Discrete Comput Geom (2015) 54:259–289
DOI 10.1007/s00454-015-9704-z

Free Edge Lengths in Plane Graphs

Zachary Abel[1] · Robert Connelly[2] · Sarah Eisenstat[1] · Radoslav Fulek[3] · Filip Morić[4] · Yoshio Okamoto[5] · Tibor Szabó[6] · Csaba D. Tóth[7]

Received: 8 August 2014 / Revised: 23 April 2015 / Accepted: 30 April 2015 / Published online: 28 May 2015

Abstract We study the impact of metric constraints on the realizability of planar graphs. Let G be a subgraph of a planar graph H (where H is the "host" of G). The graph G is *free* in H if for every choice of positive lengths for the edges of G, the host H has a planar straight-line embedding that realizes these lengths; and G is *extrinsically*

Editor in Charge: János Pach

A preliminary version of this paper has appeared in the *Proceedings of the 30th Annual Symposium on Computational Geometry* (*SoCG*), 2014, ACM Press, pp. 426–435.

Csaba D. Tóth
cdtoth@acm.org

Zachary Abel
zabel@math.mit.edu

Robert Connelly
connelly@math.cornell.edu

Sarah Eisenstat
seisenst@mit.edu

Radoslav Fulek
radoslav.fulek@gmail.com

Filip Morić
filip.moric@epfl.ch

Yoshio Okamoto
okamotoy@uec.ac.jp

Tibor Szabó
szabo@math.fu-berlin.de

1 Massachusetts Institute of Technology, Cambridge, MA, USA

2 Cornell University, Ithaca, NY, USA

free in H if all constraints on the edge lengths of G depend on G only, irrespective of additional edges of the host H. We characterize the planar graphs G that are free in every host H, $G \subseteq H$, and all the planar graphs G that are extrinsically free in every host H, $G \subseteq H$. The case of cycles $G = C_k$ provides a new version of the celebrated carpenter's rule problem. Even though cycles C_k, $k \geq 4$, are not extrinsically free in all triangulations, it turns out that "nondegenerate" edge lengths are always realizable, where the edge lengths are considered degenerate if the cycle can be flattened (into a line) in two different ways. Separating triangles, and separating cycles in general, play an important role in our arguments. We show that every star is free in a 4-connected triangulation (which has no separating triangle).

Keywords Geometric graph · Graph embedding · Bar-and-joint framework

1 Introduction

Representing graphs in Euclidean space such that some or all of the edges have given lengths has a rich history. For example, the rigidity theory of bar-and-joint frameworks, motivated by applications in mechanics, studies edge lengths that guarantee a unique (or locally unique) representation of a graph. Our primary interest lies in simple *combinatorial* conditions that guarantee realizations for all possible edge lengths. We highlight two well-known results similar to ours: (1) Jackson and Jordán [4,12] gave a combinatorial characterization of graphs that are generically globally rigid (i.e., admit unique realizations for *arbitrary* generic edge lengths). (2) Connelly et al. [5] showed that a cycle C_k, $k \geq 3$, embedded in the plane can be continuously unfolded into a convex polygon (i.e., the configuration space of the planar embeddings of C_k is connected), solving the so-called carpenter's rule problem.

We consider straight-line embeddings of planar graphs where some of the edges can have arbitrary lengths. A *straight-line embedding* (for short, *embedding*) of a planar graph is a realization in the plane where the vertices are mapped to distinct points, and the edges are mapped to line segments between the corresponding vertices such that any two edges can intersect only at a common endpoint. By Fáry's theorem [9], every planar graph admits a straight-line embedding with *some* edge lengths. However, it is NP-hard to decide whether a planar graph can be embedded with prescribed edge lengths [8], even for planar 3-connected graphs with unit edge lengths [3], but it is decidable in linear time for triangulations [7] and near-triangulations [3]. Finding a straight-line embedding of a graph with prescribed edge lengths involves a fine interplay between topological, metric, combinatorial, and algebraic constraints. Determining the impact of each of these constraints is a challenging task. In this paper,

3 Columbia University, New York, NY, USA

4 École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

5 University of Electro-Communications, Tokyo, Japan

6 Freie Universität Berlin, Berlin, Germany

7 California State University Northridge, Los Angeles, CA, USA

we characterize the subgraphs for which the metric constraints on the straight-line embedding remain independent from any topological, combinatorial, and algebraic constraints. Such subgraphs admit arbitrary positive edge lengths in an appropriate embedding of the host graph. This motivates the following definition.

Definition 1 Let $G = (V, E)$ be a subgraph of a planar graph H (the *host* of G). We say that

- G is *free* in H when, for every length assignment $\ell \colon E \to \mathbb{R}^+$, there is a straight-line embedding of H in which every edge $e \in E$ has length $\ell(e)$;
- G is *extrinsically free* in H when, for every length assignment $\ell \colon E \to \mathbb{R}^+$, if G has a straight-line embedding with edge lengths $\ell(e)$, $e \in E$, then H also has a straight-line embedding in which every edge $e \in E$ has length $\ell(e)$.

Intuitively, if G is free in H, then there is no restriction on the edge lengths of G; and if G is *extrinsically free* in H, then all constraints on the edge lengths depend on G alone, rather than the edges in $H \setminus G$. Clearly, if G is free in H, then it is also extrinsically free in H. However, an extrinsically free subgraph G of H need not be free in H. For example, K_3 is not free in any host since the edge lengths have to satisfy the triangle inequality, but it is an extrinsically free subgraph in K_4. It is easily verified that every subgraph with exactly two edges is free in every host (every pair of lengths can be attained by a suitable affine transformation); but a triangle K_3 is not free in any host (due to the triangle inequality).

Results We characterize the graphs G that are free as a subgraph of *every* host H, $G \subseteq H$.

Theorem 1 *A planar graph $G = (V, E)$ is* free *in* every *planar host H, $G \subseteq H$, if and only if G consists of isolated vertices and*

- *a matching, or*
- *a forest with at most* 3 *edges, or*
- *the disjoint union of two paths, each with* 2 *edges.*

Separating 3- and 4-cycles in triangulations play an important role in our argument. A *star* is a graph $G = (V, E)$, where $V = \{v, u_1, \dots, u_k\}$ and $E = \{vu_1, \dots, vu_k\}$. We present the following result for stars in 4-connected triangulations.

Theorem 2 *Every star is free in a* 4*-connected triangulation.*

If a graph G is free in H, then it is extrinsically free, as well. We completely classify graphs G that are extrinsically free in every host H.

Theorem 3 *Let $G = (V, E)$ be a planar graph. Then G is* extrinsically free *in* every *host H, $G \subseteq H$, if and only if G consists of isolated vertices and*

- *a forest as listed in Theorem 1 (a matching, a forest with at most* 3 *edges, the disjoint union of two paths, each with* 2 *edges), or*
- *a triangulation, or*
- *a triangle and one additional edge (either disjoint from or incident to the triangle).*

When $G = C_k$ is a cycle with prescribed edge lengths, the realizability of a host H, $C_k \subset H$, leads to a variant of the celebrated carpenter's rule problem. Even though cycles on four or more vertices are not extrinsically free, all nonrealizable length assignments are *degenerate* in the sense that the cycle C_k, $k \geq 4$, decomposes into four paths of lengths (a, b, a, b) for some $a, b \in \mathbb{R}^+$. Intuitively, a length assignment on a cycle C_k is degenerate if C_k has two noncongruent embeddings in the line (that is, in 1-dimensions) with prescribed edge lengths. We show that every host H, $C_k \subset H$, is realizable with prescribed edge lengths on C_k, that is, H admits a straight-line embedding in which every edge of C_k has its prescribed length, if the length assignment of C_k is nondegenerate.

Theorem 4 *Let H be a planar graph that contains a cycle $C = (V, E)$. Let $\ell : E \to \mathbb{R}^+$ be a length assignment such that C has a straight-line embedding with edge lengths $\ell(e)$, $e \in E$. If ℓ is nondegenerate, then H admits a straight-line embedding in which every edge $e \in E$ has length $\ell(e)$.*

Organization. Our negative results (i.e., a planar graph G is *not always* free) are confirmed by finding specific hosts H, $G \subseteq H$, and length assignments that cannot be realized (Sect. 2). We give a constructive proof that every matching is free in all planar graphs (Sect. 3). In fact, we prove a slightly stronger statement: the edge lengths of a matching G can be chosen arbitrarily in every *plane* graph H with a fixed combinatorial embedding (that is, the edge lengths *and* the outer face can be chosen arbitrarily). The key tools are edge contractions and vertex splits, reminiscent of the technique of Fáry [9]. Separating triangles pose technical difficulties: we should realize the host H even if one edge of a separating triangle has to be very short, and an edge in its interior has to be very long. Similar problems occur when two opposite sides of a separating 4-cycle are short. We use grid embeddings and affine transformations to construct embeddings recursively for all separating 3- and 4-cycles (Sect. 3.2). All other subgraphs listed in Theorem 1 have at most 4 edges. We show directly that they are free in every planar host (Sect. 4). We extend our methods to stars in 4-connected triangulations (Sect. 5) and extrinsically free graphs (Sect. 6). In Sect. 7 we show that for cycles with prescribed edge lengths any host H is realizable if the length assignment is nondegenerate. We conclude with open problems in Sect. 8.

Related Problems. As noted above, the embeddability problem for planar graphs with given edge lengths is NP-hard [3,8], but efficiently decidable for near-triangulations [3,7]. Patrignani [16] also showed that it is NP-hard to decide whether a straight-line embedding of a *subgraph* G (i.e., a *partial embedding*) can be extended to an embedding of a host H, $G \subset H$. For *curvilinear* embeddings, this problem is known as planarity testing for partially embedded graphs (PEP), which is decidable in polynomial time [2]. Recently, Jelínek et al. [13] gave a combinatorial characterization for PEP via a list of forbidden substructures. Sauer [17,18] considers similar problems in the context of structural Ramsey theory of metric embeddings: For an edge labeled graph G and a set $\mathcal{R} \subset \mathbb{R}^+$ that contains the labels, he derived conditions that ensure the existence of a metric space M on $V(G)$ that realizes the edge labels as distances between the endpoints.

Definitions. A *triangulation* is an edge-maximal planar graph with $n \geq 3$ vertices and $3n - 6$ edges. Every triangulation has well-defined faces where all faces are triangles, since every triangulation is a 3-connected polyhedral graph for $n \geq 4$. A *near-triangulation* is a 3-connected planar graph in which all faces are triangles with at most one exception (which is typically the outer face). A 3-cycle t in a near-triangulation T is called a *separating triangle* if the vertices of t form a 3-cut in T. A triangulation T has no separating triangles if and only if T is 4-connected.

Tools from Graph Drawing. To show that a graph $G = (V, E)$ is free in every host H, $G \subseteq H$, we design algorithms that, for every length assignment $\ell \colon E \to \mathbb{R}^+$, construct a desired embedding of H. Our algorithms rely on several classic building blocks developed in the graph drawing community.

By Tutte's *barycenter embedding* method [20], every 3-connected planar graph admits a straight-line embedding in which the outer face is mapped to an arbitrarily prescribed convex polygon with the right number of vertices. Hong and Nagamochi [11] extended this result and proved that every 3-connected planar graph admits a straight-line embedding in which the outer face is mapped to an arbitrarily prescribed star-shaped polygon with the right number of vertices.

A *grid embedding* of a planar graph is an embedding in which the vertices are mapped to points in some small $h \times w$ section of the integer lattice $\mathbb{Z}^2$. For an n-vertex planar graph, the dimensions of the bounding box are $h, w \in O(n)$ [6,19], which is the best possible [10]. The *angular resolution* of a straight-line embedding of a graph is the minimum angle subtended by any two adjacent edges. It is easy to see that the angular resolution of a grid embedding, where $h, w \in O(n)$, is $\Omega(n^{-2})$. By modifying an incremental algorithm by de Fraysseix et al. [6], Kurowski [14] constructed grid embeddings of n-vertex planar graphs on a $3n \times \frac{3}{2}n$ section of the integer lattice with angular resolution at least $\sqrt{2}/3\sqrt{5}n \in \Omega(1/n)$. Kurowski's algorithm embeds an n-vertex triangulation T with a given face (a, b, c) such that $a = (0, 0)$, $b = (3n, 0)$, and $c = (\lfloor 3n/2 \rfloor, \lfloor 3n/2 \rfloor)$. It has the following additional property used in our argument. When vertex c is deleted from the triangulation T, we are left with a 2-connected graph with an outer face $(a = u_1, u_2, \ldots, u_k = b)$. In Kurowski's embedding, as well as in [6], the path $(a = u_1, u_2, \ldots, u_k = b)$ is x-monotone and the slope of every edge in this path is in the range $(-1, 1)$.

2 Subgraphs with Constrained Edge Lengths

It is clear that a triangle is *not free*, since the edge lengths have to satisfy the triangle inequality in every embedding (they cannot be prescribed arbitrarily). This simple observation extends to all cycles.

