A Flexible Sphere

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It was conjectured for a long time that a closed polyhedral surface in Euclidean space E^3, with hinges along the edges, could not be continuously deformed to give non-congruent surfaces, as long as each face remained congruent to itself ("remained rigid"). In 1813 Cauchy proved that every *convex* polyhedral surface, with rigid natural faces, is inflexible.

The flexibility of a polyhedral surface with triangular faces is equivalent to the flexibility of the framework of rigid rods along its edges, flexibly attached at their common end points. In 1897 Bricard constructed flexible octahedral rod frameworks. However, filling in all flat triangles of such a flexible "octahedron" gives self-intersections, and not a flexible surface. I finally refuted the conjecture with a counter-example. What follows is a modified version of my construction of a flexible polyhedral sphere. The modification is due to N.H. Kuiper and Pierre Deligne.

Construction

The surface will be composed of three parts, a bottom part, a top part, and a crinkle.

To understand how these pieces flex it is helpful to describe the flexible octahedra of Bricard. Start with a 4-gon $aba'b'$ in 3-space with equal opposite sides

$$ab = a'b', \; a'b = ab'.$$

Assuming a, b, a', b' are not all on one line, one can easily prove that there is a unique line α meeting the diagonals aa' and bb' in their centers such that rotation by 180° about α leaves the 4 gon invariant by interchanging a with a' and b with b'.

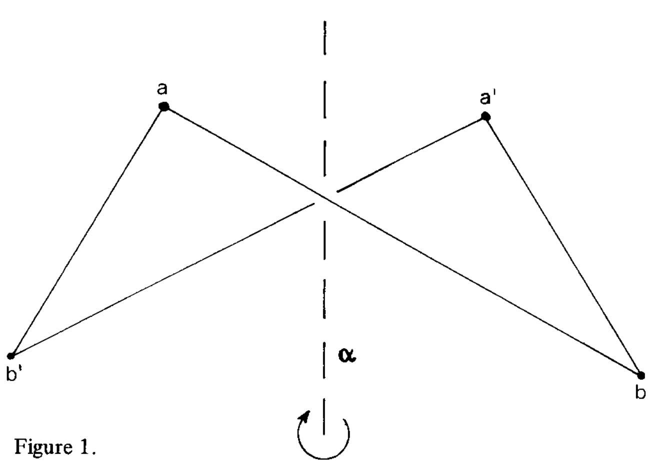

Figure 1.

Choose a point c not on the line α. The framework $c(aba'b')$ obtained by joining c with the vertices of the 4-gon will be flexible if $aba'b'$ are not coplanar. So flex it and join to it at each instant the congruent framework $c'(a'b'ab)$ obtained from the first by rotation by 180° about α The union is one of the flexible octahedra of Bricard. It is easy to make with straws and strings.

In Figure 2 α is vertical and we have an orthogonal projection onto a horizontal plane H. Here the points a and a' are at a height 0, the points b and b' at height $\epsilon > 0$ and the points c and c' at height $\delta > \epsilon$ above H. cc' is parallel to ab.

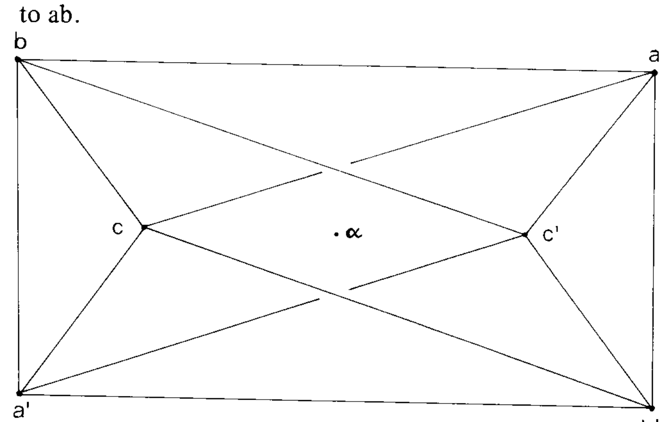

Figure 2.

