \omega_{ji} \le 0$ if $\{i, j\}$ is a strut (no condition on bars).

Outline of the Rest of the Paper. Section 2 contains the description of our cabling method, which leads to applying the rigidity theory to determine the finite stability of discrete sphere coverings in E^d with unit balls. We prove Theorem 1, which basically says that moving a finite subset of a sphere covering with unit balls and maintaining the covering property is equivalent to moving a finite subframework in one of the associated cabled framework. In Section 3, using Theorem 1, we show that the thinnest cubic lattice covering of E^d with unit balls is not finitely stable (Theorem 2) and so is not uniformly stable. For the sake of comparison, in Section 4 we mention without proofs (as they are the subject of another paper under preparation) some analogous results concerning the stabilities of sphere packings. We also state some open problems (Remarks 1 and 5).

2. Finite Stability of Sphere Coverings

Associating an Infinite Cabled Framework with a Delone Triangulation. Let C be a discrete sphere covering of E^d with unit balls. Let D be one of the Delone triangulations associated with the covering. We mentioned before that each Delone simplex of this triangulation D is inscribed in a sphere of radius ≤ 1. Let S_d be the collection of those possible degenerated d-simplices of the Delone triangulation D which are inscribed in a unit sphere and contain (either inside or on the boundary) the center of their circumscribed sphere. Let the cablestar CS_d be the family of $(d + 1)$ cables of length 1 that connect the vertices of a d-simplex of S_d to the center of the circumscribed sphere. We illustrate the planar case in Fig. 2, which shows a Delone triangle of circumradius 1 and shows how a triplet of cables (called cablestar) is attached to the vertices of the Delone triangle. Finally, consider the cabled framework, whose vertices are the vertices of the d-simplices of S_d together with the centers of the circumscribed spheres of the d-simplices in S_d, and whose cables are the cables of CS_d. This infinite cabled framework is referred to as the *Delone cabled framework* associated to the Delone triangulation D generated by the sphere covering C.

We show that:

Theorem 1. *A discrete sphere covering C of E^d $(d \geq 2)$ with unit balls is finitely stable if and only if all Delone cabled frameworks associated to Delone triangulations generated by the sphere covering are finitely rigid.*

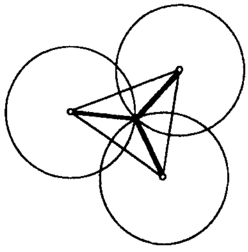

Fig. 2. A Delone triangle in the plane with a cablestar.

Proof. We show first that if the covering is finitely stable, then all Delone cabled frameworks are finitely rigid. Suppose that there is a Delone cabled framework which has a nontrivial flex. Then this flex is a flex of the covering, a contradiction.

We show next that if all of the Delone cabled frameworks are finitely rigid, then the covering is finitely stable. Assume that the discrete sphere covering C is not finitely stable. Then there is a finite subset of balls in C which can be moved and maintain the covering property. Reversing this motion we have that there is a sequence C_n of sphere coverings that tend to the covering C. Let D_n (resp. D) be the Delone tiling of C_n (resp. C). Let P be an arbitrary Delone cell of D whose circumradius is 1. Obviously, P can be obtained as the limit of the union of the Delone cells, say, $P_{1,n}, P_{2,n}, \dots, P_{m,n}$, where n tends to infinity and $P_{i,n}$ denotes a Delone cell of D_n and m is fixed. Let DS_n (resp. DS) be a Delone triangulation generated by the sphere covering C_n (resp. C) such that the limit configuration of DS_n is DS. Then we have a triangulation of P that triangulates the union of the possible lower-dimensional polytopes $\lim P_{1,n}, \lim P_{2,n}, \dots, \lim P_{m,n}$.

Finally, we take those possible degenerated d-simplices of the triangulation of P that contain the circumscribed center of P. We repeat this for other Delone cells of D whose circumradii are 1. This way we get a Delone cabling of C that has a discrete flex. Finally, it is easy to see that any discrete flex of a cabled framework can be extended to a continuous flex, a contradiction. To see that this is enough, take four points A_0, A_1, B_0, and B_1 and assume that the points with indices 0 and the points with indices 1 are connected with cables of equal lengths. Denote by $A_{1/2}$ and $B_{1/2}$ the midpoints of the segments A_0A_1 and B_0B_1. It is an elementary geometrical fact that $A_{1/2}B_{1/2} \le \max(A_0B_0, A_1B_1)$ which implies that the distance between the points $A_t = A_0 + t\overline{A_0A_1}$ and $B_t = B_0 + t\overline{B_0B_1}$, $t \in [0,1]$, is also $\le \max(A_0B_0, A_1B_1)$. In other words $A_tB_t, t \in [0,1]$, can be considered a continuous motion between the two given cables. □

As a simple application we point out:

Corollary 1. *The thinnest lattice covering of E^3 with unit balls is finitely stable.*

Proof. Recall the following result of Connelly [C1]: Given a connected pinned tensegrity framework $G(p)$ with only cables and a proper stress nonzero on each cable, $G(p)$ is uniquely realizable and rigid. Then notice that the Delone cabled framework associated to the thinnest lattice covering of E^3 with balls has a positive stress that is nonzero on each cable. □

Remark 1. It is very natural to conjecture that the thinnest lattice covering of E^3 with unit balls is uniformly stable.