Observation 1 *No cycle is free in any planar graph.*

Proof Let C be a cycle with $k \geq 3$ edges in a planar graph H. If the first $k - 1$ edges of C have unit length, then the length of the k-th edge is less than $k - 1$ by repeated applications of the triangle inequality. □

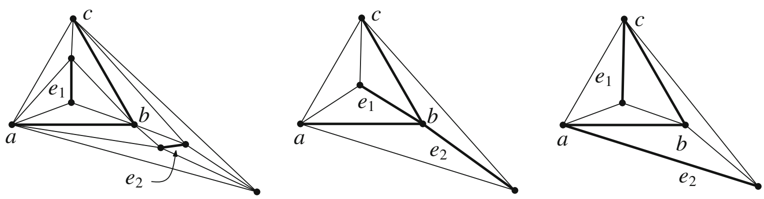

Fig. 1 Triangulations containing a bold subgraph G with edges ab, bc, e_1, and e_2. In every embedding, one of e_1 and e_2 lies in the interior of triangle abc, and so $\min\{\ell(e_1), \ell(e_2)\} < \ell(ab) + \ell(bc)$. *Left* G has four edges, two of which are adjacent. *Middle* G is a star. *Right* G is a path

Observation 2 *Let T be a triangulation with a separating triangle abc that separates edges e_1 and e_2. Then the subgraph G with edge set $E = \{ab, bc, e_1, e_2\}$ is not free in T. (See Fig. 1.)*

Proof Since abc separates e_1 and e_2, in every embedding of T, one of e_1 and e_2 lies in the interior of abc. If ab and bc have unit length, then all edges of abc are shorter than 2 in every embedding (by the triangle inequality), and hence the length of e_1 or e_2 has to be less than 2. $\square$

Based on Observations 1 and 2, we can show that most planar graphs G are not free in some appropriate triangulations T, $G \subseteq T$.

Theorem 5 *Let $G = (V, E)$ be a forest with at least 4 edges, at least two of which are adjacent, such that G is not the disjoint union of two paths P_2. Then there is a triangulation T that contains G as a subgraph and G is not free in T.*

Proof We shall augment G to a triangulation T such that Observation 2 is applicable. Specifically, we find four edges, $ab, bc, e_1, e_2 \in E$, such that either e_1 and e_2 are in distinct connected components of G or the (unique) path from e_1 to e_2 passes through a vertex in $\{a, b, c\}$. If we find four such edges, then G can be triangulated such that abc is a triangle (by adding edge ac), and it separates edges e_1 and e_2. See Fig. 1 for examples. We distinguish several cases based on the maximum degree $\Delta(G)$ of G.

Case 1: $\Delta(G) \geq 4$. Let b be a vertex of degree at least 4 in G, with incident edges ab, bc, e_1, and e_2. Then e_1 and e_2 are in the same component of G, and the unique path between them contains b.

Case 2: $\Delta(G) = 3$. Let b be a vertex of degree 3, and let e_1 be an edge not incident to b. If e_1 and b are in the same connected component of G, then let ba be the first edge of the (unique) path from b to e_1; otherwise let ba be an arbitrary edge incident to b. Denote the other two edges incident to b by bc and e_2. This ensures that if e_1 and e_2 are in the same component of G, the unique path between them contains b.

Case 3: $\Delta(G) = 2$. If G contains a path with four edges, then let the edges of the path be (e_1, ab, bc, e_2). Now the (unique) path between e_1 and e_2 clearly contains a, b, and c, so we are done in this case. If a maximal path in G has three edges, then let these edges be (ab, bc, e_1), and pick e_2 arbitrarily from another component. Finally,

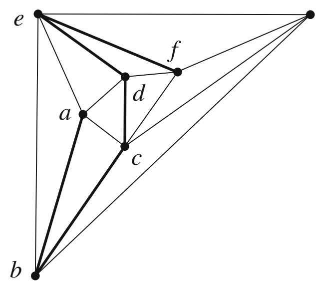

Fig. 2 A separating quadrilateral Q $= (b, c, d, e)$

if the maximal path in G has two edges, then let these edges be (ab, bc), and pick e_1 and e_2 from two distinct components (this is possible since G is not the edge-disjoint union of two paths P_2). □

Remark. Not only separating triangles impose constraints on the edge lengths: Consider the 4-connected triangulation T in Fig. 2, with a bold path (a, b, c, d, e, f). Note that $Q = (b, c, d, e)$ is a *separating quadrilateral*: In every embedding of T, either a or f lies in the interior of the polygon Q. In every embedding of T, the diameter of Q is less than $\ell(bc) + \ell(cd) + \ell(de)$. Hence, $\min\{\ell(ab), \ell(ef)\} < \ell(bc) + \ell(cd) + \ell(de)$, which is a nontrivial constraint for the edge lengths in G.

3 Every Matching is Free

In this section, we show that every matching $M = (V, E)$ in every planar graph H is free. Given an arbitrary length assignment for a matching M of H, we embed H with the specified edge lengths on M. Our algorithm is based on a simple approach, which works well when M is "well-separated" (defined below). In this case, we contract the edges in M to obtain a triangulation $\widehat{H}$, embed $\widehat{H}$ on a grid $c\mathbb{Z}^2$ for a sufficiently large $c > 0$, and then expand the edges of M to the prescribed lengths. If $c > 0$ is large enough, then the last step is only a small "perturbation" of $\widehat{H}$, and we obtain a valid embedding of H with prescribed edge lengths. If, however, some edges in M appear in separating 3- or 4-cycles, then a significantly more involved machinery is necessary.

3.1 Edge Contraction and Vertex Splitting Operations

A *near-triangulation* is a 3-connected planar graph in which all faces are triangles with at most one exception (which is typically considered to be the outer face). Let M be a matching in a planar graph H with a length assignment $\ell\colon M \to \mathbb{R}^+$. We may assume, by augmenting H if necessary, that H is a near-triangulation. Let D be an embedding of H where all the bounded faces are triangles. We shall construct a new embedding of H with the *same* vertices on the outer face where every edge $e \in M$ has length $\ell(e)$.

Edge contraction is an operation for a graph $G = (V, E)$ and an edge $e = v_1v_2 \in E$: Delete v_1 and v_2 and all incident edges, add a new vertex $\hat{v}_e$, and for every vertex

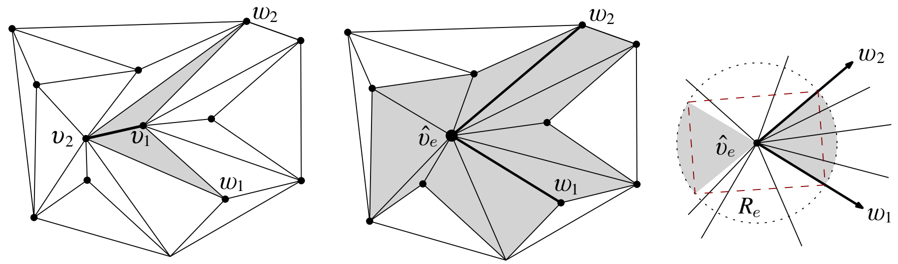

Fig. 3 *Left* An edge $e = v_1v_2$ of a near-triangulation incident to the *shaded triangles* $v_1v_2w_1$ and $v_1v_2w_2$. *Middle* e is contracted to a vertex $\hat{v}_e$. The triangular faces incident to $\hat{v}_e$ form a *star-shaped polygon*. *Right* We position edge e such that it contains $\hat{v}_e$, and lies in the shaded double wedge, and in the kernel of the *star-shaped polygon* centered at $\hat{v}_e$. For simplicity, we consider only part of the double wedge, lying in a rectangle R_e of diameter 2ε

$u \in V \setminus \{v_1, v_2\}$ adjacent to v_1 or v_2, add a new edge $u\hat{v}_e$. Suppose G is a near-triangulation and v_1v_2 does not belong to a separating triangle. Then v_1v_2 is incident to at most two triangle faces, say $v_1v_2w_1$ and $v_1v_2w_2$, and so there are at most two vertices adjacent to both v_1 and v_2. The cyclic sequence of neighbors of $\hat{v}_e$ is composed of the sequence of neighbors of v_1 from w_1 to w_2 and that of v_2 from w_2 to w_1 (in counterclockwise order). The inverse of an edge contraction is a *vertex split* operation that replaces a vertex $\hat{v}_e$ by an edge $e = v_1v_2$. See Fig. 3.

Suppose that we are given an embedding of a triangulation, and we would like to split an interior vertex $\hat{v}_e$ into an edge $e = v_1v_2$ such that (1) all other vertices remain at the same location; and (2) the common neighbors of v_1 and v_2 are w_1 and w_2 (which are neighbors of $\hat{v}_e$). Note that the bounded triangles incident to $\hat{v}_e$ form a star-shaped polygon, whose kernel contains $\hat{v}_e$ in the interior. We position $e = v_1v_2$ in the kernel of this star-shaped polygon such that the line segment e contains the point $\hat{v}_e$, and vertices w_1 and w_2 are on opposite sides of the supporting line of e. Therefore, e must lie in the double wedge between the supporting lines of $\hat{v}_e w_1$ and $\hat{v}_e w_2$ (Fig. 3, right). In Sect. 3.2, we position $e = v_1v_2$ such that its midpoint is $\hat{v}_e$, and in Sect. 4, we place either v_1 or v_2 at $\hat{v}_e$ and place the other vertex in the appropriate wedge incident to $\hat{v}_e$.

3.2 A Matching with Given Edge Lengths

We now recursively prove that every matching in every planar graph is free. In one step of the recursion, we construct an embedding of a subgraph in the interior of a separating triangle (resp., a separating 4-cycle), where the length of one edge is given (resp., the lengths of two edges are given). The work done for a separating triangle or 4-cycle is summarized in the following lemma.

Lemma 6 *Let $H = (V, E)$ be a near-triangulation and let $M \subset E$ be a matching with a length assignment $\ell\colon M \to \mathbb{R}^+$.*

(a) *Suppose that a 3-cycle (v_1, v_2, v_3), where $v_1v_2 \in M$, is a face of H. There is an $L > 0$ such that for every triangle abc with side length $|ab| = \ell(v_1v_2)$, $|bc| > L$,* *and $|ca| > L$, there is an embedding of H with prescribed edge lengths where the outer face is abc and v_1, v_2 and v_3 are mapped to a, b and c, respectively.*

(b) *Suppose that a* 4-*cycle* (v_1, v_2, v_3, v_4), *where* $v_1v_2 \in M$ *and* $v_3v_4 \in M$, *is a face of* H. *There is an* $L > 0$ *such that for every convex quadrilateral abcd with side lengths* $|ab| = \ell(v_1v_2)$, $|cd| = \ell(v_3v_4)$, $|ac| > L$, *there is an embedding of* H *with prescribed edge lengths where the outer face is abcd and* v_1, v_2, v_3, *and* v_4 *are mapped to a, b, c, and d, respectively.*

Proof We proceed by induction on the size of the matching M. We may assume, by applying an appropriate scaling, that $\min\{\ell(e) : e \in M\} = 1$.

(a) Consider an embedding D of H where $v_1v_2 \in M$ is an edge of the outer face, and let $M' = M \setminus \{v_1v_2\}$. Let $C_1, \ldots, C_k$ be the maximal separating triangles that include some edge from M', and the chordless separating 4-cycles that include two edges from M' (more precisely, we consider all such separating triangles and separating chordless 4-cycles and among them we choose those that are not contained in the interior of any other such separating triangle or chordless 4-cycle). Let H_0 be the subgraph of H obtained by deleting all vertices and incident edges lying in the interiors of the cycles $C_1, \ldots, C_k$. Let $M_0 \subseteq M'$ denote the subset of edges of M' contained in H_0. Let

$$\lambda_0 = \max\{\ell(e) : e \in M_0\}. \tag{1}$$

For $i = 1, \ldots, k$, let H_i denote the subgraph of H that consists of the cycle C_i and all vertices and edges that lie in C_i in the embedding D; and let $M_i \subset M'$ be the subset of edges of M' in H_i. Applying induction for H_i and M_i, there is an $L_i > 0$ such that H_i can be embedded with the prescribed lengths for the edges of M_i in every triangle (resp., convex quadrilateral) with two edges of lengths at least L_i. Let $L' = \max\{L_i : i = 1, \ldots, k\}$.

By construction, M_0 is a well-separated matching in H_0 (recall that v_1v_2 is not in M_0). Successively contract every edge $e = uv \in M_0$ to a vertex $\hat{v}_e$. We obtain a planar graph $\widehat{H}_0 = (\widehat{V}_0, \widehat{E}_0)$ on at most n (and at least 3) vertices.

Let $\widehat{D}_0$ be a grid embedding of $\widehat{H}_0$ constructed by the algorithm of Kurowski [14], where the outer face is a triangle with vertices $(0, 0)$, $(3n-7, 0)$, and $(\lfloor \frac{3n-7}{2} \rfloor, \lfloor \frac{3n-7}{2} \rfloor)$; the only horizontal edge is the base of the outer triangle; and the angular resolution of $\widehat{D}_0$ is $\varrho \geq \sqrt{2}\big/3\sqrt{5}n \in \Omega(1/n)$. The minimum edge length is 1, since all vertices have integer coordinates. There is an $\varepsilon \in \Omega(1/n)$ such that if we move each vertex of $\widehat{D}_0$ by at most ε, then the directions of the edges change by an angle less than $\varrho/2$, and thus we retain an embedding. We could split each vertex $\hat{v}_e$, $e \in M$, into an edge e that lies in the ε-disk centered at $\hat{v}_e$, and in the double wedge determined by the edges between $\hat{v}_e$ and the common neighbors of the endpoints of e (Fig. 3, right). However, we shall split the vertices $\hat{v}_e$, $e \in M$, only after applying the affine transformation α that maps the outer triangle of $\widehat{D}_0$ to a triangle abc such that $\alpha(v_1) = a$, $\alpha(v_2) = b$ and $\alpha(v_3) = c$. (The affine transformation α would distort the prescribed edge lengths if we split the vertices now.)

