Discrete Comput Geom 32:101–106 (2004)
DOI: 10.1007/s00454-004-0831-1

Discrete & Computational
Geometry

The Kneser–Poulsen Conjecture for Spherical Polytopes*

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Abstract. If a finite set of balls of radius $\pi/2$ (hemispheres) in the unit sphere $\mathbb{S}^n$ is rearranged so that the distance between each pair of centers does not decrease, then the (spherical) volume of the intersection does not increase, and the (spherical) volume of the union does not decrease. This result is a spherical analog to a conjecture by Kneser (1954) and Poulsen (1955) in the case when the radii are all equal to $\pi/2$.

1. Introduction

Let $\mathbb{S}^n$ be the unit sphere in Euclidean $(n+1)$-dimensional space, and let $X(\mathbf{p})$ be a finite intersection of balls of radius $\pi/2$ (closed hemispheres) in $\mathbb{S}^n$ whose configuration of centers is $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$. We say that another configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is a *contraction* of $\mathbf{p}$ if, for all $1 \le i < j \le N$, the spherical distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ is not less than the spherical distance between $\mathbf{q}_i$ and $\mathbf{q}_j$. We denote n-dimensional spherical volume by $\mathrm{Vol}_n[\ ]$. Our main result is the following.

Theorem 1. *If* $\mathbf{q}$ *is a configuration in* $\mathbb{S}^n$ *that is a contraction of the configuration* $\mathbf{p}$, *then*

$$\mathrm{Vol}_n[X(\mathbf{p})] \le \mathrm{Vol}_n[X(\mathbf{q})].$$

* The first author was partially supported by the Hungarian National Science Foundation (OTKA), Grant No. T029786. The second author was partially supported as an Erdös Visiting Professor at Eötvös University, Budapest, and the National Science Foundation Grant Number DMS-0209595.

This result is to be compared with our previous result [2], where the ambient space is Euclidean space $\mathbb{E}^n$, and the set X is the intersection of balls of arbitrary radius, but for $n \geq 3$ we need to assume that there is a piecewise-analytic monotone motion of the configuration in $\mathbb{E}^{n+2}$. This uses the results of Csikós in [4] and [5], where the ambient space $\mathbb{E}^n$, and the balls which make up the space X, are of arbitrary radius, but there must be a continuous contraction of the configuration $\mathbf{p}$ to the configuration $\mathbf{q}$ in $\mathbb{E}^n$. For a history of earlier related results and the conjectures of Kneser [9] and Poulsen [10] as well as related conjectures, see our paper [2].

The method we use here follows the same general outline as in our paper [2]. We use a leapfrog lemma to move one configuration to the other in an analytic and monotone way, but only in higher-dimensions. Then the higher-dimensional balls have their combined volume (their intersections or unions) change monotonically, a fact that we prove using Schäfli differential formula. Then we apply an integral formula to relate the volume of the higher dimensional object (or the volume of the boundary of the higher dimension set in our previous case) to the volume of the lower-dimensional object, obtaining the volume inequality for the more general discrete motions.

The following are some corollaries following from Theorem 1 and its proof. The first one is an immediate set theoretic consequence.

Corollary 1. *If a finite set of balls of radius $\pi/2$ in the unit sphere $\mathbb{S}^n$ is rearranged so that the distance between each pair of centers does not increase, then the (spherical) volume of the union does not increase.*

Corollary 2. *Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ be N points on a hemisphere of $\mathbb{S}^2$ (resp., points in $\mathbb{E}^2$), and let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ be a contraction of $\mathbf{p}$ in $\mathbb{S}^2$ (resp., in $\mathbb{E}^2$). Then the perimeter of the convex hull of $\mathbf{q}$ is less than or equal to the perimeter of the convex hull of $\mathbf{p}$.*

Proof. This theorem in $\mathbb{S}^2$ follows from taking the spherical polar of the convex hull of the configurations $\mathbf{p}$ and $\mathbf{q}$, which are the intersections of the hemispheres with centers at the vertices of the corresponding configurations. In this case the area of the polar and the perimeter of the convex hull sum up to 2π, and the result follows. Finally, the claim in $\mathbb{E}^2$ follows easily from the spherical version via a standard limiting procedure. □

The Euclidean part of Corollary 2 has been proved independently by Alexander [1], Capoyleas and Pach [3], and Sudakov[11].

