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Realizability of Graphs*

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Abstract. A graph is *d-realizable* if, for every configuration of its vertices in $\mathbb{E}^N$, there exists a another corresponding configuration in $\mathbb{E}^d$ with the same edge lengths. A graph is 2-realizable if and only if it is a partial 2-tree, i.e., a subgraph of the 2-sum of triangles in the sense of graph theory. We show that a graph is 3-realizable if and only if it does not have K_5 or the 1-skeleton of the octahedron as a minor.

1. Introduction

A basic problem in discrete geometry is to determine when a graph with prescribed edge lengths can be realized in $\mathbb{E}^d$. A *graph* G is a finite set of vertices $V(G) = \{1, \ldots, n\}$ and a finite set of edges $E(G)$, where each edge is a set containing exactly two vertices. The graphs we consider do not contain loops or multiple edges. The standard way to draw a graph is to draw a point for each vertex, and to draw a line segment between two vertices for each edge. The *complete graph on n vertices*, denoted by K_n, is the graph with n pairwise adjacent vertices. A good reference on graph theory is [D].

A *realization* of a graph G is a function which assigns to each vertex i of G a point p_i in some Euclidean space. When we draw a realization, we generally also draw the edges between vertices as straight lines. Note that a realization is different from an embedding, since the word embedding is usually reserved for the case when there are no self-intersections. For example, two vertices may be assigned to the same point in a realization and edges may intersect and even overlap.

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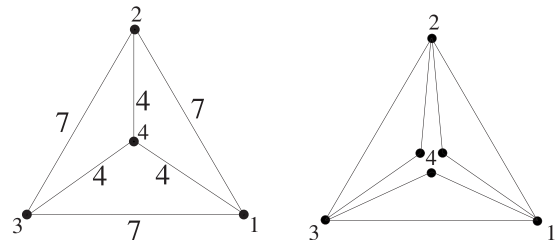


Fig. 1. A weighted graph that satisfies the triangle inequality but cannot be realized in any dimension. The first picture gives the weighted graph; the second attempts to realize the weighted graph but fails. In the second picture, vertex 4 is represented by four points.

A *weighted graph* (G, λ) is a graph G together with a vector of weights (or lengths) $\lambda = (\ldots, \lambda_{ij}, \ldots)$, where $\lambda_{ij} \geq 0$ is the weight assigned to the edge $\{i, j\}$. A *realization* $\mathbf{p} = (p_1, \ldots, p_n)$ *of a weighted graph* is a realization of the graph where each edge $\{i, j\}$ has length λ_{ij}.

The *Molecule Problem* is to determine whether a given weighted graph has a realization in $\mathbb{E}^d$, and if so to construct the realization. It is easy to construct examples of weights λ for a graph G such that (G, λ) does not have a realization in $\mathbb{E}^d$ for any d. For example, if G is a triangle with edge lengths not satisfying the triangle inequality, then (G, λ) cannot be realized in any Euclidean space. There are also examples of weighted graphs with the triangle inequalities satisfied such that all proper subgraphs have realizations in $\mathbb{E}^N$, but there is no realization of the whole graph in any Euclidean space of any dimension. For example, consider the graph K_{d+1}. Assign a weight of 1 to each edge of a K_d subgraph so that it has a realization in $\mathbb{E}^{d-1}$ as a d-simplex. Each remaining edge connects the final vertex to one of the vertices of the K_d. Let x be the distance from each vertex of the d-simplex to the center of the d-simplex. Assign a weight less than x on each remaining edge, but large enough so that the final vertex and any $d - 1$ vertices form a d-simplex that has a realization in $\mathbb{E}^{d-1}$. Then the weighted graph K_{d+1} does not have a realization in any dimension, but every subgraph of d vertices has a realization as a d-simplex. Figure 1 shows a picture of this situation for $d = 3$.

See [H] for a discussion of the molecule problem including an algorithm for solving it when there are sufficiently many edges in G. In a general setting, Crippen and Havel [CH] describe an empirical algorithm for solving the molecule problem.