For the sake of completeness we state (omitting the simple proof) an equivalent condition for the finite stability of sphere coverings of E^d with unit balls:

Remark 2. A sphere covering C of E^d with unit balls is finitely stable if and only if, for any finite subset C' of the balls of C, an $\varepsilon_{C'} > 0$ exists such that the balls of C' cannot be rearranged such that each ball is moved a distance less than $\varepsilon_{C'}$ and the rearranged balls together with the rest of the covering form a covering different from C.

3. Finite and Uniform Stability of Cubic-Lattice Coverings

As an application of Theorem 1 we show:

Theorem 2. *The thinnest cubic-lattice covering of E^d with unit balls is not finitely stable and so is not uniformly stable for $d \geq 2$.*

Proof. In view of the structure of the cubic lattice in E^d, it is enough to show Theorem 2 for $d = 2$. Notice that the Delone tiling associated to the thinnest square-lattice covering of the plane with circles is square-lattice tiling. Each square tile can be triangulated in two ways, thus, we have two possibilities of attaching a pair of triads of cables in each square tile. By choosing one for each intersection point we generate the family of cabled frameworks. Consider the one shown in Fig. 3(a). Among its properties we emphasize that two double cables and two single cables are alternatively arranged along each diagonal.

Recall the following result proved in [C2]: If $G(p)$ is a rigid framework with at least one cable, then $G(p)$ has a nonzero proper stress. In view of this theorem, in order to see that six circles centered at the vertices of two neighboring square tiles allow a covering preserving motion, it is enough to verify that the pinned cable framework shown in Fig. 3(b) has only the zero stress. In fact this follows immediately from the equilibrium equations discussed in the first section. □

4. Some Analogous Results Concerning Sphere Packings

Omitting the very simple proof we mention the following equivalent condition for the finite stability of sphere packings of E^d with unit balls.

Remark 3. A sphere packing P of E^d $(d > 1)$ with unit balls is finitely stable if and only if, for any finite subset P' of the balls of P, an $\varepsilon_{P'} > 0$ exists such that the balls

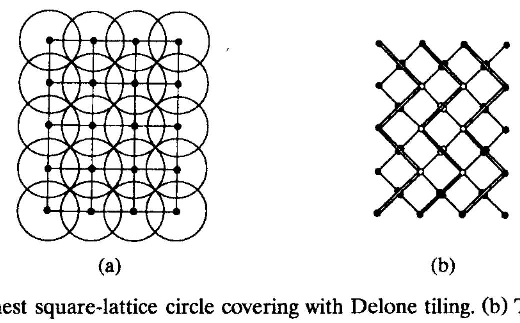

Fig. 3. (a) The thinnest square-lattice circle covering with Delone tiling. (b) The cabling associated with one of the Delone triangulations.

of P' cannot be rearranged such that each ball is moved a distance less than $\varepsilon_{P'}$ and the rearranged balls together with the rest of the packing form a packing different from P.

Let P be a packing of E^d ($d > 1$) with unit balls. It is quite natural to define *the graph* of P as the graph of E^d whose vertices are the centers of the unit balls of P and whose edges are the segments connecting the centers of contiguous balls. In view of the paragraph under "Frameworks", labeling the edges struts, this graph is also a tensegrity framework, called the *tensegrity framework generated by the packing*. Though the following equivalent condition for the finite stability of sphere packings can be proved in a very trivial way it has several applications.

Remark 4. A packing P of E^d ($d > 1$) with unit balls is finitely stable if and only if the tensegrity framework generated by the graph of the packing P is finitely rigid.

In [BBC] among others we show that:

(i) The densest cubic-lattice packing of E^3 with unit balls is finitely stable.
(ii) The densest cubic-lattice packing of E^d with unit balls is not uniformly stable, but it is finitely stable.

Finally, let us note:

Remark 5. It may be conjectured that the densest lattice packing of E^3 with unit balls is uniformly stable.

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Received February 13, 1994, and in revised form September 28, 1994.