Recall that the *extreme points* of a compact convex set X are the points that do not lie on the relative interior of a segment joining two other points of X. The following is a result of a compactness argument applied to a sequence of points that are dense in X.

Corollary 3. *If there is a contraction of the extreme points of a compact convex set X in an open hemisphere of $\mathbb{S}^n$, then the polar of X has volume no larger than the volume of the polar of the convex hull of the image of the extreme points.*

2. Leapfrog Lemmas

We repeat here one of our previous lemmas from [2], the Leapfrog Lemma.

Lemma 1. *Suppose that* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *are two configurations in* $\mathbb{E}^n$. *Then the following is a continuous motion* $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ *in* $\mathbb{E}^{2n}$, *that is analytic in* t, *such that* $\mathbf{p}(0) = \mathbf{p}$, $\mathbf{p}(1) = \mathbf{q}$ *and for* $0 \le t \le 1$, $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ *is monotone*:

$$\mathbf{p}_i(t) = \left(\frac{\mathbf{p}_i + \mathbf{q}_i}{2} + (\cos \pi t)\frac{\mathbf{p}_i - \mathbf{q}_i}{2}, (\sin \pi t)\frac{\mathbf{p}_i - \mathbf{q}_i}{2}\right), \qquad 1 \le i < j \le N. \quad (1)$$

We need to apply this to a sphere, rather than Euclidean space. Here we consider the unit spheres $\mathbb{S}^n \subset \mathbb{S}^{n+1} \subset \mathbb{S}^{n+2} \cdots$ in such a way that each $\mathbb{S}^n$ is the set of points that are a unit distance from the origin in $\mathbb{E}^{n+1}$. So we need the following.

Corollary 4. *Suppose that* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *are two configurations in* $\mathbb{S}^n$. *Then there is a monotone analytic motion from* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *to* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *in* $\mathbb{S}^{2n+1}$.

Proof. Apply Lemma 1 to each configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ with the origin as an additional configuration point for each. So for each t, the configuration $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ lies at a unit distance from the origin in $\mathbb{E}^{2n+2}$, which is just $\mathbb{S}^{2n+1}$. □

3. The Differentiable Case

We look at the case when there is a smooth motion of the configuration $\mathbf{p}(t)$ in $\mathbb{S}^n$. More precisely we consider the family $X(t) = X(\mathbf{p}(t))$ of convex spherical n-polytopes in $\mathbb{S}^n$ having the same combinatorial face structure with facet hyperplanes being differentiable in the parameter t. The following classical theorem of Schläfli (see for example [8]) describes how the volume of $X(t)$ changes as a function of its dihedral angles and the volume of its $(n-2)$-dimensional faces.

Theorem 2 (Schläfli). *For each* $(n-2)$*-face* $F_{ij}(t)$ *of the convex spherical* n*-polytope* $X(t)$ *in* $\mathbb{S}^n$, *let* $\alpha_{ij}(t)$ *represent the dihedral angle between the two facets* $F_i(t)$ *and* $F_j(t)$ *meeting at* $F_{ij}(t)$. *Then the following holds*:

$$\frac{d}{dt}\mathrm{Vol}_n[X(t)] = \frac{1}{n-1}\sum_{F_{ij}} \frac{d\alpha_{ij}(t)}{dt}\mathrm{Vol}_{n-2}[F_{ij}(t)], \quad (2)$$

to be summed over all $(n-2)$*-faces*.

