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The polyhedral Tammes problem

By

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Abstract. The central question of this paper is the following somewhat more general version of the well-known Tammes problem, which we call the polyhedral Tammes problem. For given integers $n \geqq d+1 \geqq 3$ find the maximum of the shortest distance between any two of the n vertices of a convex d-polytope with diameter 1 in $\mathbf{E^d}$. We study this question for large n, as well as for some small values of n for fixed d.

1. Introduction and Results. Independently, Larman and Tamvakis [12], Reinhardt [13] and Vincze [17] proved the following interesting result. If $n = m2^l$, where m is an odd integer greater than 1 and l is a nonnegative integer, then among all convex n-gons of diameter 1, those with the largest perimeter are exactly the ones which have equal sides and are inscribed in a Reuleaux polygon of constant width 1, so that the vertices of the Reuleaux polygon are also vertices of the polygon of maximum perimeter. Somewhat surprisingly the solution of this problem for $n = 2^s, s > 2$ remains open (see [9]). The problem of Bateman and Erdős [1] raises a related discrete question. Of all sets of n points with mutual distances at least 1 in d-dimensional Euclidean space $\mathbf{E^d}$, which set has the minimum diameter $g(n,d)$? The solution of this problem in the plane is known up to 8 points (see [3] for the case of 8 points in the plane and for a survey on planar results). The value $g(d+2,d) = \sqrt{2}\left(\frac{\lceil \frac{d}{2}\rceil}{\lceil \frac{d}{2}\rceil + 1} + \frac{\lfloor \frac{d}{2}\rfloor}{\lfloor \frac{d}{2}\rfloor + 1}\right)^{-\frac{1}{2}}, d \geqq 2$, has been independently determined by several people [2], [4], [10], [14], [15]. Moreover, it is known [14] that $g(6,3) = \sqrt{2}$. Motivated by these results, we raise the following questions.

Problem A. Find the minimum diameter $f(n,d)$ of convex d-polytopes with n vertices having edge lengths $\geqq 1$ in $\mathbf{E^d}$.

Problem B. Find the minimum diameter $F(n,d)$ of convex d-polytopes with n vertices having pairwise distances $\geqq 1$ in $\mathbf{E^d}$.

It is obvious that $1 \leqq f(n,d) \leqq F(n,d)$ and $1 \leqq g(n,d) \leqq F(n,d)$ for all $n \geqq d+1 \geqq 3$. Moreover, the above results imply that for $d > 2$, Problems A and B have a new character,

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and one can obtain only the following short list of extremal values as an immediate corollary: $f(d+1,d) = F(d+1,d) = g(d+1,d) = 1$, $f(d+2,d) = F(d+2,d) = g(d+2,d), d \geqq 3$ and $g(6,3) = F(6,3) = \sqrt{2}$. In this paper, we prove the following results, the first of which is a counterexample to Conjecture 6 in [16], and the second of which is an anologue to the main theorem of [18] (see also [6]).

Theorem 1.1. $f(n,d) \leqq 3$ *for all* $n \geqq d+1 \geqq 4$.

In contrast to Theorem 1.1, the next theorem says that the limiting shape of the extremal configurations for Problem B is spherical as $n \to \infty$. In order to phrase this properly, we introduce a bit of notation. Let $\mathscr{P}_{n,d}$ be the family of convex d-polytopes with n vertices having pairwise distances $\geqq 1$ which have the smallest possible diameter in $\mathbf{E^d}$. Let B^d denote the closed d-dimensional ball of radius 1 centered at the origin $\mathbf{o}$ in $\mathbf{E^d}$. Moreover, if P is a convex d-polytope in $\mathbf{E^d}$, then let $\lambda(P)$ denote its *sphericity ratio*, defined by

$$\lambda(P) = \min_{\mathbf{x} \in \mathbf{E^d}, r, R} \left\{ \frac{R}{r} \,\middle|\, \mathbf{x} + rB^d \subset P \subset \mathbf{x} + RB^d \right\}.$$

Finally, we define the smallest ratio $\lambda_{n,d}$ of the radii of two concentric $(d-1)$-dimensional spheres between which the boundary of any convex d-polytope $P \in \mathscr{P}_{n,d}$ can be squeezed, as follows

$$\lambda_{n,d} = \max_{P \in \mathscr{P}_{n,d}} \lambda(P).$$

Theorem 1.2. $\lim_{n \to \infty} \lambda_{n,d} = 1$ *for all* $d \geqq 3$.

