A Flexible Sphere

Robert Connelly

It was conjectured for a long time that a closed polyhedral surface in Euclidean space \( E^3 \), with hinges along the edges, could not be continuously deformed to give non-congruent surfaces, as long as each face remained congruent to itself ("remained rigid"). In 1813 Cauchy proved that every convex polyhedral surface, with rigid natural faces, is inflexible.

The flexibility of a polyhedral surface with triangular faces is equivalent to the flexibility of the framework of rigid rods along its edges, flexibly attached at their common end points. In 1897 Bricard constructed flexible octahedral rod frameworks. However, filling in all flat triangles of such a flexible "octahedron" gives self-intersections, and not a flexible surface. I finally refuted the conjecture with a counter-example. What follows is a modified version of my construction of a flexible polyhedral sphere. The modification is due to N.H. Kuiper and Pierre Deligne.

**Construction**

The surface will be composed of three parts, a bottom part, a top part, and a crinkle.

To understand how these pieces flex it is helpful to describe the flexible octahedra of Bricard. Start with a 4-gon \( aba'b' \) in 3-space with equal opposite sides

\[
ab = a'b', a'b = ab'.
\]

Assuming \( a, b, a', b' \) are not all on one line, one can easily prove that there is a unique line \( \alpha \) meeting the diagonals \( aa' \) and \( bb' \) in their centers such that rotation by \( 180^\circ \) about \( \alpha \) leaves the 4-gon invariant by interchanging \( a \) with \( a' \) and \( b \) with \( b' \).

Choose a point \( c \) not on the line \( \alpha \). The framework \( c(aba'b') \) obtained by joining \( c \) with the vertices of the 4-gon will be flexible if \( aba'b' \) are not coplanar. So flex it and join to it at each instant the congruent framework \( c'(a'b'ab) \) obtained from the first by rotation by \( 180^\circ \) about \( \alpha \). The union is one of the flexible octahedra of Bricard. It is easy to make with straws and strings.

In Figure 2 \( \alpha \) is vertical and we have an orthogonal projection onto a horizontal plane \( H \). Here the points \( a \) and \( a' \) are at a height \( 0 \), the points \( b \) and \( b' \) at height \( e > 0 \), and the points \( c \) and \( c' \) at height \( \delta > e \) above \( H \). \( cc' \) is parallel to \( ab \).

![Figure 2](image)

Seen from above \( ac \) is under \( bc' \), and \( a'c' \) is under \( b'c \).

The bottom part of the surface is obtained by deleting \( c \) and the four rods incident to \( c \) and filling in the 4 remaining flat triangles except \( abc' \). The bottom surface is completed with an upside down bottomless tetrahedron on \( abc' \), leaving a three-dimensional hole for later use.

![Figure 3](image)

The upper part of the surface is obtained by first deleting \( c' \) and the 4 rods incident to it together with the rod \( ac \) from Figure 2. Next consider the pyramids \( d(bac) \) and \( d'(b'ac) \) with \( d \) and \( d' \) chosen well above \( H \) so that \( da = dc = d'a = d'c \). From these pyramids take the flat triangular faces \( dab, dcb, d'ab', d'cb' \) and obtain the upper surface.
Then the frameworks $d(cefa)$ and $d'(cefa)$ flex in conjunction, while $cefa$ remains coplanar. The flexible union of the triangular faces $dce$, $def$, $d'fa$, $d'ef$, $d'fa$ is crinkle, another Bricard octahedron, but with the two triangular faces $cd'a$ and $cda$ removed. Thus the distance from $a$ to $c$ remains constant during the flex. When this surface is fitted into the hole of Figure 4 the final flexible surface appears, Figure 6. The last point chosen, $f$, fits neatly into the pit left for it in the bottom surface.

This flexible triangulated sphere has 11 vertices and 18 faces. Subsequent to my construction a flexible sphere with a smaller number of vertices was found by Klaus Steffen. It has 9 vertices and is constructed as shown in Figure 7. The arrows indicate which edges are glued and the following choice of the edge lengths works well:

$$u = 6, v = 5, w = 2.5, x = 5.5, y = 8.5.$$