In the grid embedding $\widehat{D}_0$, the central angle of such a double wedge is at least $\varrho \in \Omega(1/n)$, i.e., the angular resolution of $\widehat{D}_0$. The boundary of the double wedge intersects the boundary of the ε-disk in four vertices of a rectangle that we denote by R_e. Note that the center of R_e is $\hat{v}_e$, and its diameter is $2\varepsilon \in \Omega(1/n)$. Hence, the aspect ratio of each R_e, $e \in M_0$, is at least $\tan(\varrho/2) \in \Omega(1/n)$, and so the width of R_e is $\Omega(1/n^2)$.

We show that if $L = \max\{10n(L' + 2\lambda_0 + |ab|), \xi n^3 \lambda_0\}$, for some constant $\xi > 0$, then the affine transformation α defined above satisfies the following two conditions—the first condition allows splitting the vertices $\hat{v}_e$, $e \in M$, into edges of desired lengths, and the second one ensures that the existing edges remain sufficiently long after the vertex splits:

(i) every rectangle R_e, $e \in M_0$, is mapped to a parallelogram $\alpha(R_e)$ of diameter at least λ_0 (defined in (1));
(ii) every nonhorizontal edge in $\widehat{D}_0$ is mapped to a segment of length at least $L' + 2\lambda_0$.

For (i), note that α maps a grid triangle of diameter $3n - 7 < 3n$ into triangle abc of diameter more than L. Hence, it stretches every vector parallel to the preimage of the diameter of abc by a factor of at least $L/(3n)$. Since the width of a rectangle R_e, $e \in M_0$, is $\Omega(1/n^2)$, the diameter of $\alpha(R_e)$ is at least $\Omega(L/n^3)$. If $L \in \Omega(n^3\lambda_0)$ is sufficiently large, then the diameter of every $\alpha(R_e)$ is at least λ_0.

For (ii), we may assume w.l.o.g. that the triangle abc is positioned such that $a = (0, 0)$ is the origin, $b = (|ab|, 0)$ is on the positive x-axis, and c is above the x-axis (i.e., it has a positive y-coordinate). Then, the affine transformation α is a linear transformation with an upper triangular matrix:

$$\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cy \end{bmatrix},$$

where $A, C > 0$, and by symmetry we may assume $B \ge 0$. We show that if $L \ge 10n(L' + 2\lambda_0 + |ab|)$, then α maps every nonhorizontal edge of $\widehat{D}_0$ to a segment of length at least $L' + 2\lambda_0$.

A nonhorizontal edge in the grid embedding $\widehat{D}_0$, directed upward, is an integer vector (x, y) with $x \in [-3n + 7, 3n - 7]$ and $y \in [1, \frac{3n-7}{2}]$. It is enough to show that $(Ax + By)^2 + (Cy)^2 > (L' + 2\lambda_0)^2$ for $x \in [-3n, 3n]$ and $y \in [1, \frac{3}{2}n]$. Since α maps the right corner of the outer grid triangle $(3n - 7, 0)$ to $b = (|ab|, 0)$, we have $A = |ab|/(3n - 7)$. Since $|ac| > L$, where $a = (0, 0)$ and $c = \alpha\left((\lfloor\frac{3n-7}{2}\rfloor, \lfloor\frac{3n-7}{2}\rfloor)\right)$, we have

$$\left(A \cdot \tfrac{3n-7}{2} + B \cdot \tfrac{3n-7}{2}\right)^2 + \left(C \cdot \tfrac{3n-7}{2}\right)^2 = |ac|^2 > L^2 \ge 100n^2(L' + 2\lambda_0 + |ab|)^2. \tag{2}$$

We distinguish two cases based on which term is dominant in the left-hand side of (2):

Case 1: $(C \cdot \frac{3n-7}{2})^2 \ge 50n^2(L' + 2\lambda_0 + |ab|)^2$. In this case, we have $C^2 > (L' + 2\lambda_0)^2$, and so $(Cy)^2 > (L' + 2\lambda_0)^2$ since $y \ge 1$.

Case 2: $(A \cdot \frac{3n-7}{2} + B \cdot \frac{3n-7}{2})^2 > 50n^2(L' + 2\lambda_0 + |ab|)^2$. In this case, we have $A \cdot \frac{3n-7}{2} + B \cdot \frac{3n-7}{2} > 7n(L' + 2\lambda_0 + |ab|)$. Combined with $A = |ab|/(3n - 7)$, this gives $B > 4(L' + 2\lambda_0 + |ab|)$. It follows that $(Ax + By)^2 > (L' + 2\lambda_0)^2$, as claimed, since $|Ax| \le |ab|$ and $y \ge 1$.

We can now reverse the edge contraction operations, that is, split each vertex $\hat{v}_e$, $e \in M_0$, into an edge e of length $\ell(e)$ within the parallelogram $\alpha(R_e)$. By (i), we obtain an embedding of H_0. Each cycle C_i, $i = 1, \dots, k$, is a triangle (resp., quadrilateral) where the edges of M_0 have prescribed lengths, and any other edge has length at least $L' = \max\{L_i : i = 1, \dots, k\}$ by (ii). By induction, we can insert an embedding of H_i with prescribed lengths on the matching M_i into the embedding of the cycle C_i, for $i = 1, \dots, k$. We obtain the required embedding of H.

(b) The proof for the case when the outer face of H is a 4-cycle follows the same strategy as for (a), with some additional twists.

Suppose we are given a convex quadrilateral $abcd$ as described in the statement of the lemma. Denote by q the intersection of its diagonals. We show that ($|aq|$ and $|bq|$ are both at least $L/3$) or ($|cq|$ and $|dq|$ are both at least $L/3$) if $L > 9\max(|ab|, |cd|)$. Indeed, we have $|ac| > |bc| - |ab| > \frac{8}{9}L$ from the triangle inequality for abc. Since $|ac| = |aq| + |cq|$, we have $|aq| > \frac{4}{9}L$ or $|cq| > \frac{4}{9}L$. If $|aq| > \frac{4}{9}L$, then $|bq| > |aq| - |ab| > \frac{1}{3}L$ from the triangle inequality for abq; otherwise $|dq| > |cq| - |cd| > \frac{1}{3}L$. We may assume without loss of generality that $|aq| > L/3$ and $|bq| > L/3$. In the remainder of the proof, we embed H such that almost all vertices lie in the triangle abq, and the vertices v_1, v_2, v_3, and v_4 are mapped to a, b, c, and d, respectively.

Similarly to (a), we define H_0 as the graph obtained by deleting all vertices and incident edges lying in the interior of maximal separating triangles or chordless 4-cycles, containing an edge from $M \setminus \{v_1v_2\}$. Define L' as before, by using the inductive hypothesis in the separating cycles. Contract successively all remaining edges of $M \setminus \{v_1v_2\}$ that are in H_0 (including edge v_3v_4) to obtain a graph $\hat{H}_0$. Denote by $\hat{v}_3$ the vertex of $\widehat{H}_0$ corresponding to $v_3v_4 \in M$, and consider an embedding of $\widehat{H}_0$ with the outer face $v_1v_2\hat{v}_3$.

We again use the embedding $\widehat{D}_0$ of Kurowski [14] such that v_1, v_2, and $\hat{v}_3$ are mapped to $(0, 0)$, $(3n-7, 0)$, and $(\lfloor\frac{3n-7}{2}\rfloor, \lfloor\frac{3n-7}{2}\rfloor)$, respectively. We first split vertex $\hat{v}_3$ into two vertices v_3 and v_4, exploiting the fact that $\hat{v}_3$ is a boundary vertex in $\widehat{D}_0$ and some special properties of the embedding in [14] (described below), and then split all other contracted vertices of $\hat{H}_0$ similarly to (a).

Denote the neighbors of $\hat{v}_3$ in $\widehat{D}_0$ in counterclockwise order by $v_1 = u_0, u_1, \dots, u_k = v_2$ (Fig. 4, left). The grid embedding in [14] has the following property (mentioned in Section 1): the path $u_0, \dots, u_k$ is x-monotone and the slope of every edge is in the range $(-1, 1)$. Let $p = (\lfloor\frac{3n-7}{2}\rfloor, 2\lfloor\frac{3n-7}{2}\rfloor)$, and note that the slope of every line between p and $u_1, \dots, u_k$ is outside of the range $(-2, 2)$. Similarly, if we place the points v_3 (resp., v_4) on the ray emitted by p in direction $(1, 2)$ (resp., $(-1, 2)$), then the slope of every line between v_3 (resp., v_4) and $u_1, \dots, u_k$ is outside of $(-2, 2)$.

We can now split vertex $\hat{v}_3$ as follows. Refer to Fig. 4. Let α be the affine transformation that maps the triangle v_1v_2p to abq such that $\alpha(v_1) = a$, $\alpha(v_2) = b$, and $\alpha(p) = q$. Since the diagonals ac and bd intersect at q, the segments $v_1\alpha^{-1}(c)$ and $v_2\alpha^{-1}(d)$ intersect at p. We split vertex $\hat{v}_3$ into $v_3 = \alpha^{-1}(c)$ and $v_4 = \alpha^{-1}(d)$. By the above observation, the edges incident to v_3 and v_4 remain above the x-monotone path $u_0, \dots, u_k$. (Note, however, that the angles between edges incident with v_3 or v_4 may be arbitrarily small.)

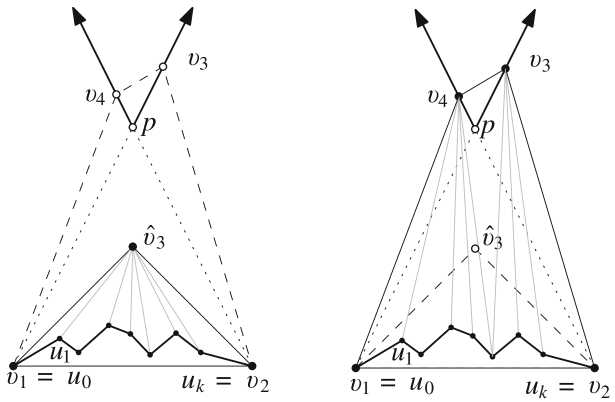

Fig. 4 *Left* The embedding $\widehat{D}_0$ into a triangle $v_1v_2\hat{v}_3$, and the x-monotone path $v_1 = u_0, u_1, \ldots, u_k$ formed by the neighbors of $\hat{V}_3$. A point p lies above $\hat{v}_3$, and the rays emitted by p in directions $(1, 2)$ and $(-1, 2)$. *Right* Vertex $\hat{v}_3$ is split into v_3 and v_4 on the two rays emitted by p

With a very similar computation as for (a), we conclude that for a large enough $L \in \Omega(L' + \lambda_0)$ we can guarantee the same two properties we needed in (a), that is, α maps every small rectangle R_e to a parallelogram $\alpha(R_e)$ whose diameter is at least λ_0, and every nonhorizontal edge to a segment of length at least $L' + 2\lambda_0$. Hence, every remaining contracted vertex v_e in $\widehat{D}_0$ can be split within the parallelogram $\alpha(R_e)$ as in (a). To finish the construction, it remains to apply the inductive hypothesis to fill in the missing parts in the maximal separating triangles or 4-cycles. □

We are now ready to prove the main result of this section.

Theorem 7 *Every matching in a planar graph is free.*

Proof Let $H = (V, E)$ be a planar graph, and let $M \subseteq E$ be a matching with a length assignment $\ell \colon M \to \mathbb{R}^+$. We may assume, by augmenting H with new edges if necessary, that H is a triangulation. Consider an embedding of H such that an edge $e \in M$ is on the outer face. Now Lemma 6 completes the proof. □

4 Graphs with Three or Four Edges

By Theorems 5 and 7, a graph G with at least five edges is free in every host H if and only if G is a matching. For graphs with four edges, the situation is also clear except for the case of the disjoint union of two paths of two edges each. In this section, we show that every forest with three edges, as well as the disjoint union of two paths of length two, is always free.

We show (Lemma 9) that it is enough to consider hosts H in which G is a spanning subgraph, that is, $V(G) = V(H)$. For a planar graph $G = (V, E)$, the *triangulation of* G is an edge-maximal planar graph T, $G \subset T$, on the vertex set V. (The following lemma holds for every graph G, including matchings. However, it would not simplify the argument in that case.)