Turning to extremal configurations in $\mathbf{E^3}$, recall that a nice result of Schütte ([14]) says that $g(6,3) = \sqrt{2}$, which is achieved by the regular octahedron of edge length 1. Consequently, we have $F(6,3) = \sqrt{2}$. In connection with this we prove the following somewhat stronger statement.

Theorem 1.3. $f(6,3) = \sqrt{2}$ *and the only extremal polyhedra are the regular octahedron of edge length 1 and the right prism of height 1 with regular triangle base of side length 1.*

Finally, it seems useful to rephrase Problem B in the following way. One can view Problem B as a somewhat more general version of the well-known Tammes problem, which asks for the maximum of the shortest spherical distance among n points on the unit sphere (for a short survey on Tammes problem, see [8], for example). Thus, Problem B is equivalent to the following problem.

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From the many seemingly interesting open cases, we mention here, the following two only. For $d = 2$ and any $n \neq 2^s, s > 2$, the Polyhedral Tammes Problem follows easily from the results in [12], [13], [17]. But, it seems to be open for the remaining integers, that is, for all $n = 2^s, s > 2$ in $\mathbf{E^2}$. In $\mathbf{E^3}$, it is natural to conjecture that among convex polyhedra of diameter 1 and of 12 vertices, the regular icosahedron has the largest minimum distance occuring between pairs of vertices.

2. Proof of Theorem 1.1. We explain our construction in $\mathbf{E^3}$ only. The higher dimensional construction is essentially the same.

Let the positive integer $n \geqq 4$ be given. In what follows, we inductively define n points

$$\{\mathbf{a}, \mathbf{b}, \mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{n-3}\} \subset \mathbf{E}^3$$

in terms of their Cartesian coordinates such that they form the vertices of a convex polyhedron of diameter 3 having edge lengths > 1.

Step 0. Let $\mathbf{a} = (0,0,0), \mathbf{b} = (3,0,0)$ and $\mathbf{c}_0 = (2,1,0)$.

Step 1. Let $0 < \varepsilon_1 < \frac{1}{2}$ be an arbitrary real number, and let $0 < \delta_1$ be sufficiently small. Then let

$$\mathbf{c}_1 = (1, \varepsilon_1, \delta_1).$$

Obviously, the convex hull $P_1 = \operatorname{conv}\{\mathbf{a}, \mathbf{b}, \mathbf{c}_0, \mathbf{c}_1\}$ is a tetrahedron of diameter 3 having edge lengths > 1.

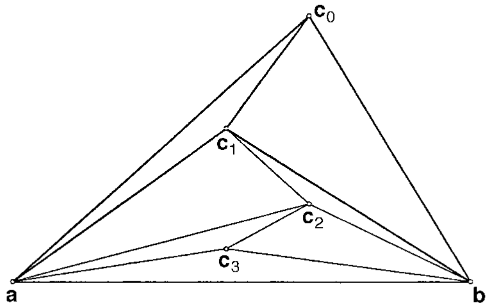

Figure 1.

Step 2. Let $\mathbf{c}_2 = (2, \varepsilon_2, \delta_2)$ with ε_2, δ_2 chosen as follows. On the one hand, $0 < \varepsilon_2 < \frac{1}{2}\varepsilon_1$, and on the other hand, $0 < \delta_2$ is chosen such that the convex polyhedron $P_2 = \operatorname{conv}(\{\mathbf{c}_2\} \cup P_1)$ compared to P_1 has an additional new vertex namely, $\mathbf{c}_2$ and three more edges namely, the line segments connecting $\mathbf{c}_2$ to $\mathbf{c}_1$ as well as to $\mathbf{a}$ and $\mathbf{b}$. Obviously, P_2 is a convex polyhedron with 5 vertices having edge lengths > 1 and diameter 3.