Given a weighted graph (G, λ), the problem to decide whether there exists a corresponding configuration of points in a Euclidean space of any dimension, which is a realization of the weighted graph, is called the "Euclidean distance matrix completion problem" (EDM) in Section 4 of [L1]. In Section 5 of [L1] it is stated that there is no known efficient algorithm for EDM in general, but that the problem becomes easy if one allows approximations. Indeed, the approximation problem is a special instance of semidefinite programming problems. In the paper here we regard the realization of a graph G in some high-dimensional Euclidean space as given, and then proceed. With this in mind, we make the following definition.

Definition 1. A graph G is *d-realizable* if, given any realization $p_1, \ldots, p_n$ of the graph in some finite-dimensional Euclidean space, there exists a realization $q_1, \ldots, q_n$ in $\mathbb{E}^d$ with the same edge lengths: $|p_i - p_j| = |q_i - q_j|$ for all $\{i, j\} \in E(G)$.

Note that d-realizability is a property of graphs—for G to be d-realizable, *every* realizable (G, λ) must have a realization in $\mathbb{E}^d$. (It has been suggested that we could use the word "universally d-realizable" instead of the word d-realizable. This is descriptive, but we feel that using the word d-realizable will not create any confusion and is simpler.)

Note also that our definitions allow edges to have length 0. If we required edges to positive length, then it would not change which graphs are d-realizable, which will be explained later.

Examples.

1. A path is 1-realizable, because we can arrange the vertices in order on a line with the appropriate distance between any two consecutive points.
2. Similarly, a tree (a connected graph containing no cycles) is also 1-realizable.
3. A triangle is not 1-realizable, because the triangle with all edge lengths 1 can only be realized in $\mathbb{E}^2$ but not in $\mathbb{E}^1$.

In this paper we look at the question of which graphs are d-realizable for $d \leq 3$ and obtain the following results.

Theorem 1. *A graph G is* 1*-realizable if and only if it does not have K_3 as a minor* (*i.e.*, *G is a forest*).

Theorem 2. *A graph G is* 2*-realizable if and only if it does not have K_4 as a minor.*

Theorem 3 (Main Theorem). *A graph G is* 3*-realizable if and only if it does not have either K_5 or the* 1*-skeleton of the octahedron as a minor.*

In this paper we only prove that a graph is 3-realizable if and only if it does not have either K_5 or the 1-skeleton of the octahedron as a minor assuming that the graphs V_8 and $C_5 \times C_2$ are 3-realizable (see Fig. 3 for the definitions of these graphs). The graphs V_8 and $C_5 \times C_2$ were recently shown to be 3-realizable by Sloughter [Sl] using techniques of stress theory, but not assuming any results of this paper. The basic idea is to break up the given graph G as a 3-sum (see Definition 4) of smaller pieces forming a partial 3-tree. For many of these pieces, they automatically span a three-dimensional space, and each piece can be "folded" into a given three-dimensional space. However for the two exceptional graphs above, an additional "stretching" operation is used, where certain pairs of points are pushed apart, and this flattens the configuration enough to force it into a three-dimensional space. There is one exceptional case, though, for $C_5 \times C_2$, where the stretching operation has to be performed twice.

2. Low-Dimensional Results

Our discussion of 1-realizable graphs leads us to the following theorem.

Theorem 1. *A graph is* 1*-realizable if and only if it is a forest* (*a disjoint collection of trees*).

Proof. Clearly, any forest with any specified edge lengths can be realized in one dimension. If a graph is not a forest, then it contains a cycle as a subgraph. This cycle can be realized in the Euclidean plane with three edges of length 1 and with the remaining edges having length 0. There is no realization in the line with the same edge lengths. Thus, a graph containing a cycle is not 1-realizable. □

Observe, in the above proof, it was helpful to consider a subgraph to show that a graph was not 1-realizable. In general if a graph G is d-realizable, then any subgraph of G is also d-realizable.