Lemma 8 *If G is a subgraph of a triangulation H with $0 < |V(G)| < |V(H)|$, then there is an edge in H between a vertex in $V(H)$ and a vertex in $V(H) - V(G)$ that does not belong to any separating triangle of H.*

Proof Let $V = V(G)$ denote the vertex set of G and $U = V(H) \setminus V$. Let $E(U, V)$ be the set of edges in H between U and V. Since H is connected, $E(U, V)$ is nonempty. Consider an arbitrary embedding of H (with arbitrary edge lengths). For every edge $uv \in E(U, V)$, let $k(uv)$ denote the maximum number of vertices of H that lie in the interior of a triangle (u, v, w) of H, where $w \in V(H)$. Let $uv \in E(U, V)$ be an edge that minimizes $k(uv)$. If $k(uv) = 0$, then uv does not belong to any separating triangle, as claimed. For the sake of contradiction, suppose $k(uv) > 0$, and let (u, v, w) be a triangle in H that contains exactly $k(uv)$ vertices of H. Since H is a triangulation, there is a path between u and v via the interior of (u, v, w). Since $u \in U$ and $v \in V$, one edge of this path must be in $E(U, V)$, say $u'v' \in E(U, V)$. Note that any triangle (u', v', w') of H lies inside the triangle (u, v, w), and hence contains strictly fewer vertices than (u, v, w). Hence $k(u'v') < k(u, v)$ contradicting the choice of edge uv. □

Lemma 9 *If a planar graph G is (extrinsically) free in every triangulation of G, then G is (extrinsically) free in every planar host H, $G \subseteq H$.*

Proof Let $G = (V, E)$ be a planar graph with a length assignment $\ell : E \to \mathbb{R}^+$. It is enough to prove that G is (extrinsically) free in every triangulation H, $G \subset H$. We proceed by induction on $n' = |V(H)| - |V(G)|$, the number of extra vertices in the host H. If $n' = 0$, then H is a triangulation *of* G, and G is free in H by assumption. Consider a triangulation H, $G \subset H$, and assume that the claim holds for all smaller triangulations H', $G \subseteq H'$.

By Lemma 8, there is an edge $e = uv$ in H between $v \in V(G)$ and $u \in V(H) - V(G)$ that does not belong to any separating triangle. Contract e into a vertex $\hat{v}_e$ to obtain a triangulation H', $G \subset H'$. By induction, H' admits a straight-line embedding in which the edges of G have prescribed lengths. Since e is not part of a separating triangle of H', we can split vertex $\hat{v}_e$ into u and v such that v is located at point $\hat{v}_e$, and u lies in a sufficiently small neighborhood of $\hat{v}_e$ (refer to Fig. 3). Thus, we obtained a straight-line embedding of H in which edges of G have prescribed lengths. □

The next theorem finishes the characterization of free graphs.

Theorem 10 *Let G be a subgraph of a planar graph H, such that G is*

(1) *the star with three edges, or*
(2) *the path with three edges, or*
(3) *the disjoint union of a path with two edges and a path with one edge, or*
(4) *the disjoint union of two paths with two edges each.*

Then G is free in H.

Proof By Lemma 9, it is enough to prove the theorem in the case when G is a spanning subgraph of H. We can also assume that H is a triangulation.

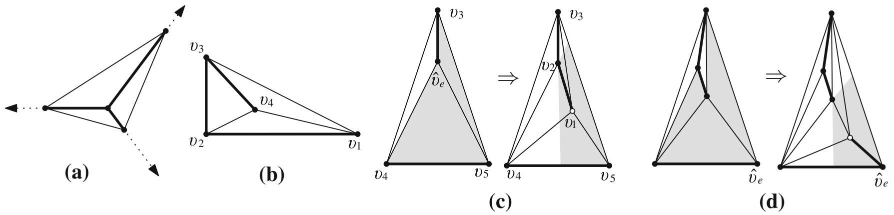

Fig. 5 **a** Embedding of a star with three leaves. **b** Embedding of a path of three edges. **c** Graph H' and regions that are used for splitting $\hat{v}_e$. **d** Graph H' and its spanning subgraph G whose edges have prescribed lengths

(1) If G is the star with three edges, then H is K_4. Embed the center of the star at the origin. Place the three leaves on three rotationally symmetric rays emitted by the origin, at prescribed lengths from the origin (Fig. 5a). The remaining three edges are embedded as straight line segments on the convex hull of the three leaves.

(2) Let G be the path (v_1, v_2, v_3, v_4) with $\ell(v_1v_2) \geq \ell(v_3v_4)$. Embed v_2 at the origin, place v_1 and v_3 on the positive x- and y-axis, respectively, at prescribed distance from v_2. Note that $\Delta = \text{conv}(v_1, v_2, v_3)$ is a right triangle whose diameter (hypotenuse) is larger than the other two sides (Fig. 5a). Thus we can embed v_4 at a point in the interior of Δ at distance $\ell(v_3v_4)$ from v_3. Since the four vertices have a triangular convex hull, $H = K_4$ embeds as a straight-line graph.

(3) Suppose that $G = (V, E)$ is the disjoint union of path (v_1, v_2, v_3) and (v_4, v_5). Since H has five vertices, there exists at most one separating triangle in H. Thus, the path of G with two edges contains an edge, say $e = v_1v_2$, that does not belong to any separating triangle. Contract edge e to a vertex $\hat{v}_e$, obtaining a triangulation $H' = K_4$ on four vertices, and a perfect matching $G' \subset H'$. Let us embed the two edges of G' with prescribed lengths such that one lies on the x-axis, the other lies on the orthogonal bisector of the first edge at distance $\ell(e)$ from the x-axis. This defines a straight-line embedding of H', as well. We obtain a desired embedding of H by splitting vertex $\hat{v}_e$ into edge e such that v_2 is embedded at point $\hat{v}_e$ and v_1 is mapped to a point in the kernel of the appropriate star-shaped polygon (c.f. Fig. 3). By the choice of our embedding of H', the diameter of this kernel is more than $\ell(e)$, and we can split $\hat{v}_e$ without introducing any edge crossing (Fig. 5c)

(4) Assume that G is the disjoint union of two paths P_1 and P_2, each with two edges. Since G is a spanning subgraph of H, neither path can span a separating triangle. Moreover, since there exist at most two separating triangles in H, one of the paths, say P_1, contains an edge e that is not part of a separating triangle. Contract edge e to $\hat{v}_e$, obtaining a triangulation H' and a subgraph G'. Similarly to the case (3), embed H' respecting the lengths of all the edges of G' such that all edges between the two components of G' have length at least $\ell(e)$. By the choice of our drawing of H', the kernel of the appropriate star-shaped polygon has diameter at least $\ell(e)$. Therefore, we can split e into two vertices such that the middle vertex of P_1 remains at $\hat{v}_e$, and the endpoint of P_1 is embedded at distance $\ell(e)$ from $\hat{v}_e$ (Fig. 5d). □

5 Stars are Free in 4-Connected Triangulations

In this section, we prove Theorem 2.

Theorem 2 *Every star is free in a* 4*-connected triangulation.*

Proof Let $T = (V, E)$ be a 4-connected triangulation. Let $S \subset E$ be a star centered at $v_0 \in V$ with $k \geq 3$ edges $S = \{v_0v_1, \dots, v_0v_k\}$ labeled cyclically around v_0. Let $\ell\colon S \to \mathbb{R}^+$ be a length assignment. We embed T such that every edge $e \in S$ has length $\ell(e)$. Refer to Fig. 6 (upper-left).

Since T is a triangulation, every two consecutive neighbors of v_0 are adjacent. Nonconsecutive neighbors of v_0 are nonadjacent; otherwise they would create a separating triangle. In other words, the vertices $\{v_1, \dots, v_k\}$ induce a cycle $C = (v_1, \dots, v_k)$ in T. Consider an embedding D of T in which v_0 is an interior vertex.

Note that the outer face is a triangle, since T is a triangulation. If two neighbors of v_0 are incident on the outer triangle, they must be consecutive neighbors of v_0, since T is 4-connected. Therefore, if v_0 has 3 neighbors on the outer face, then $T = K_4$, and it obviously has an embedding with prescribed edge lengths. We can distinguish two cases:

Case 1: at most one neighbor of v_0 is incident to the outer face in D. Refer to Fig. 6. Then every edge of C is an interior edge of D. The two triangles adjacent to every edge v_iv_{i+1} form a quadrilateral incident to v_0 and some other vertex, which is nonadjacent to v_0 (otherwise there would be a separating triangle in T). Consider all 4-cycles incident to v_0 and to a vertex nonadjacent to v_0. Such a cycle is called *maximal* if it is not contained in any other such 4-cycle. Let $C_1, C_2, \dots, C_m$ be a collection of maximal 4-cycles, each incident to v_0 and to some nonadjacent vertex, in counterclockwise order around v_0. Denote by u_i, $i = 1, \dots, m$, the vertex in C_i that is not adjacent to v_0. It is clear that every triangle incident to v_0 is contained in one of the cycles C_i, every cycle C_i passes through exactly two edges of the star H, and the number of cycles is at least $m \geq 2$. For $i = 1, \dots, m$, the consecutive cycles C_i and C_{i+1} (with $m + 1 = 1$) share exactly one edge, by their maximality, which is denoted by $v_0v_{\kappa(i)}$.

We construct a triangulation $\widehat{T} = (\widehat{V}, \widehat{E})$ in two steps: First delete all vertices and incident edges in the interiors of the cycles $C_1, \dots, C_m$ (Fig. 6, upper-right) and then successively contract the remaining edges $v_0v_{\kappa(1)}, \dots, v_0v_{\kappa(m)}$ of H (Fig. 6, lower-left). The vertices $v_0, v_1, \dots, v_k$ merge into a single vertex $\hat{v}_0$ in $\widehat{T}$, and the cycles C_i collapse into distinct edges incident to $\hat{v}_0$.

Consider an embedding $\widehat{D}$ of $\widehat{T}$ in which $\hat{v}_0$ is an interior vertex. The triangles incident to $\hat{v}_0$ form a star-shaped polygon P, whose vertices include $u_1, \dots, u_m$, and vertex $\hat{v}_0$ lies in the interior of the kernel of P. There is an $\varepsilon > 0$ such that the ε-neighborhood of $\hat{v}_0$ also lies in the kernel of P. We may assume (by scaling) that $\varepsilon = \max\{\ell(e) : e \in M\}$.

Embed the star center v_0 at $\hat{v}_0$, and for $j = 1, \dots, m$, embed vertex $v_{\kappa(j)}$ at the point on the segment v_0u_j at distance $\ell(v_0v_{\kappa(j)})$ from the center (Fig. 6, lower-right). Hence each cycle $C_j = (v_0, v_{\kappa(j-1)}, u_j, v_{\kappa(j)})$ is embedded in (weakly) convex position, where edges $v_0v_{\kappa(j)}$ and $v_{\kappa(j)}u_j$ are collinear.

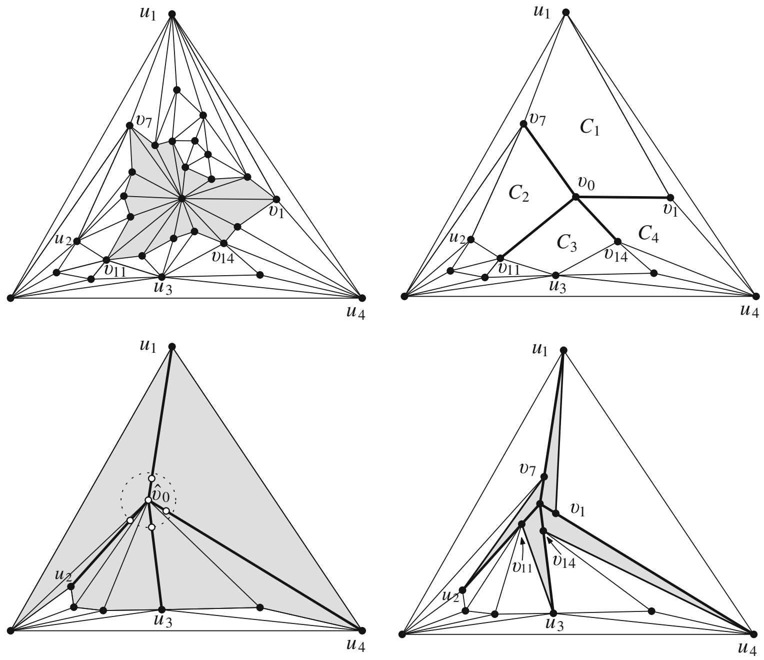

Fig. 6 *Upper-left* A star centered at v_0 in a 4-connected triangulation. The maximal 4-cycles $C_1, \dots, C_4$ incident to v_0 and to some nonadjacent vertex are defined by vertices $u_1, \dots, u_4$. *Upper-right* The graph obtained after deleting all vertices and incident edges in the interior of the cycles $C_1, \dots, C_4$. *Lower-left* The remaining edges of the star are contracted to $\hat{v}_0$. The cycles $C_1, \dots C_4$ collapse to bold edges. The faces incident to $\hat{v}_0$ form a star-shaped polygon, whose kernel contains an ε-neighborhood of $\hat{v}_0$. *Lower-Right* v_0 is embedded at $\hat{v}_0$, and vertices $v_{\kappa(j)}$ of cycles C_j are embedded on the segment $\hat{v}_0 u_j$ for $j = 1, \dots, 4$

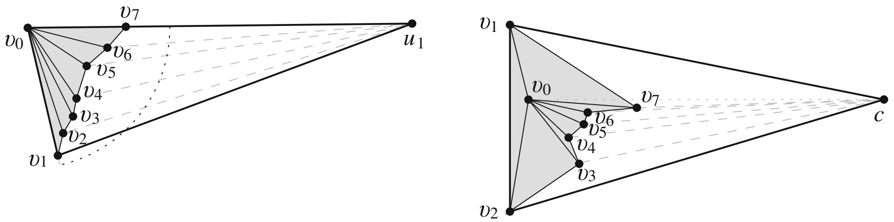

Fig. 7 *Left* Vertices $v_i, i = 2, \dots, 6$, are successively embedded in the triangles $v_0 v_{i-1} u_1$. *Right* Triangle $v_0 v_1 v_2$ is embedded such that $v_0 v_1$ and $v_0 v_2$ are almost collinear; then v_i, $i = 3, \dots, k$, are successively embedded in the triangles $v_0 v_{i-1} c$

For $j = 1, \dots, m$, the vertices $v_{\kappa(j)+1}, \dots, v_{\kappa(j+1)-1}$ should lie in the interior of the triangle $\Delta_j = v_0 u_j v_{\kappa(j+1)}$. Embed successively $v_{\kappa(j)+1}, \dots, v_{\kappa(j+1)-1}$ at the prescribed distance from the center v_0 in the sequence of nested triangles $v_0 u_j v_{i-1}$ (Fig. 7, left). As a result, the path $(v_{\kappa(j)}, \dots, v_{\kappa(j+1)})$ partitions the triangle Δ_j into two star-shaped polygons, with star centers v_0 and u_j, respectively. All remaining vertices in the interior of C_j can be embedded in the latter star-shaped polygon by the Hong–Nagamochi theorem [11].