Now, after performing Step $n - 4$, we arrive at the points $\mathbf{a}, \mathbf{b}, \mathbf{c}_0, \ldots, \mathbf{c}_{n-4}$ with the property that the convex polyhedron $P_{n-4} = \operatorname{conv}\{\mathbf{a}, \mathbf{b}, \mathbf{c}_0, \ldots, \mathbf{c}_{n-4}\}$ has $n - 1$ vertices and edge lengths > 1 moreover, $\operatorname{conv}\{\mathbf{a}, \mathbf{b}, \mathbf{c}_{n-4}\}$ is a triangular face of P_{n-4}. (Figure 1 shows the "top view" of P_3.) The following step then completes our construction in $\mathbf{E}^3$.

Step $n - 3$. Case (i): n is odd. Let $\mathbf{c}_{n-3} = (2, \varepsilon_{n-3}, \delta_{n-3})$ with $\varepsilon_{n-3}, \delta_{n-3}$ chosen as follows. On the one hand, $0 < \varepsilon_{n-3} < \frac{1}{2}\varepsilon_{n-4}$, and on the other hand, $0 < \delta_{n-3}$ is chosen such that the convex polyhedron $P_{n-3} = \operatorname{conv}(\{\mathbf{c}_{n-3}\} \cup P_{n-4})$ compared to P_{n-4} has an additional new vertex namely, $\mathbf{c}_{n-3}$ and three more edges namely, the line segments connecting $\mathbf{c}_{n-3}$ to $\mathbf{c}_{n-4}$ as well as to $\mathbf{a}$ and $\mathbf{b}$. The convex polyhedron P_{n-3} obtained in this way is the desired one namely, it is of diameter 3 having n vertices and edge lengths > 1.

Case (ii): n is even. In this case, let $\mathbf{c}_{n-3} = (1, \varepsilon_{n-3}, \delta_{n-3})$ with $\varepsilon_{n-3}, \delta_{n-3}$ chosen exactly in the same way as it is described in *Case (i)*.

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2. Let K be a convex body in $\mathbf{E^d}$ (i.e. a compact convex set with nonempty interior in $\mathbf{E^d}$). Let $\varepsilon > 0$, and let $n(K, \varepsilon)$ be the maximum number of boundary points of K with pairwise distances $\geqq \varepsilon$. Then let $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{n(K,\varepsilon)}\} \subset \mathrm{bd}K$ be boundary points of K such that $\|\mathbf{q}_i - \mathbf{q}_j\| \geqq \varepsilon$ for all $1 \leqq i < j \leqq n(K, \varepsilon)$, where of course, $\|\dots\|$ stands for the Euclidean norm of the given vector. Finally, let $B^d[\mathbf{c}, r]$ denote the closed d-dimensional ball of radius $r > 0$ centered at the point $\mathbf{c}$ in $\mathbf{E^d}$. In particular, let $B^d = B^d[\mathbf{o}, 1]$.

Definition 3.1. The balls $B^d[\mathbf{q}_i, \frac{\varepsilon}{2}]$, $1 \leqq i \leqq n(K, \varepsilon)$ generate a packing on the boundary $\mathrm{bd}K$ of K, the *density* of which we define by

$$\delta[\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{n(K,\varepsilon)}] = \frac{\sum_{i=1}^{n(K,\varepsilon)} \mathrm{Surf}_{d-1}(B^d[\mathbf{q}_i, \frac{\varepsilon}{2}] \cap \mathrm{bd}K)}{\mathrm{Surf}_{d-1}(\mathrm{bd}K)},$$

where $\mathrm{Surf}_{d-1}(\dots)$ refers to the proper surface area measure.

Definition 3.2. Let $\delta(L)$ be the largest *(upper) density* of packings of translates of the convex body L in $\mathbf{E^d}$.

Lemma 3.1. $\lim_{\varepsilon \to 0^+} \delta[\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{n(K,\varepsilon)}] = \delta(B^{d-1})$.

Proof. As Lemma 3.1 can be regarded as a variant of the following theorem of Hlawka ([11]), we sketch only the main steps of the proof of Lemma 3.1.

Lemma 3.2 (Hlawka [11]). *For a given convex body $L \subset \mathbf{E^d}$, we denote by $m(L, M)$ the maximum number of bodies $\mathbf{x} + L, \mathbf{x} \in \mathbf{E^d}$ which can be packed into the convex body $M \subset \mathbf{E^d}$. Then*

$$\lim_{\sigma \to +\infty} \frac{m(L, \sigma M)\mathrm{Vol}_d(L)}{\mathrm{Vol}_d(\sigma M)} = \delta(L)$$

exists and does not depend on M, where $\mathrm{Vol}_d(\dots)$ stands for the standard d-dimensional volume measure in $\mathbf{E^d}$.