It was also helpful to consider a realization where some edges had length 0. However, if we required edges to have positive length, it would not change which graphs are d-realizable. Let G be a graph, and let $v = |V(G)|$ and $e = |E(G)|$. Consider the function $f : \mathbb{R}^{dv} \to \mathbb{R}^e$ which takes a realization of G in $\mathbb{E}^d$ and returns the length of each edge of G. The image of f applied to a closed ball of radius M is a compact set in $\mathbb{R}^e$, since f is continuous. Thus, the set of edge lengths which cannot be realized in $\mathbb{E}^d$ inside a closed ball of radius M is an open set in $\mathbb{R}^e$ (as it is the complement of a compact set). Since every list of edges with a realization in $\mathbb{E}^d$ has a realization inside a closed ball with sufficiently large radius M, the set of edge lengths which cannot be realized in $\mathbb{E}^d$ is an open set in $\mathbb{R}^e$. If a graph G has a realization $\mathbf{p} = (p_1, \dots, p_n)$ in $\mathbb{E}^N$ with some 0 length edges that is not realizable in $\mathbb{E}^d$ with the same edge lengths, then a sufficiently small perturbation of $\mathbf{p} = (p_1, \dots, p_n)$ to a configuration with no 0 length edges in $\mathbb{E}^N$ will still not be realizable with the same edge lengths in $\mathbb{E}^d$, since the set of edge lengths that cannot be realized is open.

The following is a standard definition from graph theory.

Definition 2. A *minor* of a graph G is any graph obtained from G by a sequence of

- edge deletions and
- edge contractions (identify the two vertices belonging to an edge and then remove any loops or multiple edges).

Theorem 4. *If a graph G is d-realizable and H is a minor of G, then H is d-realizable.*

Proof. Zero length edges are allowed. □

A graph property is called *minor monotone* if it is closed under the operation of taking minors. Minor monotone graph properties are interesting, because of the graph minor theorem of Robertson and Seymour [RS2].

Theorem 5 (The Graph Minor Theorem). *Every minor monotone graph property has a finite list of forbidden minors*; *i.e. there exits a finite list of graphs $G_1, \dots, G_n$ such that a graph G satisfies the graph property if and only if G does not have any G_i as a minor.*

The survey paper [T] by Thomas provides many examples of graph properties and their corresponding forbidden minors.

We do not need Theorem 5 in order to prove our theorem about forbidden minors. This result simply predicts that there will be a finite list of forbidden minors for our problem, while it provides no help in finding them.

The forbidden minor for 1-realizability is K_3. For d-realizability, the graph K_{d+2} is a forbidden minor (but not necessarily the only minimal forbidden minor), because it can be realized as the 1-skeleton of a $(d + 1)$-simplex.

The following definition will be helpful in characterizing 2-realizable graphs.

Definition 3. A graph is *series parallel* if it is a subgraph of a graph that is constructed from a K_2 by repeatedly attaching subdivided edges to two adjacent vertices.

Wagner [W] classified series parallel graphs in terms of minors. See [D] for a more recent proof.

Theorem 6 [W]. *A graph G is series parallel if and only if G does not contain K_4 as a minor*; *i.e.*, *K_4 is the only forbidden minor for series parallel graphs*.

We are now ready to classify 2-realizable graphs.

Theorem 2. *A graph is 2-realizable if and only if it does not have K_4 as a minor.*

Proof. First, suppose that a graph G does not have K_4 as a minor. Then by Theorem 6, G is series parallel. We can assume that G is maximally series parallel (if any edge is added to the graph, it is no longer series parallel), since subgraphs of d-realizable graphs are d-realizable. A maximally series parallel graph can be constructed from K_2 by attaching subdivided edges with exactly one subdivision between two adjacent vertices.

We will proceed by induction. The graph K_2 is 2-realizable. If we attach a subdivided edge to adjacent vertices with edge lengths satisfying the triangle inequality to a graph that is realized in $\mathbb{E}^2$, the resulting graph can also be realized in $\mathbb{E}^2$. By induction, all maximally series parallel graphs are 2-realizable.