Case 2: exactly two neighbors of v_0 are on the outer triangle. Without loss of generality the outer face of D is the triangle v_1v_2c. Refer to Fig. 7 (right). Embed triangle $v_0v_1v_2$ such that v_0v_1 and v_0v_2 have prescribed edge lengths, and v_1v_2 has length $\ell(v_0v_1) + \ell(v_0v_2) - \varepsilon$ for a small $\varepsilon > 0$. Embed vertex c at distance $2\max\{\ell(v_0v_i) : 1 \le i \le k\}$ from v_0 such that the line v_0c is orthogonal to v_1v_2.

Embed successively $v_3, \dots, v_k$ in the sequence of nested triangles $v_0v_{i-1}c$. Then the path $(v_2, v_3 \dots, v_m, v_1)$ partitions the triangle v_1v_2c into two star-shaped polygons, with star centers v_0 and c, respectively. All remaining vertices in the interior of T can be embedded in the latter star-shaped polygon by the Hong–Nagamochi theorem [11]. □

6 Extrinsically Free Subgraphs

In this section, we prove Theorem 3. A forest $G = (V, E)$ has a straight-line embedding with every length assignment $\ell : E \to \mathbb{R}^+$, that is, it has no intrinsic constraints on the edge lengths. Theorems 5 and 7 classify all *forests* G that are extrinsically free in every host H. It remains to classify all planar graphs G that contain cycles, except for the cases that G itself is a cycle with $k \ge 4$ vertices. Our positive results are limited to two types of graphs.

Lemma 11 *A subgraph G of a planar graph H is extrinsically free if G is*

(1) *a triangulation, or*
(2) *a triangle and one additional edge* (*either disjoint from or incident to the triangle*).

Proof We may assume that H is a triangulation, by augmenting H with dummy edges if necessary.

(1) Let $G = (V, E)$ be a triangulation with a length assignment $\ell : E \to \mathbb{R}^+$ such that G has an embedding D_0 with edge length $\ell(e)$, $e \in E$, and let D_1 be an arbitrary embedding of H. Since every triangulation has a unique combinatorial embedding, D_0 is combinatorially equivalent to the restriction of D_1 to G. Partition the vertices into the vertex sets of the connected components of $H \setminus G$, each lying in a face of G. By Tutte's barycenter method [20], we can embed each vertex class within the corresponding triangular face of G_0.
(2) Let abc and e be a triangle and an edge in G. Consider an embedding of H such that ab is an edge of the outer face and e lies outside of abc. An argument analogous to Lemma 6(1) shows that H has an embedding such that the edges of the triangle abc are mapped to a given triangle and in the exterior of that triangle edge e has prescribed length. (Recall that in Lemma 6(1), the outer face was mapped to a given triangle such that disjoint edges in the interior of the triangle had prescribed lengths.) □

In the remainder of this section, we show that no other planar graph G is extrinsically free in all hosts H if G contains a cycle. We start by observing that it is enough to consider graphs with at most two components.

Observation 3 *Let $G = (V, E)$ be extrinsically free in every host H, $G \subset H$. If G contains a cycle, then G has at most two connected components.*

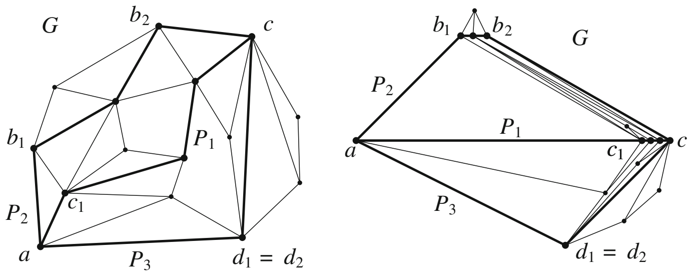

Fig. 8 *Left* Embedding D_0 of G, where path P_1 separates P_2 from P_3. *Right* Embedding D_1 of G where path P_1 lies on the x-axis, and $P_2 \cup P_3$ lies in an ε-neighborhood of the parallelogram ab_1cd_1

Proof Suppose to the contrary that G has at least three connected components, denoted by G_1, G_2, and G_3. Assume, without loss of generality, that G_1 contains a cycle C. Augment G to a triangulation in which C is a separating cycle, separating G_2 and G_3. Let $\ell\colon E \to \mathbb{R}^+$ be a length assignment that assigns a total length of 1 to the edges of C, and length at least 1 to every edge of G_2 and G_3, respectively. In every embedding of H, one of G_2 and G_3 lies in the interior of the simple polygon C; however, every edge in G_2 and G_3 is longer than the diameter of C. Hence H cannot be embedded with the prescribed edge lengths, and so G is not extrinsically free. □

The following technical lemma is the key tool for treating the remaining cases except for when G is a cycle.

Lemma 12 *Let $G = (V, E)$ be a planar graph such that two vertices $a, c \in V$ are connected by three independent paths P_1, P_2, and P_3; the paths P_2 and P_3 have some interior vertices; and the interior vertices of P_2 and P_3 are in distinct components of $G - P_1$. Then G is not extrinsically free in some host H, $G \subset H$.*

Proof We may assume that H is a triangulation, by augmenting H with dummy edges if necessary. Furthermore, we may also assume that P_1, P_2, and P_3 are three independent paths with the above properties that have the minimum total number of vertices in G. We start with a brief overview of the proof: We shall describe an embedding D_1 of G; then define $H = (V, E \cup \{uv\})$ by augmenting G with a single new edge uv (to be determined), and finally show that no embedding of G with the same edge lengths as in D_1 can be augmented to an embedding of H.

Refer to Fig. 8 (left). Denote by ac_1 the edge of P_1 incident to a, with possibly $c_1 = c$. Denote by ab_1 and cb_2 the edges of P_2 incident to its endpoints (possibly $b_1 = b_2$); and by ad_1 and cd_2 the edges of P_3 incident to its endpoints (possibly $d_1 = d_2$).

Consider an embedding D_0 of G such that P_1 lies inside the cycle $P_2 \cup P_3$. The vertices of G that are incident to none of the three paths P_1, P_2, P_3 can be partitioned as follows: Let V_b^- and V_d^- denote the vertices lying in the interior of cycles $P_1 \cup P_2$ and $P_1 \cup P_3$, respectively. The vertices in the exterior of cycle $P_2 \cup P_3$ can be partitioned into two sets, since G contains no path between interior vertices of P_2 and P_3: Let V_b^+ and V_d^+ denote the vertices lying in the exterior of cycle $P_2 \cup P_3$ and joined with P_2 or P_3, respectively, by a path in $G - P_1$.

We now construct a straight-line embedding D_1 combinatorially equivalent to D_0 as follows (refer to Fig. 8, right): Let $0 < \varepsilon < 1/(2|E|)$ be a small constant. Embed path P_1 on the positive x-axis with $a = (0, 0)$, $c = (3, 0)$, and if $c_1 \neq c$, then $c_1 = (0, 3 - \varepsilon)$. Let $b_1 = (1, 1)$, and if $b_1 \neq b_2$, then embed the vertices of P_2 between b_1 and b_2 on a horizontal segment between $b_1 = (1, 1)$ and $b_2 = (1 + \varepsilon, 1)$. Similarly, let $d_1 = (2, -1)$, and if $d_1 \neq d_2$, then embed the vertices of P_3 between d_1 and d_2 on a horizontal segment between $d_1 = (2, -1)$ and $d_2 = (2 + \varepsilon, -1)$. The vertices in V_b^- (resp., V_d^-) can be embedded by Tutte's barycenter method [20], since the cycle $P_1 \cup P_2$ (resp., $P_1 \cup P_3$) is embedded as a (weakly) convex polygon, and the path P_2 (resp., P_3) has no shortcut edges by the choice of P_1, P_2, and P_3. The vertices of V_b^+ (resp., V_d^+) can be embedded by the Hong–Nagamochi theorem in an ε-neighborhood of b_1 (resp., d_1), since the region above P_2 (resp., below P_3) is star-shaped. This completes the description of D_1.

Let $H = (V, E \cup \{uv\})$, where u (resp., v) is a vertex of the outer face strictly above (resp., below) the x-axis in the embedding D_1. Note that vertices u and v are in an ε-neighborhood of b_1 and d_1, respectively, and so cannot be connected by a straight edge in D_1.

Let D_2 be an embedding of G in which every edge has the same length as in D_1. Assume, without loss of generality, that $a = (0, 0)$ and ac_1 lies on the positive x-axis, and b_1 is above the x-axis. By the length constraints, path P_1 lies in the ε-neighborhood of edge ac_1. Path P_2 lies in the 2ε-neighborhood of its position in D_1. Vertex d_1 must be below the x-axis, otherwise b_2c and ad_1 would cross. Hence path P_3 is also in the 2ε-neighborhood of its location in D_1. All interior vertices of P_2 and all vertices in V_b^+ are in the $(\varepsilon|E|)$-neighborhood of b_1. Similarly, all interior vertices of P_3 and all vertices in V_d^+ are in the $(\varepsilon|E|)$-neighborhood of d_1. Since $\varepsilon|E| < \frac{1}{2}$, the line segment uv crosses ac_1, and so H cannot be embedded with the prescribed edge lengths. This confirms that G is not extrinsically free in H. $\square$

Lemma 13 *Let $G = (V, E)$ be extrinsically free in every host H, $G \subset H$. If G contains a cycle C with $k \geq 4$ vertices such that all vertices of C are incident to a common face in some embedding of G, then C is a 2-connected component of G.*

Proof Consider an embedding D_0 of G in which all vertices of C are incident to common face F. Assume without loss of generality that F is a bounded face that lies inside C. We first show that C must be a chordless cycle. Indeed, if C has an (exterior) chord ac, then G would not be extrinsically free by applying Lemma 12 with the three paths $P_1 = \{ac\}$, and letting P_2 and P_3 be the two arcs of C between a and c.

Let a and c be two nonadjacent vertices of C (Fig. 9). If there is a path P between a and c (via the interior or exterior of C), then again G would not be extrinsically free by applying Lemma 12 for P and the two arcs of C between a and c. By *Menger's theorem*, G has a 2-cut that separates a and c. Such a 2-cut necessarily consists of two vertices of C, say b and d. If $\{b, d\}$ is a 2-cut, then both b and d are incident to both F and the outer face of D_0. Applying the same argument for every two nonconsecutive vertices of C, we conclude that all vertices of C are incident to both F and the outer face in D_0. $\square$

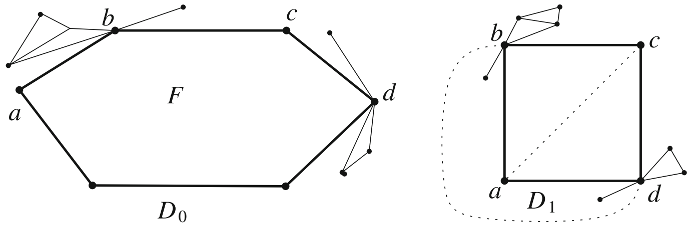

Fig. 9 *Left* The embedding D_0 of G. Every vertex of C is incident to both F and the outer face. *Right* The embedding D_1 of G in which $abcd$ is a square

It remains to consider planar graphs G that are cycles or contain some triangulations as subgraphs. First, we deal with triangulations.