By packing a maximum number of $(d-1)$-dimensional balls of radii $\frac{\varepsilon}{2}$ into the facets of a convex d-polytope $K \subset \mathbf{E^d}$, one can easily see that Lemma 3.2 implies Lemma 3.1 for that K. The general case can be obtained from this as follows. Let $K \subset \mathbf{E^d}$ be a convex body that is contained by the convex d-polytope T and contains the convex d-polytope I. The convexity of these sets implies that $n(I, \varepsilon) \leqq n(K, \varepsilon) \leqq n(T, \varepsilon)$ for all $\varepsilon > 0$. Even more is true, however. Namely, it is not hard to see that for any given $\rho > 0$, there exists a positive real number τ depending on K only, such that for the above convex d-polytopes with $\mathrm{dist}_H(K, I) < \tau, \mathrm{dist}_H(K, T) < \tau$, we have that

$$\delta[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n(I,\varepsilon)}] - \rho < \delta[\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n(K,\varepsilon)}] < \delta[\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{n(T,\varepsilon)}] + \rho$$

for all sufficiently small $\varepsilon > 0$, where $\mathrm{dist}_H(\dots, \dots)$ stands for the Hausdorff distance ([5]) between the given convex bodies. From this, Lemma 3.1 follows in a rather straightforward way. $\square$

Now, we turn to the proof of Theorem 1.2. For $P_n \in \mathscr{P}_{n,d}$, let $R(P_n)$ and $r(P_n)$ denote the radii of the concentric closed d-dimensional balls, the larger of which contains P_n and the smaller of which is in P_n such that

$$\frac{R(P_n)}{r(P_n)} = \min_{\mathbf{x}\in\mathbf{E^d}}\left(\min_{\mathbf{x}+rB^d\subset P_n\subset\mathbf{x}+RB^d}\left(\frac{R}{r}\right)\right).$$

Obviously, $\frac{1}{R(P_n)}P_n \subset B^d$. Now, assume that $\lim_{n\to\infty}\lambda_{n,d} \neq 1$ for some $d \geqq 3$. Then Blaschke's selection theorem [5] implies that there exists a sequence $P_{n_i} \in \mathscr{P}_{n_i,d}, i = 1, 2, \ldots$ such that

$$\lim_{i\to\infty}\frac{1}{R(P_{n_i})}P_{n_i} = Q,$$

where $Q \subset B^d$ is a non-spherical compact convex set in $\mathbf{E^d}$. Hence, based on Bieberbach's theorem [5], let $r > 0$ such that

$$\operatorname{diam}(rB^d) \leqq \alpha\cdot\operatorname{diam}(Q) \text{ and } \operatorname{Surf}_{d-1}(rB^d) \geqq \beta\cdot\operatorname{Surf}_{d-1}(Q)$$

with properly chosen $0 < \alpha < 1, 1 < \beta$. Thus, if i is sufficiently large, then

$$\operatorname{diam}(R(P_{n_i})rB^d) < \alpha'\cdot\operatorname{diam}(P_{n_i}) \text{ and } \operatorname{Surf}_{d-1}(R(P_{n_i})rB^d) > \beta'\cdot\operatorname{Surf}_{d-1}(P_{n_i})$$