Seen from above ac is under bc', and a'c' is under b'c.

The bottom part of the surface is obtained by deleting c and the four rods incident to c and filling in the 4 remaining flat triangles except abc'. The bottom surface is completed with an upside down bottomless tetrahedron on abc', leaving a three-dimensional hole for later use.

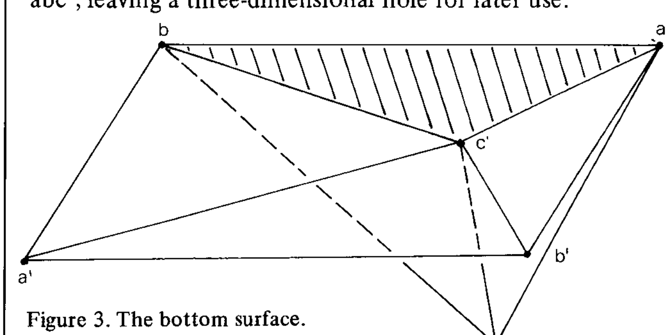

Figure 3. The bottom surface.

The upper part of the surface is obtained by first deleting c' and the 4 rods incident to it together with the rod ac from Figure 2. Next consider the pyramids d(bac) and d'(b'ac) with d and d' chosen well above H so that da = dc = d'a = d'c. From these pyramids take the flat triangular faces dab, dcb, d'ab', d'cb' and obtain the upper surface.

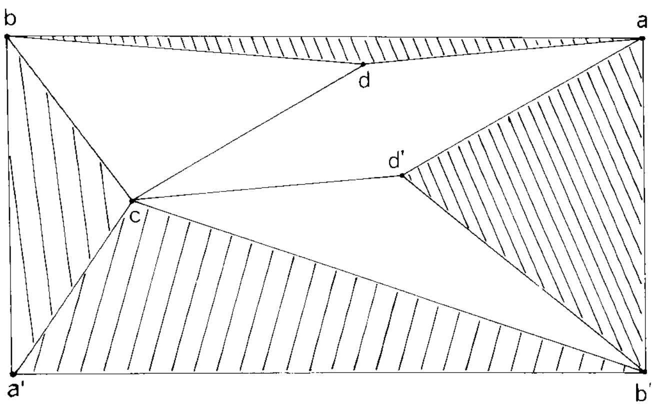

Figure 4a. The upper surface.

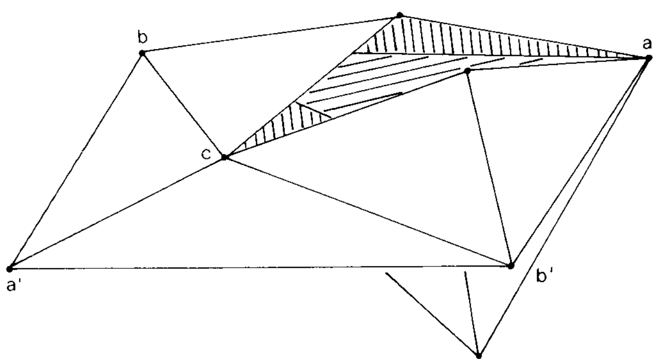

Figure 4b. The upper and lower surfaces glued together.

This is glued to the bottom surface along the 4-gon aba′b′ and yields a surface with boundary which is flexible and keeps the distance ac fixed.