Now, suppose that a graph G is 2-realizable. Note that K_4 is not 2-realizable, because there are realizations of K_4 in $\mathbb{E}^3$ as the 1-skeleton of a tetrahedron. Thus, G cannot contain K_4 as a minor. □

3. Tree Decompositions

It will be helpful to be able to create examples of d-realizable graphs. In creating some examples of d-realizable graphs, we want a generalization of trees and series parallel graphs. Trees are created by joining paths together along vertices. Series parallel graphs are created by attaching a subdivided edge to two adjacent vertices and possibly taking a subgraph. The generalization we need is provided by tree decompositions, which were used extensively by Robertson and Seymour [RS1].

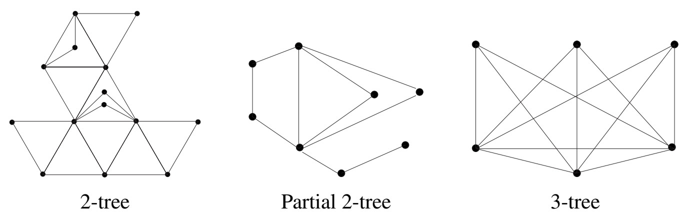


Fig. 2. Examples of partial k-trees.

Definition 4. Let G_1 and G_2 be two graphs, both containing a K_k as a subgraph. The *k-sum* of G_1 and G_2, denoted $G_1 \oplus_k G_2$, is the graph obtained by identifying the two K_k's.

Note that $G_1 \oplus_k G_2$ is uniquely defined once the correspondence between the vertices in the copies of K_k in G_1 and G_2 is determined.

Definition 5. A graph is a *k-tree* if it can by obtained through a sequence of k-sums of K_{k+1}'s. A graph is a *partial k-tree* if it is a subgraph of a k-tree.

Clearly, a graph is a partial 2-tree if and only if it is a series parallel graph. Figure 2 shows an example of a 2-tree, a partial 2-tree, and a 3-tree.

Suppose G_1 and G_2 are both d-realizable and both contain a K_d subgraph. We can realize both G_1 and G_2 in $\mathbb{E}^d$ and then attach the two realizations along the common K_d subgraph to create a realization of $G_1 \oplus_d G_2$ in $\mathbb{E}^d$. Thus, $G_1 \oplus_d G_2$ is also d-realizable.

Forests are equivalent to partial 1-trees, so 1-realizable graphs are partial 1-trees. Series parallel graphs are equivalent to partial 2-trees, so 2-realizable graphs are partial 2-trees. Clearly, all partial d-trees are d-realizable.

4. Which Graphs Are 3-Realizable?

Arnborg et al. [APC] have determined the forbidden minors of partial 3-trees.

Theorem 7 [APC]. *The forbidden minors for partial* 3*-trees are* K_5, *the* 1*-skeleton of the octahedron* ($K_{2,2,2}$), V_8, *and* $C_5 \times C_2$ (*see Fig*. 3).

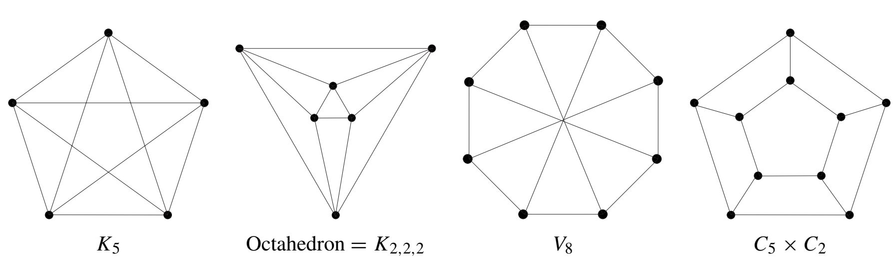


Fig. 3. Forbidden minors for partial 3-trees.