with some $\alpha < \alpha' < 1$ and $1 < \beta' < \beta$ independent from i. Consequently, if i is sufficiently large, then Lemma 3.1 and $\operatorname{Surf}_{d-1}(R(P_{n_i})rB^d) > \beta'\cdot\operatorname{Surf}_{d-1}(P_{n_i})$ imply that $n(R(P_{n_i})rB^d, 1) \geqq n(P_{n_i}, 1)$ and $\operatorname{diam}(R(P_{n_i})rB^d) < \operatorname{diam}(P_{n_i})$, a contradiction. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3. Let $\mathbf{S^{d-1}} = \{\mathbf{x}\in\mathbf{E^d} \mid \operatorname{dist}_E(\mathbf{o},\mathbf{x}) = 1\}$ be the $(d-1)$-dimensional unit sphere centered at the origin $\mathbf{o}$ of $\mathbf{E^d}$, where $\operatorname{dist}_E(\ldots,\ldots)$ stands for the Euclidean distance between two points of $\mathbf{E^d}$. Recall that a subset K of $\mathbf{S^{d-1}}$ is called *spherically convex* if no two points of K are antipodal moreover, for any two points of K the shorter great circular arc connecting the two points on $\mathbf{S^{d-1}}$ belongs to K. The intersection of all spherically convex sets that contain a finite pointset lying on an open $(d-1)$-dimensional hemisphere of $\mathbf{S^{d-1}}$ is called a *convex spherical polytope*. A *vertex* of a convex spherical polytope $P \subset \mathbf{S^{d-1}}$ is a boundary point of P that lies on a supporting $(d-2)$-dimensional great sphere of P intersecting P at the given boundary point only. Any convex spherical polytope is the spherical convex hull of its vertices, that is, any spherically convex set that contains all the vertices of a convex spherical polytope contains the convex spherical polytope itself. Note that the vertex set is the smallest set with this property. As usual, we measure the *spherical distance* $\operatorname{dist}_S(\mathbf{u},\mathbf{v})$ between the points $\mathbf{u},\mathbf{v}$ of $\mathbf{S^{d-1}}$ by the central angle $\angle\mathbf{uov}$.

Lemma 4.1. *Let P be a convex spherical polytope on $\mathbf{S^{d-1}}, \mathbf{d} \geqq 3$, and let λ be the largest spherical distance between any two vertices of P. If $\lambda < \frac{\pi}{2}$, then the spherical diameter* $\operatorname{diam}_S(P) = \max\{\operatorname{dist}_S(\mathbf{x},\mathbf{y}) \mid \mathbf{x}\in P, \mathbf{y}\in P\}$ *of P is equal to λ.*

Proof. Suppose that the claim does not hold. Then there exist $\mathbf{p}_1 \in P, \mathbf{p}_2 \in P$ such that $\operatorname{dist}_S(\mathbf{p}_1,\mathbf{p}_2) > \lambda$. Let $M = \{\mathbf{x}\in\mathbf{S^{d-1}} \mid \operatorname{dist}_S(\mathbf{p}_1,\mathbf{x}) \leqq \lambda\}$. Since $\lambda < \frac{\pi}{2}$, the set M is spherically convex. Note that since $\operatorname{dist}_S(\mathbf{p}_1,\mathbf{p}_2) > \lambda$, the point $\mathbf{p}_2$ is not in M. Thus, there exists a vertex $\mathbf{v}_1$ of P which does not lie in M. (One can see this as follows. Suppose that all vertices of P lie in M. Then, as M is spherically convex and P is the convex hull of its vertices, we get that $P \subset M$, but $\mathbf{p}_2 \in P$ and $\mathbf{p}_2 \notin M$, a contradiction.) Now, let $M' = \{\mathbf{x}\in\mathbf{S^{d-1}} \mid \operatorname{dist}_S(\mathbf{v}_1,\mathbf{x}) \leqq \lambda\}$.

Note that $\mathbf{p}_1 \notin M'$, and by the same argument, there exists a vertex $\mathbf{v}_2$ of P such that $\mathbf{v}_2 \notin M'$. But then $\mathrm{dist}_S(\mathbf{v}_1, \mathbf{v}_2) > \lambda$, a contradiction. $\square$

Remark 4.1. If $\lambda > \frac{\pi}{2}$, then Lemma 4.1 does not hold. (For example, take an isosceles spherical triangle of side lengths $\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} + \varepsilon, \varepsilon$ on $\mathbf{S^2}$, where $0 < \varepsilon$ is a small real number.) Moreover, using the same argument, it is easy to extend Lemma 4.1 for $\lambda = \frac{\pi}{2}$.