Corollary 5. *If the configuration* $\mathbf{p}(t)$ *is a differentiable contraction in* t, *then with the notation of Theorem* 2,

$$\frac{d}{dt}\mathrm{Vol}_n[X(t)] \geq 0.$$

Proof. As the distance between $\mathbf{p}_i(t)$ and $\mathbf{p}_j(t)$ is decreasing, the derivative of the dihedral angle $d\alpha_{ij}(t)/dt \geq 0$. The result then follows from (2). □

4. Integral Formulas

The last piece of information that we need before we get to the main result is a way of relating higher-dimensional volumes to lower-dimensional volumes. Let X be any integrable set in $\mathbb{S}^n$. Recall that we regard

$$X \subset \mathbb{S}^n = \mathbb{S}^n \times \{\mathbf{0}\} \subset \mathbb{E}^{n+1} \times \mathbb{E}^{k+1}.$$

Consider

$$\{\mathbf{0}\} \times \mathbb{S}^k \subset \mathbb{E}^{n+1} \times \mathbb{E}^{k+1}.$$

Let $X * \mathbb{S}^k$ be the subset of $\mathbb{S}^{n+k+1}$ consisting of the union of the geodesic arcs from each point of X to each point of $\{\mathbf{0}\} \times \mathbb{S}^k$. (So, in particular, $\mathbb{S}^n * \mathbb{S}^k = \mathbb{S}^{n+k+1}$.)

Theorem 3. *For any integrable subset* X *of* $\mathbb{S}^n$,

$$\mathrm{Vol}_{n+k+1}[X * \mathbb{S}^k] = \frac{\kappa_{n+k+1}}{\kappa_n}\mathrm{Vol}_n[X],$$

where $\kappa_n = \mathrm{Vol}_n[\mathbb{S}^n]$, *and* $\kappa_{n+k+1} = \mathrm{Vol}_{n+k+1}[\mathbb{S}^n * \mathbb{S}^k] = \mathrm{Vol}_{n+k+1}[\mathbb{S}^{n+k+1}]$.

Proof. Since the $*$ operation (a kind of spherical join) is associative, we only need to consider the case when $k = 0$. Regard $\{\mathbf{0}\} \times \mathbb{S}^0 = \{N, S\}$, the north pole and the south pole of S^{n+1}. We use polar coordinates centered at N to calculate the $(n+1)$-dimensional volume of $X * \mathbb{S}^0$. Let $X(z) = X * \mathbb{S}^0 \cap [\mathbb{E}^{n+1} \times \{z\}]$, and let θ be the angle that a point in $\mathbb{S}^{n+1}$ makes with N, the north pole in $\mathbb{S}^{n+1}$. So $z = z(\theta) = \cos\theta$. Then the volume element for $S^n(z)$ is $dV_n(z) = (\sin^n\theta)\, dV_n(0)$ because $\mathbb{S}^n(z)$ is obtained from $\mathbb{S}^n(0)$ by a dilation by $\sin\theta$. Then

$$\begin{aligned}
\mathrm{Vol}_{n+1}[X * \mathbb{S}^0] &= \int_{X*\mathbb{S}^0} dV_n(z)\, d\theta \\
&= \int_0^\pi \int_{X(z(\theta))} dV_n(z)\, d\theta \\
&= \int_0^\pi (\sin^n\theta)\, \mathrm{Vol}_n[X]\, d\theta \\
&= \mathrm{Vol}_n[X] \int_0^\pi (\sin^n\theta)\, d\theta \\
&= \mathrm{Vol}_n[X]\frac{\kappa_{n+1}}{\kappa_n}.
\end{aligned}$$

The last integral can be seen by taking $X = \mathbb{S}^n$, or by performing the integral explicitly. □

5. Proof of Theorem 1

Let the configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ be a contraction of the configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ in $\mathbb{S}^n$. By Corollary 4, there is an analytic motion $\mathbf{p}(t)$, in $\mathbb{S}^{2n+1}$ for $0 \le t \le 1$, where $\mathbf{p}(0) = \mathbf{p}$, and $\mathbf{p}(1) = \mathbf{q}$, and all the pairwise distances between the points of $\mathbf{p}(t)$ vary monotonically in t.