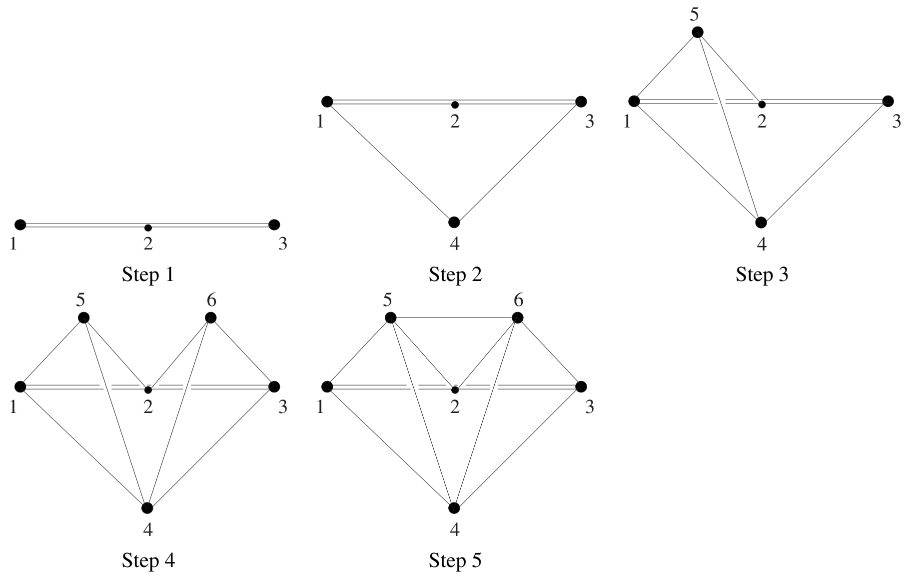


Fig. 4. Steps 1–5 of the proof of Theorem 8.

Given the above theorem, it is reasonable to ask which graphs in Fig. 3 are 3-realizable. If any of these graphs is not 3-realizable, then it is a forbidden minor for 3-realizability. We already know that K_5 is not 3-realizable. The following theorem shows that the octahedron is not 3-realizable.

Theorem 8. *The* 1*-skeleton of the octahedron* ($K_{2,2,2}$) *is not* 3*-realizable*.

Proof. We construct a realization of the octahedron in $\mathbb{E}^4$ that cannot be realized in $\mathbb{E}^3$. Figure 4 shows the construction.

Step 1: We start with a degenerate triangle with edge lengths 1, 1, and 2. This is the only way to realize these three points with the given lengths (up to congruence, which includes reflections and translations). We label these vertices 1, 2, and 3 in order.

Step 2: Now we attach vertex 4 to this degenerate triangle at vertices 1 and 3 with edge lengths $\sqrt{2}$ and $\sqrt{2}$. This is again the only way to realize this graph with these edge lengths (up to congruence).

Step 3: Now we attach vertex 5 to vertices 1, 2, and 4. We place this vertex in the third dimension above the plane Π determined by vertices 1, 2, 3, and 4. We make all of the new edges have length 1. This is the only way to realize this graph with these edge lengths (up to congruence).

Step 4: We now attach vertex 6 to vertices 2, 3, and 4. In three dimensions we place it either above or below the plane Π. We make all of the new edges have length 1. Note that in $\mathbb{E}^3$ there are only two possible realizations. However, in $\mathbb{E}^4$, there are infinitely many possible realizations. Vertex 6 can rotate around plane Π without changing any of the edge lengths.

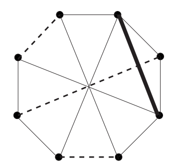


Fig. 5. Graphs of V_8 with an added edge contract to K_5.

Step 5: There is one final edge to add between vertices 5 and 6. In $\mathbb{E}^3$ this edge has only two possible lengths ($\sqrt{2}$ and 2 for the given edge lengths), but in $\mathbb{E}^4$ this edge can be any length in between.