Lemma 14 *Let $G = (V, E)$ be a planar graph in which every 2-connected component is a triangulation or a single edge. Suppose that G satisfies one of the following conditions:*

(i) *a maximal 2-connected component of G is a triangulation with at least 4 vertices, and G contains at least one additional edge;*
(ii) *a maximal 2-connected component of G is a triangle abc, and G contains two cut edges incident to abc;*
(iii) *a maximal 2-connected component of G is a triangle abc, and G contains a path of two edges incident to abc;*
(iv) *G is the disjoint union of a triangle and either a path P_2 or another triangle.*

Then there is a host H, $G \subseteq H$, such that G is not extrinsically free in H.

Proof In all four cases, we augment G to a triangulation H such that G contains at least two edges of a separating triangle abc, and abc separates two other edges e_1, e_2 of G. Then we construct a valid length assignment in which the diameter of abc is less than 2 and $\ell(e_1) = \ell(e_2) = 2$. This will show that G is not extrinsically free in H.

(i) Let T be a maximal 2-connected component of G that is a triangulation with at least 4 vertices. If G contains two components, then let G_1 be the component containing T, including a triangle abc, and let $G_2 = G - G_1$. If G is connected, then G contains an edge ad incident to T at vertex a, and a is incident to a triangle abc in T. Note that a is a cut vertex, since T is a maximal 2-connected subgraph. Therefore G decomposes into two subgraphs that intersect in vertex a only: G_1 contains T and G_2 contains ad.

In both cases, let H_1 be a triangulation of G_1 in which abc is a face, and let H_2 be an arbitrary triangulation of G_2. Now let H be a triangulation of $H_1 \cup H_2$ (identifying vertex a if G is connected) such that G_2 lies inside the triangle abc. Consider an embedding D_1 of H_1 in which abc is the outer face and it is a regular triangle with unit sides. Let G_2 be an embedding of H_2 such that a is a vertex of the outer face, and $|ad| = 2$. The union of D_1 and D_2 gives an embedding of H (hence G), identifying a if G is connected. However, H does not have an embedding in which every edge of G has the same length as in D. Indeed, the edge lengths in T completely determine

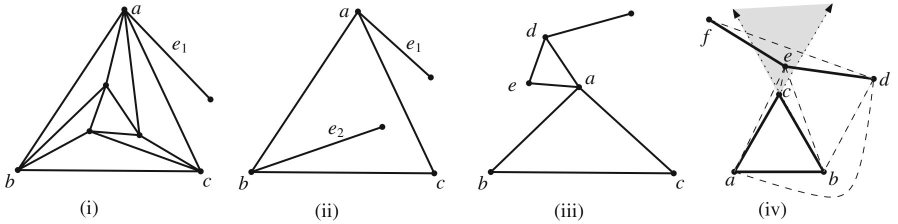

Fig. 10 Illustrations for cases (i), (ii), (iii), and (iv)

the embedding (we cannot interchange the interior and exterior of any face), and the edge ad is longer than the diameter of the outer face abc.

(ii) Let e_1 and e_2 be cut edges of G incident to abc (Fig. 10). Decompose G into three subgraphs, every two of which intersect in one vertex only: G_1 contains e_1, G_2 contains e_2, and G_3 is triangle abc. Let H_1 and H_2 be arbitrary triangulations of G_1 and G_2, respectively. Let H be a triangulation of $H_1 \cup H_2 \cup abc$, identifying the common vertices, such that abc separates H_1 from H_2. We show that G has an embedding where abc is a regular triangle with unit sides and $\ell(e_1) = \ell(e_2) = 2$. Consider an embedding D_1 of H_1 such that the vertex incident to abc is on the outer face, which is a regular triangle, and $\ell(e_1) = 2$. Let D_2 be an embedding of H_2 with analogous properties. The union of D_1, D_2 and the unit triangle abc readily gives the required embedding of G, after identifying the shared vertices.

(iii) We may assume without loss of generality that G contains a path (a, d, e) incident to abc at vertex a. Note that a is a cut vertex (since abc is maximal 2-connected), and so G can be decomposed into two subgraphs that intersect in vertex a only: G_1 contains abc, and G_2 contains the path (a, d, e). Let H_1 be a triangulation of G_1 such that the outer face is abc, and let H_2 be a triangulation of G_2 such that ad is an edge of the outer face. Now let H be a triangulation of $H_1 \cup H_2$, with vertex a identified, such that d is adjacent to b and c, and H_2 lies in the triangle abd. Clearly, abd is a separating triangle, separating edges ac and de.

We show that G has an embedding where $\ell(ab) = \ell(ad) = 1$ and $\ell(ac) = \ell(de) = 2$. Consider an embedding of H_1 where a is a vertex of the outer face and abc is a bounded face, and edges ab and ac have prescribed lengths (the length constraints can be met by an affine transformation). Similarly, H_2 has an embedding such that d is on the outer face and ad and de have prescribed lengths. Identifying a in the embeddings of H_1 and H_2, drawn on two sides of a line, gives an embedding of $G = G_1 \cup G_2$ with the desired edge lengths.

(iv) Let G consist of a triangle abc and either a path (d, e, f) or another triangle def. Clearly, G has an embedding with unit length edges. Let H be the triangulation on the vertex set $V = \{a, b, d, c, e, f\}$, containing both triangles abc and def, and the edges $\{ae, be, ce, ad, bd\}$. See Fig. 10. Suppose that H can be embedded such that $G \subset H$ has unit length edges. Assume without loss of generality that edge ab is on the x-axis, and c is above the x-axis. Note that triangle abc cannot contain def, since $\ell(de) = \ell(ef) = 1$ is the diameter of abc. So the two adjacent triangles, abe and bce, are outside of abc, and e has to be in the wedge between $\overrightarrow{ac}$ and $\overrightarrow{bc}$ above vertex c. Vertex d is in the exterior of abc, at distance 1 from e, and so it is also above the x-axis.

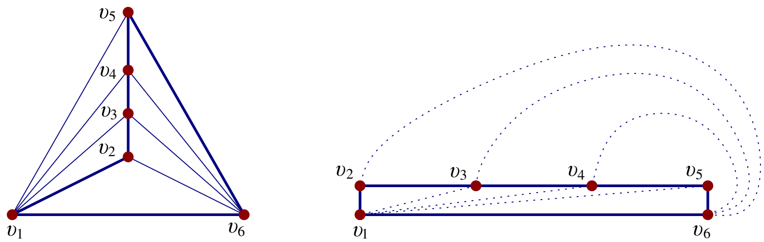

Fig. 11 *Left* A straight-line embedding of H_6. *Right* A straight-line embedding of C_6 with prescribed edge lengths

However, triangle abd requires d to be below the x-axis. We derived a contradiction, which shows that H cannot be embedded with the prescribed edge lengths. □

Finally, we prove that cycles on more than three vertices are not extrinsically free. For an integer $k \geq 4$, we define the graph H_n on the vertex set $\{v_1, v_2, \ldots, v_n\}$ as a union of a Hamilton cycle $C_n = (v_1, v_2, \ldots, v_n)$, and two spanning stars centered at v_1 and v_n, respectively. Note that H_n is planar: the two stars can be embedded in the interior and the exterior of an arbitrary embedding of C_n. Fig. 11 (left) depicts a straight-line embedding of H_6.

We show that C_n is not extrinsically free in the host H_n. Consider the following length assignment on the edges of C_n: let $\ell(v_1v_2) = \ell(v_{n-1}v_n) = \frac{1}{4}$, $\ell(v_i, v_{i+1}) = 1$ for $i = 2, 3, \ldots, n-2$, and $\ell(v_1v_n) = n-3$. Fig. 11 shows a straight-line embedding of C_n with the prescribed edge lengths.

Suppose that H_n admits a straight-line embedding that realizes the prescribed lengths on the edges of the cycle C_n. We may assume, by applying a rigid transformation if necessary, that $v_1 = (0, 0)$, $v_n = (n-3, 0)$, vertex v_2 lies on or above the x-axis, and the vertices $v_1, v_2, \ldots, v_n$ are ordered clockwise around C_n. Denote the coordinates of vertex v_i in this embedding by (x_i, y_i).

Claim 1 *Vertices $v_3, \ldots, v_{n-2}$ lie strictly above the x-axis.*

Proof For $i = 2, \ldots, n-1$, the distance of v_i from v_1 and v_n, respectively, is bounded by

$$|v_1v_i| \leq \sum_{j=1}^{i-1} \ell(v_jv_{j+1}) = i - \frac{7}{4},$$
$$|v_iv_n| \leq \sum_{j=i}^{n-1} \ell(v_jv_{j+1}) = n - i - \frac{3}{4}.$$

That is, v_i lies in the intersection R_i of the disk centered at v_1 of radius $i - \frac{3}{4}$ and a disk centered at v_n of radius $n - i - \frac{3}{4}$. See Fig. 12 (left). The orthogonal projection

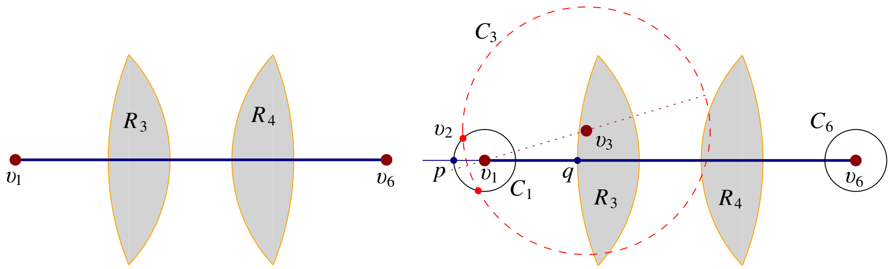

Fig. 12 *Left* The regions R_3 and R_4 for $n = 6$. *Right* Vertices v_2 (resp., v_5) is at distance 1/4 from v_1 (resp., v_n)

of R_i to the x-axis is the interval $[i - \frac{9}{4}, i - \frac{7}{4}]$, which is contained in segment $v_1 v_n$. Hence,

$$x_i \in \Big[i - \frac{9}{4}, i - \frac{7}{4}\Big], \qquad \text{for } i = 2, \ldots, n-1. \tag{3}$$

For $i = 3, \ldots, n-2$, the orthogonal projection of v_i to the x-axis lies on $v_1 v_n$. Recall that v_2 lies on or above the x-axis by assumption. Since v_3 cannot be on the edge $v_1 v_n$ in an embedding, it is strictly above the x-axis.

For $i = 3, \ldots, n-3$, the orthogonal projections of both v_i and v_{i+1} to the x-axis lie on $v_1 v_n$; hence the projection of the segment $v_i v_{i+1}$ is contained in $v_i v_{i+1}$. Since $v_i v_{i+1}$ cannot cross $v_i v_n$, both endpoints are on the same side of the x-axis. Therefore, $v_3, \ldots, v_{n-2}$ all lie strictly above the x-axis. □

Claim 2 *Vertices v_2 and v_{n-1} lie strictly above the x-axis.*

Proof We argue about vertex v_2 (the case of v_{n-1} is analogous). Vertex v_2 is at an intersection point of the circle C_1 of radius $\ell(v_1 v_2) = \frac{1}{4}$ centered at v_1 and the circle C_3 of radius $\ell(v_2 v_3) = 1$ centered at v_3. See Fig. 12 (right). The circles C_1 and C_3 intersect in two points, lying on opposite sides of the symmetry axis $v_1 v_3$ of the two circles. Vertex cannot be at the intersection points in $C_1 \cap C_3$ below the line $v_1 v_3$ because the line segment between that point at v_3 crosses the segment $v_1 v_n$. Hence, v_2 must be the point in $C_1 \cap C_3$ that lies above the $v_1 v_3$.

Suppose now that $v_2 \in C_1 \cap C_3$ is on or below the x-axis. Then the halfcircle of C_1 above the x-axis lies in the closed disk bounded by C_3. In particular, point $p = (-\frac{1}{4}, 0) \in C_1$ must be on or in the interior of C_3, which has radius 1. The only point in the region R_3 within distance 1 from p is $q = (\frac{3}{4}, 0)$. However, q lies on the segment $v_1 v_n$, and so $v_3 \neq q$. Therefore, v_2 lies strictly above the x-axis. □

Claim 3 *The convex hull of C_n is a triangle $\Delta(v_1, v_n, v_i)$, where v_i is a vertex with maximal y-coordinate.*

Proof By Claims 1-2, vertices $v_2, \ldots, v_{n-1}$ are strictly above the x-axis. Let v_i, $1 < i < n$, be a vertex with maximal y-coordinate. Suppose vertex v_j, for some $j \neq i$, lies outside of the triangle $\Delta(v_1, v_n, v_i)$. Refer to Fig. 13 (left). Without loss of generality,

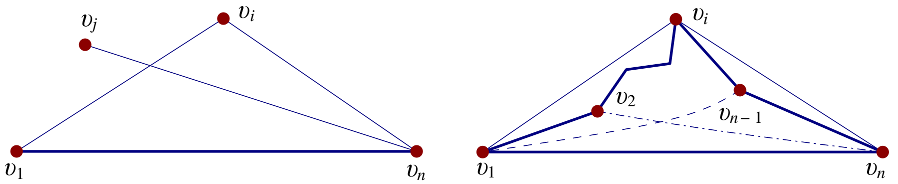

Fig. 13 *Left* If v_j lies to the left of $\Delta(v_1, v_n, v_i)$, then v_1v_i crosses v_jv_n. *Right* If $2 < i < n-1$, then v_1v_{n-1} and v_2v_n are both internal diagonals of C_n

assume that v_j lies to the left of the vertical line through v_i. Then edge v_1v_i crosses v_jv_n, contrary to the assumption that we have a plane embedding of H_n. $\square$

Claim 4 *The convex hull of C_n is either $\Delta(v_1, v_n, v_2)$ or $\Delta(v_1, v_n, v_{n-1})$.*

Proof Suppose the v_i is a vertex with maximal y-coordinate for some $2 < i < n-1$. Refer to Fig. 13 (right). Then both v_1v_{n-1} and v_2v_n are internal diagonals of the cycle C_n; hence they cross, contradicting our assumption that we have a plane embedding of H_n. $\square$

By symmetry, we may assume that the convex hull of C_n is $\Delta(v_1, v_n, v_{n-1})$. We say that a polygonal chain $(p_1, p_2, \ldots, p_k)$ is *monotone* in the direction of a nonzero vector $\mathbf{u}$ if the inner products $\langle \overrightarrow{p_i p_{i+1}}, \mathbf{u}\rangle$ are positive for $i = 1, \ldots, k-1$.