Without loss of generality we may assume that $X^n(\mathbf{p}(0))$ is a convex spherical n-polytope in S^n. Since $\mathbf{p}(t)$ is analytic in t, the intersection of the hemispheres in $\mathbb{S}^{2n+1}$, $X^{2n+1}(\mathbf{p}(t))$, is a convex spherical $(2n+1)$-polytope with a constant combinatorial structure, except for a finite number of points in the interval [0, 1]. By Corollary 5, $\mathrm{Vol}_{2n+1}[X^{2n+1}(\mathbf{p}(t))]$ is monotone increasing in t.

Recall that $X^n(\mathbf{p})$ and $X^n(\mathbf{q})$ are the intersections of the hemispheres centered at the points of $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{S}^n$. From the definition of the spherical join $*$,

$$X^n(\mathbf{p}) * \mathbb{S}^n = X^{2n+1}(\mathbf{p}) = X^{2n+1}(\mathbf{p}(0)),$$
$$X^n(\mathbf{q}) * \mathbb{S}^n = X^{2n+1}(\mathbf{q}) = X^{2n+1}(\mathbf{p}(1)).$$

Hence by Theorem 3,

$$\mathrm{Vol}_n[X^n(\mathbf{p})] = \frac{\kappa_n}{\kappa_{2n+1}}\mathrm{Vol}[X^{2n+1}(\mathbf{p}(0))] \le \frac{\kappa_n}{\kappa_{2n+1}}\mathrm{Vol}[X^{2n+1}(\mathbf{p}(1))] = \mathrm{Vol}_n[X^n(\mathbf{q})].$$

This finishes the proof of Theorem 1. □

6. Comments

We present some possible generalizations of the above theorems.

Csikós [6] very recently gave a proof of the following theorem that extends Theorem 1 of our paper [2] to spherical as well as to hyperbolic space.

Theorem 4. *Let* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *be two configurations in* n*-dimensional space of constant curvature* (*i.e.*, *in Euclidean*, *spherical or hyperbolic* n*-space*) *such that* $\mathbf{q}$ *is a piecewise-analytic contraction of* $\mathbf{p}$ *in* $(n+2)$*-dimensional space of constant curvature. If* $X(\mathbf{p})$ *and* $X(\mathbf{q})$ *denote the finite intersection* (*resp.*, *union*) *of balls whose configuration of centers are* $\mathbf{p}$ *and* $\mathbf{q}$ *with the property that the balls centered at the corresponding points of* $\mathbf{p}$ *and* $\mathbf{q}$ *are congruent*, *then the* n*-dimensional volumes* $\mathrm{Vol}_n[X(\mathbf{p})]$, $\mathrm{Vol}_n[X(\mathbf{q})]$ *satisfy the inequality*

$$\mathrm{Vol}_n[X(\mathbf{p})] \le \mathrm{Vol}_n[X(\mathbf{q})]$$
$$(\textit{resp.}, \mathrm{Vol}_n[X(\mathbf{p})] \ge \mathrm{Vol}_n[X(\mathbf{q})]).$$

This theorem and our proof [2] of the Kneser–Poulsen conjecture in Euclidean plane leads us to the following conjecture.

Leapfrog Conjecture on S^2. *Suppose that* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *are two configurations in* $\mathbb{S}^2$. *Then there is a monotone piecewise-analytic motion from* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *to* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *in* $\mathbb{S}^4$.

Obviously, our Leapfrog Conjecture and the above-mentioned theorem imply the Kneser–Poulsen conjecture for circles in S^2.

Finally, we remark that Theorem 1 as well as Theorem 4 extend to flowers of balls as well. The notion of flowers of balls has been introduced in [7] by Gordon and Meyer (in set theoretic sense a flower of balls is a finite union of finite intersections of balls) and the proper technology to treat them analogously to intersections (resp., unions) of balls has been developed by Csikós in [5]. For more information on this see also our paper [2].

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Received January 17, 2003, *and in revised form October* 21, 2003. *Online publication April* 29, 2004.