This gives us infinitely many realizations in $\mathbb{E}^4$ that cannot be realized in $\mathbb{E}^3$. Thus, the octahedron is not 3-realizable. □

The graphs V_8 and $C_5 \times C_2$ are 3-realizable, as shown in [Sl]. This leaves open the possibility that there are other graphs which are not 3-realizable but do not have K_5 or the octahedron as a minor. We eliminate this possibility by showing that any graph containing V_8 or $C_5 \times C_2$ as a minor either contains K_5 or the octahedron as a minor or is 3-realizable. We need some lemmas about V_8 and $C_5 \times C_2$.

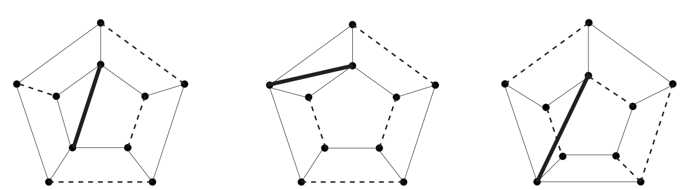
Lemma 1. *If any edge is added between nonadjacent vertices of* V_8, *the resulting graph has* K_5 *as a minor.*

Proof. There are two ways to add an edge to V_8 up to graph isomorphism. Figure 5 shows these two graphs. The solid bold edge is the added edge. If we contract the dotted edges, the resulting graph is K_5. □

Lemma 2. *If any edge is added between nonadjacent vertices of* $C_5 \times C_2$, *the resulting graph has either the octahedron or* K_5 *as a minor.*

Proof. There are three ways to add an edge to $C_5 \times C_2$ up to graph isomorphism. Figure 6 shows these three graphs. The added edge is in bold. Contracting the dotted edges produces the octahedron for the first two graphs and K_5 for the third graph. □

We say that a graph G is obtained from a graph H by *splitting a vertex* if H is obtained from G by contracting an edge e, where both ends of e have degree at least 3 in G. A



Contracts to octahedron Contracts to octahedron Contracts to K_5

Fig. 6. Graphs of $C_5 \times C_2$ with an added edge contract to either the octahedron or K_5.

graph is a *wheel* if it is obtained from a cycle on at least three vertices by adding a vertex joined to every vertex on the cycle. A graph G is 3-connected if it has at least four vertices and every graph obtained by deleting two vertices is connected.

Seymour [Se] proved the following theorem, which is a useful tool for proving forbidden minor theorems. The theorem can also be found in [T]

Theorem 9. *Let H be a 3-connected minor of a 3-connected graph G such that H is not a wheel. Then G can be obtained from H by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

Note that V_8 and $C_5 \times C_2$ are both 3-connected.

We are now ready to prove the main theorem. We thank Monique Laurent and Robin Thomas for pointing out omissions in the initial draft of this proof.

Theorem 3 (Main Theorem). *The forbidden minors for 3-realizability are K_5 and the octahedron.*

Proof. We assume that V_8 and $C_5 \times C_2$ are 3-realizable (see [Sl]).

We know that K_5 is a forbidden minor. By Theorem 8, the octahedron is a forbidden minor.

We need to show that if a graph G does not have K_5 or the octahedron as a minor, then it is 3-realizable. We can assume that G is connected, since each connected component can be realized separately.

In a somewhat similar manner, we can assume that G is 3-connected. If there is a vertex whose deletion disconnects the graph, then G is the 1-sum of two graphs, which can be realized in $\mathbb{E}^3$ separately and then joined together at the vertex. If there are two vertices whose deletion disconnects the graph, we can do essentially the same thing. The graph G is a subgraph of the 2-sum of two graphs, which can be realized separately. To get the two graphs from G:

1. Remove the two vertices u and v, disconnecting the graph into two graphs H_1 and H_2.
2. Consider the induced subgraph of G spanned by the vertices of H_1 and u and v. Do the same for H_2.
3. Add an edge between u and v (if there is not already an edge). Call the resulting graphs G_1 and G_2.

Now, G is a subgraph of $G_1 \oplus_2 G_2$ (either G is $G_1 \oplus_2 G_2$ or G is $G_1 \oplus_2 G_2$ minus the edge between u and v). Note that if G did not contain K_5 or $K_{2,2,2}$ as a minor, then G_1 and G_2 do not either.