Lemma 4.2. *Let Q be a convex polytope in $\mathbf{E^d}$, $\mathbf{d} \geqq 3$, and let μ be largest angle between any two adjacent edges of Q. If $\mu \leqq \frac{\pi}{2}$ then the angles of any triangle spanned by three vertices of Q are less than or equal to μ.*

Proof. Suppose that the claim does not hold. Then there exist three vertices say, $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ of Q, such that $\angle \mathbf{v}_2\mathbf{v}_1\mathbf{v}_3$ is strictly larger than μ. Without loss of generality, we may assume that $\mathbf{v}_1 = \mathbf{o}$. Then let P be the central projection of Q from the center point $\mathbf{o}$ onto the $(d-1)$-dimensional unit sphere $\mathbf{S^{d-1}}$. By assumption, P is a convex spherical polytope on $\mathbf{S^{d-1}}$ with the property that the spherical distance between any two of its vertices is at most μ. As $\mu \leqq \frac{\pi}{2}$, Lemma 4.1 implies that $\mathrm{diam}_S(P) \leqq \mu$. In particular, $\mathrm{dist}_S(\mathbf{u}_2, \mathbf{u}_3) \leqq \mu$, where $\mathbf{u}_2$ (resp., $\mathbf{u}_3$) denotes the central projection of the vertex $\mathbf{v}_2$ (resp., $\mathbf{v}_3$) of Q from the center point $\mathbf{o}$ onto $\mathbf{S^{d-1}}$. But, $\mathrm{dist}_S(\mathbf{u}_2, \mathbf{u}_3) = \angle \mathbf{v}_2\mathbf{v}_1\mathbf{v}_3 > \mu$, a contradiction. $\square$

Now, we are in a position to prove $f(6,3) = \sqrt{2}$ in a rather short way. Let Q be a convex polyhedron with 6 vertices having edge lengths $\geqq 1$ in $\mathbf{E^3}$. Then Q must have two adjacent edges whose angle is not an acute angle. We prove this as follows. If the angle of any two adjacent edges of Q is an acute angle, then Lemma 4.2 implies that any triangle spanned by three vertices of Q is an acute triangle. But, a theorem of Croft ([7]) states that the maximum number of points in $\mathbf{E^3}$ having the property that the triangles determined by them are all acute, is 5, a contradiction. So, there exist three vertices say, $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ of Q, such that $\frac{\pi}{2} \leqq \angle \mathbf{v}_2\mathbf{v}_1\mathbf{v}_3$. As $1 \leqq \mathrm{dist}_E(\mathbf{v}_2, \mathbf{v}_1)$ and $1 \leqq \mathrm{dist}_E(\mathbf{v}_3, \mathbf{v}_1)$, we get that $\sqrt{2} \leqq \mathrm{dist}_E(\mathbf{v}_2, \mathbf{v}_3)$ and so the diameter of Q is at least $\sqrt{2}$. This implies that $\sqrt{2} \leqq f(6,3)$. Finally, recall that the regular octahedron of edge length 1 has diameter $\sqrt{2}$. Thus, $f(6,3) = \sqrt{2}$.

Next, let Q^* be a convex polyhedron with 6 vertices having edge lengths $\geqq 1$ and diameter $\sqrt{2}$ in $\mathbf{E^3}$. Obviously, the angle of any two adjacent edges of Q^* is $\leqq \frac{\pi}{2}$, and so, Lemma 4.2 implies that there is no obtuse triangle among the triangles spanned by the vertices of Q^*. Moreover, an elegant result of Croft ([7]) says that the following are the only six-point configurations with right or acute angles and no obtuse ones spanned by triplets of six points:

(α) The vertices of two right sections of a triangular right prism, the cross sections being congruent acute- or right-angled triangles;

(β) The four vertices of a rectangle, together with two points an equal distance vertically above two opposite corners of the rectangle;

(γ) A configuration formed of two vertices of two congruent triangles in parallel planes, such that if $\Delta\,\mathbf{abc}$ be one triangle, $\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$ the orthogonal projections of the other vertices upon it, the circumcircles of $\Delta\,\mathbf{abc}$, $\Delta\,\mathbf{x}^*\mathbf{y}^*\mathbf{z}^*$ coincide, and $\mathrm{dist}_E(\mathbf{a}, \mathbf{x}^*) = \mathrm{dist}_E(\mathbf{b}, \mathbf{y}^*) = \mathrm{dist}_E(\mathbf{c}, \mathbf{z}^*)$ is the diameter of it.

If we apply the above result to the vertex set of Q^*, it is immediate that (β) is simply not possible to have. For the remaining part notice that if four vertices of Q^* are coplanar, then they must form the vertex set of a unit square. Thus, in the case of (α), Q^* must be a right prism of height 1 with regular triangle base of side length 1, and in the case of (γ), we get that Q^* is a regular octahedron of edge length 1.

This finishes the proof of Theorem 1.3.

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