Claim 5 *The polygonal chain $(v_1, v_2, \ldots, v_{n-1})$ is monotone in both directions $\overrightarrow{v_1v_n}$ and $\overrightarrow{v_1v_{n-1}}$.*

Proof For $i = 2, 3, \ldots, n-1$, we have $x_i \in [i - \frac{9}{4}, i - \frac{7}{4}]$ from (3). Combined with the assumption that the convex hull of C_n is $\Delta(v_1, v_n, v_{n-1})$, this already implies that $(v_1, v_2, \ldots, v_{n-1})$ is x-monotone. Note that $y_{n-1} \in (0, \frac{1}{4}]$ since $|v_{n-1}v_n| = \ell(v_{n-1}v_n) = \frac{1}{4}$ and $y_i \in (0, \frac{1}{4}]$ since v_{n-1} has maximal y-coordinate. Using (3), the slope of segment v_iv_{i+1}, $i = 2, \ldots, n-2$, is bounded as

$$\left|\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right| \leq \frac{\max_i y_i}{1/2} \leq \frac{1/4}{1/2} = \frac{1}{2}. \tag{4}$$

Similarly, the slope of v_1v_{n-1} is bounded by $|(y_{n-1} - y_1)/(x_{n-1} - x_1)| = y_{n-1}/x_{n-1} \leq (1/4)/(n - 11/4) \leq 1/5$. Finally, the slope of segment v_1v_2 is bounded by that of v_1v_{n-1}, since v_2 lies in the triangle $\Delta(v_1, v_n, v_{n-1})$. Hence the inner products $\langle \overrightarrow{v_iv_{i+1}}, \overrightarrow{v_1v_{n-1}}\rangle$ are positive for $i = 1, \ldots, n-2$. $\square$

Let $\gamma = (v_1, v_2, \ldots, v_{n-1})$ be the polygonal chain from v_1 to v_{n-1} along the cycle C_n. Refer to Fig. 14. Note that the slopes of the segments v_1v_i, $i = 1, \ldots, n-1$, are monotonically increasing, since these edges connect v_1 to all other vertices of γ. However, γ is not necessarily a convex chain. Rearranging the edge vectors of γ in monotonically increasing order by slope, we obtain a *convex* polygonal chain $\gamma' = (u_1, u_2, \ldots, u_{n-1})$ between v_1 and v_{n-1}.

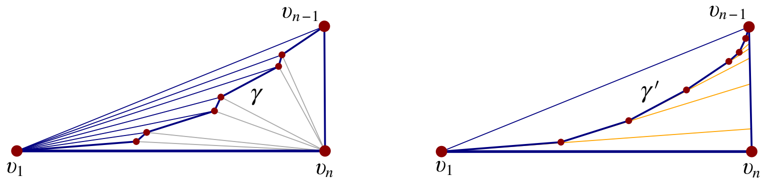

Fig. 14 *Left* The chain $(v_1, v_2, \ldots, v_{n-1})$ is monotone in both directions v_1v_n and v_1v_{n-1}, but it is not necessarily convex. *Right* The edge vectors of $(v_1, v_2, \ldots, v_{n-1})$ can be rearranged into a convex chain within $\Delta(v_1, v_n, v_{n-1})$

Claim 6 *The polygonal chain* $\gamma' = (u_1, u_2, \ldots, u_{n-1})$ *defined above lies in the triangle* $\Delta(v_1, v_n, v_{n-1})$.

Proof It is clear that γ' is monotone in both directions $\overrightarrow{v_1v_n}$ and $\overrightarrow{v_1v_{n-1}}$, since it consists of the same edge vectors as γ. Hence γ' crosses neither v_1v_n nor v_1v_{n-1}. To confirm that γ' lies in the triangle $\Delta(v_1, v_n, v_{n-1})$, we need to show that γ' does not cross $v_{n-1}v_n$.

The slope of every edge of γ is positive by Claim 5, and bounded above by $2\max_i y_i = 2y_{n-1}$ due to (4). We distinguish two cases. If $x_{n-1} \le x_n$, then v_nv_{n-1} and γ' lie in two closed halfplanes on opposite sides of the vertical line through v_{n-1}. If $x_n < x_{n-1}$, then the slope of $\overrightarrow{v_nv_{n-1}}$ is $y_{n-1}/(x_n - x_{n-1}) > y_{n-1}/\ell(v_{n-1}, v_n) = 4y_{n-1}$, that is, larger than the slope of any edge of γ'. In both cases, γ' cannot cross the segment v_nv_{n-1}. Therefore, γ' also lies in the triangle $\Delta(v_1, v_n, v_{n-1})$. □

Now γ' is a convex polygonal chain from v_1 to v_{n-1} within the triangle $\Delta(v_1, v_n, v_{n-1})$. All edges of γ' have strictly positive slopes, so γ' is disjoint from v_1v_n. By (a repeated application of) the triangle inequality, γ' is strictly shorter than the polygonal chain (v_1, v_n, v_{n-1}). However, by construction, these two chains have the same length (namely, $n - \frac{11}{4}$). We conclude that our initial assumption is false, and H_n has no straight-line embedding in which every edge $e \in E$ has length $\ell(e)$.

7 Embedding a Cycle with Nondegenerate Lengths

We say that a length assignment $\ell : E \to \mathbb{R}^+$ for a cycle $C = (V, E)$ is *feasible* if C admits a straight-line embedding with edge length $\ell(e)$ for all $e \in E$. Lenhart and Whitesides [15] showed that ℓ is feasible for C if and only if no edge is supposed to be longer than the semiperimeter $s = \frac{1}{2}\sum_{e\in E}\ell(e)$. Recall that three positive reals, a, b, and c, satisfy the triangle inequality if and only if each of them is less than $\frac{1}{2}(a+b+c)$.

By Lemma 9, it is enough to prove Theorem 4 in the case when C is a Hamilton cycle in H. Consider a Hamilton cycle C in a triangulation H. We construct a straight-line embedding of H with given nondegenerate edge lengths using the following two-step strategy: We first embed C on the boundary of a triangle T such that each edge of $H - C$ is either an internal diagonal of C or a line segment along one of the sides of the triangle T (Lemma 15). If any edge of $H - C$ overlaps with edges C, then this is not a proper embedding of H yet (Fig. 15). In a second step, we perturb the embedding of C to accommodate all edges of H (see Sect. 7.1).

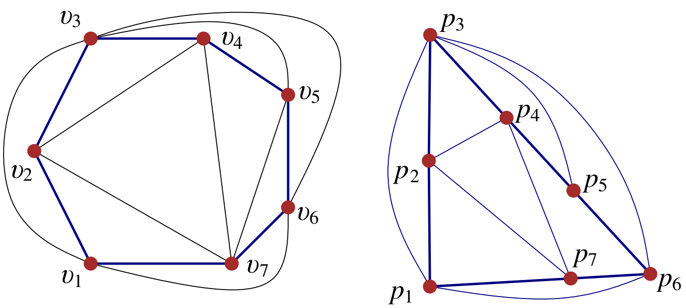

Fig. 15 *Left* A planar graph H with a Hamilton cycle C (think lines). *Right* The graph H has a 3-cycle (1,3,6) such that C admits a straight-line embedding with the same edge lengths as in the left and all edges of C are along the edges of triangle (1,3,6)

Lemma 15 *Let H be a triangulation with a Hamilton cycle $C = (V, E)$ and a feasible nondegenerate length assignment $\ell : E \to \mathbb{R}^+$. Then, there is a 3-cycle (v_i, v_j, v_k) in H such that the prescribed arc lengths of C between these vertices, i.e., the three sums of lengths of edges corresponding to these three arcs, satisfy the triangle inequality.*

Proof Consider an arbitrary embedding of H (with arbitrary lengths). The edges of H are partitioned into three subsets: edges E of the cycle C, interior chords E_{int}, and exterior chords E_{ext}. Each chord $v_i v_j \in E_{\text{int}} \cup E_{\text{ext}}$ decomposes C into two paths. If the length assignment ℓ is nondegenerate, then there is at most one chord $v_i v_j \in E_{\text{int}} \cup E_{\text{ext}}$ that decomposes C into two paths of equal length. Assume, by exchanging interior and exterior chords if necessary, that no edge in E_{ext} decomposes C into two paths of equal length.

Denote by $\delta_{ij} > 0$ the absolute value of the difference between the sums of the prescribed lengths on the two paths that an exterior chord $v_i v_j$ produces. Let $v_i v_j \in E_{\text{ext}}$ be an exterior chord that minimizes δ_{ij}. The chord $v_i v_j$ is adjacent to two triangles, say $v_i v_j v_k$ and $v_i v_j v_{k'}$, where v_k and $v_{k'}$ are vertices of two different paths determined by $v_i v_j$. Assume, without loss of generality, that v_k is part of the longer path (measured by the prescribed length). The path length between v_i and v_k (resp., v_j and v_k) cannot be less than δ_{ij}—otherwise $\delta_{kj} < \delta_{ij}$ (resp., $\delta_{ik} < \delta_{ij}$). Therefore, the three arcs between v_i, v_j, and v_k satisfy the triangle inequality. □

7.1 A Hamilton Path with Given Edge Lengths

Our main tool to "perturb" a straight-line drawing with collinear edges is the following lemma.

Lemma 16 *Let H be a planar graph with $n \geq 3$ vertices and a fixed combinatorial embedding; let $P = (V, E)$ be a Hamilton path in H with both of its endpoints incident to the outer face of H; and let $\ell : E \to \mathbb{R}^+$ be a length assignment with $L = \sum_{e \in E} \ell(e)$ and $\ell_{\min} = \min_{e \in E} \ell(e)$.*

For every sufficiently small $\varepsilon > 0$, H admits a straight-line embedding such that the two endpoints of P are at points origin $(0, 0)$ and $(0, L - \varepsilon)$ on the x-axis, and every edge $e \in E$ has length $\ell(e)$.

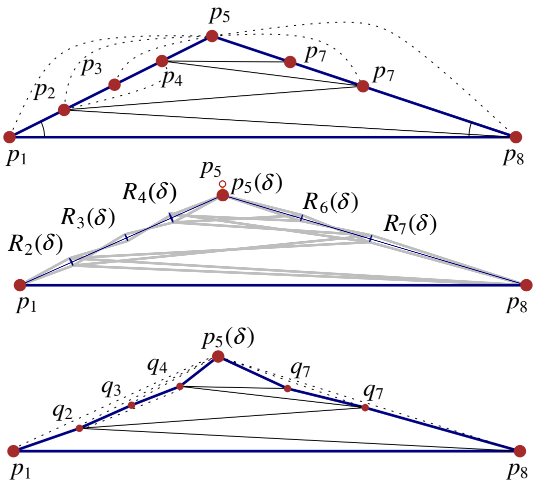

Fig. 16 *Top* A path $P = (p_1, \dots p_8)$ embedded on the boundary of a triangle (p_1, p_5, p_8) with prescribed edge lengths. The edges of $H - P$ between different sides of the triangle are represented in solid thin lines, the edges of $H - P$ between vertices of the same side of the triangle are represented in dotted lines. *Middle* When point p_5 is shifted down to $p_5(\delta)$, in any embedding of C with prescribed edge lengths, vertex p_i is located in a region $R_i(\delta)$ for $i = 2, 3, 4, 6, 7$. *Bottom* A straight-line embedding of H is obtained by embedding the subgraphs induced by $P_1 = (p_1, \dots, p_5)$ and $P_2 = (p_5, \dots, p_8)$ by induction

Proof We proceed by induction on $n = |V|$, the number of vertices of H. The base case is $n = 3$, where P consists of two edges, H is a triangle, and we can place the two endpoints of P at $(0, 0)$ and $(L - \varepsilon, 0)$. Assume now that $n > 3$ and the claim holds for all instances where H has fewer than n vertices.