Thus, we can assume that G is 3-connected, so we can use Theorem 9.

The graph G must contain either V_8 or $C_5 \times C_2$ as a minor. By Theorem 9, G can be obtained from V_8 or $C_5 \times C_2$ by repeatedly adding an edge and splitting a vertex. Thus, either G is V_8 or $C_5 \times C_2$ or G has V_8 plus an edge or $C_5 \times C_2$ plus an edge as a minor. In the second case, by Lemmas 1 and 2, G has K_5 or $K_{2,2,2}$ as a minor.

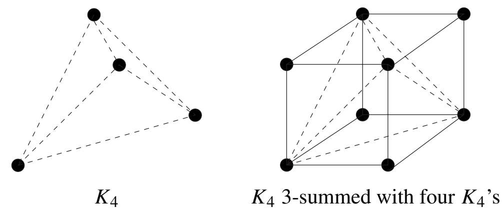


Fig. 7. The cube is a partial 3-tree.

Thus, we get that every graph that does not contain K_5 or $K_{2,2,2}$ as a minor can be constructed from partial 3-trees, V_8's, and $C_5 \times C_2$'s using 1-sum, 2-sum, and 3-sums, and is 3-realizable. □

We can also classify 3-realizable graphs based on their k-sum "building blocks." Every 3-realizable graph is a subgraph of a graph that can be obtained by a sequence of 3-sums and 2-sums involving K_4, V_8, and $C_5 \times C_2$. Since neither V_8 nor $C_5 \times C_2$ contains a K_3 as a subgraph, both of these graphs must be attached with 2-sums.

5. Examples

Example 1. *The* 1-*skeleton of the cube is a partial* 3-*tree*, *and therefore* 3-*realizable*.

Consider the 1-skeleton of the tetrahedron (K_4). Take the 3-sum of this graph with four other K_4's, one for each face of the tetrahedron. The resulting graph shown in Fig. 7 has the cube as a subgraph.

Example 2. *The graph* $K_{3,3}$ *is a partial* 3-*tree*, *and therefore* 3-*realizable*.

Consider a triangle (K_3), and 3-sum this graph with three K_4's, all being attached to the original triangle. The resulting graph shown in Fig. 2 has $K_{3,3}$ as a subgraph.

Example 3. *The Cauchy graph on* $n \geq 5$ *vertices* Ch_n (*defined below*) *is* 4-*realizable*, *but not* 3-*realizable*.

The graph Ch_n is the graph obtained from a cyclic graph by placing an edge between every other vertex. Figure 8 shows several Cauchy graphs. The graph Ch_n is a minor of Ch_{n+2}—if we label the vertices around the outer cycle $1, 2, \ldots, n+2$, then contracting edges $\{1, 3\}$ and $\{2, 4\}$ of Ch_{n+2} yields the graph Ch_n. The Cauchy graph on five vertices is K_5, so it is not 3-realizable; and the Cauchy graph on six vertices is the octahedron, so it is not 3-realizable. Thus, all Ch_n for $n \geq 5$ are not 3-realizable. However, all Cauchy graphs are partial 4-trees, and thus 4-realizable.

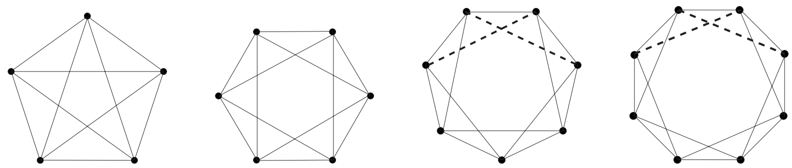


Fig. 8. The Cauchy graphs on five, six, seven, and eight vertices. Contracting the dotted edges in Ch_7 and Ch_8 produces the graphs Ch_5 and Ch_6, respectively.

6. Discussion and Open Problems

The main theorem along with [Sl] classifies all 3-realizable graphs. For higher dimensions, the problem is even harder. There are over 75 known forbidden minors for partial 4-trees [Sa1]. There is an algorithm in [Sa2] that determines whether a graph is a partial 4-tree in linear time.