The straight line ca, if drawn, would pierce the bottom surface under bc′. In order to avoid this self-intersection, we construct a new part, the crinkle, to complete the flexible surface. Choose a 4-gon ceaf in the median plane K of d and d′ with opposite equal sides ce = af, ca = ef as in Figure 5. The points must satisfy

dc = de = df = da = d′c = d′e = d′f = d′a

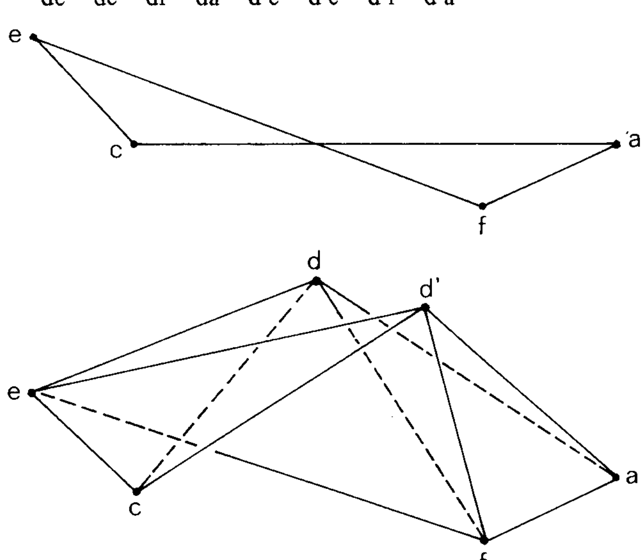

Figure 5.

Then the frameworks d(cefa) and d′(cefa) flex in conjunction, while cefa remains coplanar. The flexible union of the triangular faces dce, def, dfa, d′ce, d′ef, d′fa is crinkle, another Bricard octahedron, but with the two triangular faces cd′a and cda removed. Thus the distance from a to c remains constant during the flex. When this surface is fitted into the hole of Figure 4 the final flexible surface appears, Figure 6. The last point chosen, f, fits neatly into the pit left for it in the bottom surface.

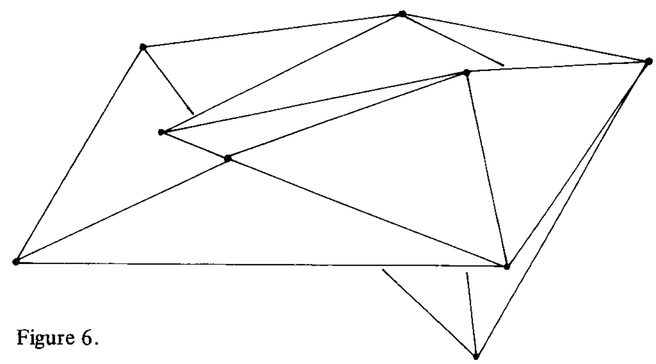
Figure 6.

This flexible triangulated sphere has 11 vertices and 18 faces. Subsequent to my construction a flexible sphere with a smaller number of vertices was found by Klaus Steffen. It has 9 vertices and is constructed as shown in Figure 7. The arrows indicate which edges are glued and the following choice of the edge lengths works well:

$u = 6, v = 5, w = 2.5, x = 5.5, y = 8.5.$

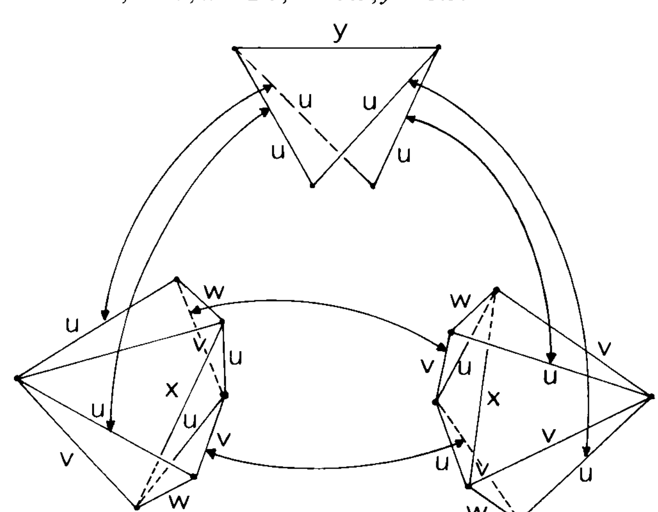

Figure 7.

References

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