Given a graph G and a dimension d, it should be possible to use techniques of algebraic geometry to determine whether G is d-realizable. Let $e = |E(G)|$ and $v = |V(G)|$, and suppose that we know that G is N-realizable (for example, N could be v). There is a polynomial function from $\mathbb{R}^{Nv}$ to $\mathbb{R}^e$ which takes a realization in $\mathbb{E}^N$ and returns the length of each edge. The image of this polynomial function is a semi-algebraic set (a set defined as a finite union of sets defined by a finite list of polynomial inequalities). There is a similar polynomial function from $\mathbb{R}^{dv}$ to $\mathbb{R}^e$. The question of whether G is d-realizable is then equivalent to the question of whether the two semi-algebraic sets are equal. This question can be solved, but the algorithm is exponential. One bound on the complexity is $(4e)^{O(Ndv^2)}$. See Chapter 13 of [BPR] for more information on determining whether two semi-algebraic sets are equal.

Another question to ask is How fast does the number of forbidden minors for d-realizability grow? What is an upper and lower bound for the number of forbidden minors? We know that K_{d+2} is a forbidden minor for all d. Also, there is an analogue of the octahedron construction for all $d \geq 3$, so there are at least two forbidden minors for all d, and probably a lot more than two. It seems reasonable to conjecture that the number of forbidden minors for d-realizability grows at a similar rate to the number of forbidden minors for partial d-trees.

Once we know which graphs are d-realizable, we would like a reasonable algorithm to realize a given weighted graph (a graph with specified edge lengths) in $\mathbb{E}^d$. The algorithm should take a weighted d-realizable graph and either return that the weighted graph cannot be realized in any dimension or return a realization in $\mathbb{E}^d$. For $d = 3$, Matoušek and Thomas showed that given a graph, a 3-tree decomposition can be determined in linear time (see [MT]). A correction to their algorithm appears in [Sa2]. Their algorithm takes a graph and either returns that the graph is not a partial 3-tree or returns a 3-tree which has the original graph as a subgraph. This algorithm could be modified to find a decomposition containing V_8's and $C_5 \times C_2$'s.

For realizing partial 3-trees, the remaining question is how to assign edge lengths to the new edges (the edges that are part of the 3-tree but not part of the original partial 3-tree). Note that it does not matter which tree decomposition we use. There may be multiple ways to make a partial 3-tree into a 3-tree. If the partial 3-tree (with given edge lengths) has a realization in some dimension, then any 3-tree decomposition also has a realization in that dimension (assign the edge lengths based on the partial 3-tree realization). Thus, if we determine that one 3-tree with the required edge lengths on the subgraph cannot be realized in dimension 3, then the original weighted graph could not be realized in dimension 3. For realizing graphs containing V_8's and $C_5 \times C_2$'s, we would need a way to assign edge lengths to new edges and we would need a way to realize V_8's and $C_5 \times C_2$'s with specified edge lengths.

The analogous question for $d = 2$ has been fully answered by Jack Snoeyink. He has given an algorithm running in linear time and space as a function of n, the number of vertices of the graph G, to determine a partial 2-tree realization.

One of the motivations for this paper is a result of Barvinok in [B1]. See also [DL] for another proof of the first statement below. The following is a special case of a more general situation considered by Barvinok for the solution of quadratic polynomial equations, but this is most relevant for us.

Theorem 10. *Any graph G with e edges is d-realizable if $e < (d+1)(d+2)/2$. Furthermore, G is still d-realizable if $e = (d+1)(d+2)/2$, and G is not the complete graph K_{d+1}.*

This last extension is in [B2]. This leads to the following conjecture:

Conjecture. *If a graph G has e edges and $e < (d+1)(d+2)/2$, then G is a partial d-tree. Furthermore, if G has $e = (d+1)(d+2)/2$, and G is not the complete graph K_{d+1}, then G is still a d-tree.*

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