We may assume, by adding dummy edges if necessary, that H is a triangulation. Denote the vertices of the path P by $(v_1, v_2, \dots, v_n)$. By assumption, the endpoints v_1 and v_n are incident to the outer face (i.e., outer triangle). Denote by v_k, $1 < k < n$, the third vertex of the outer triangle. Let $P_1 = (v_1, \dots, v_k)$ and $P_2 = (v_k, \dots, v_n)$ be two subpaths of P, with total lengths $L_1 = \sum_{i=1}^{k-1} \ell(v_i v_{i+1})$ and $L_2 = \sum_{i=k}^{n-1} \ell(v_i v_{i+1})$. We may assume without loss of generality that $L_1 \le L_2$. We may assume, by applying a reflection if necessary, that the triple (v_1, v_k, v_n) is clockwise in the given embedding of H. Let H_1 (resp., H_2) be the subgraph of H induced by the vertices of P_1 (resp., P_2), and let $E_{1,2}$ denote the set of edges of H between $\{v_1, \dots, v_{k-1}\}$ and $\{v_{k+1}, \dots, v_n\}$. In the remainder of the proof, we embed P_1 and P_2 by induction, after choosing appropriate parameters ε_1 and ε_2.

We first choose "preliminary" points p_i for each vertex v_i as follows: Let (p_1, p_k, p_n) be a triangle with clockwise orientation, where $p_1 = (0, 0)$, $p_n = (L - \varepsilon, 0)$, and the edges $p_1 p_k$ and $p_k p_n$ have lengths L_1 and L_2, respectively (see Fig. 16). Place the points $p_2, \dots p_{k-1}$ on segment $p_1 p_k$, and the points $p_{k+1}, \dots, p_{n-1}$ on segment $p_k p_n$ such that the distance between consecutive points is $|p_i p_{i+1}| = \ell(v_i v_{i+1})$ for $i = 1, \dots, n - 1$.

Note that segment $p_1 p_k$ has a positive slope, say $\overline{s}$; and $p_k p_n$ has a negative slope, $\underline{s}$. The slope of every segment $p_i p_j$, for $v_i v_j \in E_{1,2}$, is in the open interval $(\underline{s}, \overline{s})$. Let

$[\underline{r}, \overline{r}]$ be the smallest closed interval that contains the slopes of all segments $p_i p_j$ for $v_i v_j \in E_{1,2}$. Let $\overline{t} \in (\overline{r}, \overline{s})$ and $\underline{t} \in (\underline{s}, \underline{r})$ be two arbitrary reals that "separate" the sets of slopes. We shall perturb the vertices $p_2, \ldots, p_{n-1}$ such that the directions of the edges of H_2, $E_{1,2}$, and H_1 remain in pairwise disjoint intervals $(2\underline{s}, \underline{t})$, $(\underline{t}, \overline{t})$, and $(\overline{t}, 2\overline{s})$, respectively.

Suppose that we move point p_k to position $p_k(\delta) = p_k + (0, -\delta)$. In any straight-line embedding of P_1 with $v_1 = p_1$ and $v_k = p_k(\delta)$, each vertex v_i, $i = 2, \ldots, k-1$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at p_1 and $p_k(\delta)$ of radii $|p_1 p_i|$ and $|p_i p_k|$, respectively (Fig. 16). Similarly, in any straight-line embedding of P_2 with $v_k = p_k(\delta)$ and $v_n = p_n$, each vertex v_i, $i = k+1, \ldots, n-1$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at $p_k(\delta)$ and p_n of radii $|p_k p_i|$ and $|p_i p_n|$, respectively. We also define one-point regions $R_1(\delta) = \{p_1\}$, $R_k(\delta) = \{p_k(\delta)\}$, and $R_n(\delta) = \{p_n\}$. Choose a sufficiently small $\delta > 0$ such that the slope of any line intersecting $R_i(\delta)$ and $R_j(\delta)$ is in the interval

- $(\overline{t}, 2\overline{s})$ if $1 \le i < j \le k$;
- $(\underline{t}, \overline{t})$ if $v_i v_j \in E_{1,2}$;
- $(2\underline{s}, \underline{t})$ if $k \le i < j \le n$.

Embed vertices v_1, v_k, and v_n at points p_1, $p_k(\delta)$, and p_n, respectively. If H_1 (resp., H_2) has three or more vertices, embed it by induction such that the endpoints of path P_1 are p_1 and $p_k(\delta)$ (resp., the endpoints of P_2 are $p_k(\delta)$ and p_n). Each vertex v_i is embedded in a point in the region R_i for $i = 1, \ldots n$. By the choice of δ, the slopes of the edges of H_1 and H_2 are in the intervals $(\overline{t}, 2\overline{s})$ and $(2\underline{s}, \underline{t})$, respectively, while the slopes of the edges in $E_{1,2}$ are in a disjoint interval $(\underline{t}, \overline{t})$. Therefore, these edges are pairwise noncrossing, and we obtain a proper embedding of graph H. □

7.2 Proof of Theorem 4

By Lemma 9, it is enough to prove Theorem 4 in the case when C is a Hamilton cycle in H.

Theorem 17 *Let H be a planar graph that contains a cycle $C = (V, E)$. Let $\ell : E \to \mathbb{R}^+$ be a feasible nondegenerate length assignment. Then H admits a straight-line embedding in which each $e \in E$ has length $\ell(e)$.*

Proof We may assume that H is an edge-maximal planar graph, that is, H is a triangulation. By Lemma 15, H contains a 3-cycle (v_a, v_b, v_c) such that the prescribed arc lengths of C between these vertices, i.e., the three sums of lengths of edges corresponding to these three arcs, satisfy the triangle inequality (see Fig. 17).

Let P_1, P_2, and P_3 denote the paths along C between the vertex pairs (v_a, v_b), (v_b, v_c), and (v_c, v_a), and let their prescribed edge lengths be L_1, L_2, and L_3, respectively. For $j = 1, 2, 3$, let H_j be the subgraphs of H induced by the vertices of the path P_j. Denote by $E_{1,2,3}$ the set of edges of H between an interior vertex of P_1, P_2, or P_3, and a vertex not on the same path. Consider a combinatorial embedding of H (with arbitrary edge lengths) such that (v_a, v_b, v_c) is triangle in the *exterior* of C. In this embedding, all edges in $E_{1,2,3}$ are *interior* chords of C.

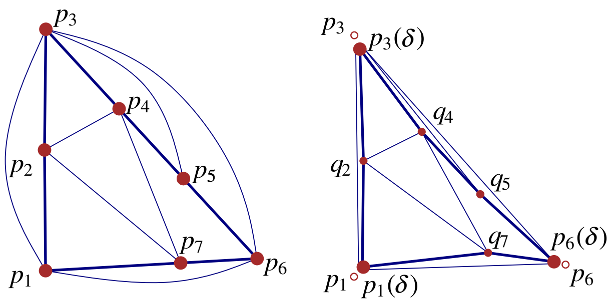

Fig. 17 *Left* A cycle $C = (p_1, \ldots p_8)$ embedded on the boundary of a triangle (p_1, p_3, p_6) with prescribed edge lengths. *Right* When the vertices of the triangle are translated by δ towards the center of the triangle, we can embed the subgraphs induced by (p_1, p_2, p_3), (p_3, p_4, p_5, p_6), and (p_6, p_7, p_1) by straight-line edges so that they do not cross any of the diagonals between different sides of the triangle

Similarly to the proof of Lemma 16, we start with a "preliminary" embedding, where the vertices v_i are embedded as follows: Let (p_a, p_b, p_c) be a triangle with edge lengths $|p_a p_b| = L_1$, $|p_b p_c| = L_2$, and $|p_c p_a| = L_3$. Place all other points p_i on the boundary of the triangle such that the distance between consecutive points is $|p_i p_{i+1}| = \ell(v_i v_{i+1})$ for $i = 1, \ldots, n-1$. Suppose, without loss of generality, that no two points have the same x-coordinate. Note that the slope of every line segment $p_i p_j$, for $v_i v_j \in E_{1,2,3}$ is *different* from the slopes of the sides of the triangle that contains p_i and p_j. Let η be the minimum difference between the slopes of two segments $p_i p_j$, with $v_i v_j \in E_{1,2,3}$.

Move points p_a, p_b, and p_c toward the center of triangle (p_a, p_b, p_c) by a vector of length $\delta > 0$ to positions $p_a(\delta)$, $p_b(\delta)$, and $p_c(\delta)$. In any straight-line embedding of C with $v_a = p_a(\delta)$, $v_b = p_b(\delta)$ and $v_c = p_c(\delta)$, each vertex v_i, $i = 2, \ldots, n$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at two vertices of the triangle $(p_a(\delta), p_b(\delta), p_c(\delta))$. Choose a sufficiently small $\delta > 0$ such that the slopes of a line intersecting $R_i(\delta)$ and $R_j(\delta)$ with $v_i v_j \in H$ is within $\eta/2$ from the slope of the segment $p_i p_j$.

Embed vertices v_i, v_j, and v_k at points $p_i(\delta)$, $p_j(\delta)$, and $p_k(\delta)$, respectively. If H_1 (resp., H_2 and H_3) has three or more vertices, embed it using Lemma 16 such that the endpoints of the path P_1 are $p_i(\delta)$ and $p_k(\delta)$ (resp., $p_j(\delta)$, $p_k(\delta)$ and $p_k(\delta)$, $p_i(\delta)$). Each vertex v_i is embedded in a point in the region R_i for $i = 1, \ldots n$. By the choice of δ, the slopes of the edges of H_1, H_2, and H_3 are in three small pairwise disjoint intervals, and these intervals are disjoint from the slopes of any edge $v_i v_j \in E_{1,2,3}$. Therefore, the edges of H are pairwise noncrossing, and we obtain a proper embedding of H. □

8 Conclusions

We have characterized the planar graphs G that are free subgraphs in every host H, $G \subseteq H$. In Sect. 3, we showed that every triangulation T has a straight-line embedding in which a matching $M \subset T$ has arbitrarily prescribed edge lengths, and the outer face is fixed. Several related questions remain unanswered:

1. Given a length assignment $\ell\colon M \to [1, \lambda]$ for a matching M in an n-vertex planar graph G, what is the minimum Euclidean *diameter* (resp., *area*) of an embedding of G with prescribed edge lengths?
2. Is there a polynomial time algorithm for deciding whether a subgraph G of a planar graph H is free or extrinsically free in H?
3. Is there a polynomial time algorithm for deciding whether a planar graph H is realizable such that the edges of a cycle $C = (V, E)$ have given (possibly degenerate) lengths?
4. What are the planar graphs G that are free in every 4-*connected* triangulation H, $G \subseteq H$? We know that stars are, but we do not have a complete characterization.

Recently, Angelini et al. [1] proved that given any two homeomorphic embeddings of a planar graph, one can continuously morph one embedding into the other in $O(n)$ successive linear morphs (in which each vertex moves with constant speed). Combined with our Theorem 1, this implies that if we are given *two* length assignments $\ell_1\colon M \to \mathbb{R}^+$ and $\ell_2\colon M \to \mathbb{R}^+$ for a matching M in an n-vertex triangulation T, then one can continuously morph an embedding with one length assignment into another embedding with the other assignment in $O(n)$ linear morphs. It remains an open problem whether fewer linear morphs suffice between two embeddings that admit two different length assignments of M.

Acknowledgments The work on this problem started at the 10th *Gremo Workshop on Open Problems* (Bergün, GR, Switzerland) and continued at the MIT-Tufts Research Group on Computational Geometry. We thank all participants of these meetings for stimulating discussions.

References

1. Angelini, P., Da Lozzo, G., Di Battista, G., Frati, F., Patrignani, M., Roselli, V.: Morphing planar graph drawings optimally. In: Proceedings of the 41st International Colloquium on Automata, Languages, and Programming (ICALP), Part I. Lecture Notes in Computer Science 8572, pp. 126–137. Springer, New York (2014)
2. Angelini, P., Di Battista, G., Frati, F., Jelínek, V., Kratochvíl, J., Patrignani, M., Rutter, I.: Testing planarity of partially embedded graphs. ACM Transactions on Algorithms **11**(4), Art No. 32 (2015)
3. Cabello, S., Demaine, E.D., Rote, G.: Planar embeddings of graphs with specified edge lengths. J. Graph Algorithms Appl. **11**(1), 259–276 (2007)
4. Connelly, R.: Generic global rigidity. Discrete Comput. Geom. **33**(4), 549–563 (2005)
5. Connelly, R., Demaine, E.D., Rote, G.: Straightening polygonal arcs and convexifying polygonal cycles. Discrete Comput. Geom. **30**(2), 205–239 (2003)
6. de Fraysseix, H., Pach, J., Pollack, R.: How to draw a planar graph on a grid. Combinatorica **10**(1), 41–51 (1990)
7. Di Battista, G., Vismara, L.: Angles of planar triangular graphs. SIAM J. Discrete Math. **9**(3), 349–359 (1996)
8. Eades, P., Wormald, N.C.: Fixed edge-length graph drawing is NP-hard. Discrete Appl. Math. **28**, 111–134 (1990)
9. Fáry, I.: On straight line representation of plane graphs. Acta. Sci. Math. Szeged **11**, 229–233 (1948)
10. Frati, F., Patrignani, M.: A note on minimum-area straight-line drawings of planar graphs. In: 15th Graph Drawing. Lecture Notes in Computer Science 4875, pp. 339–344. Springer, New York (2008)
11. Hong, S.-H., Nagamochi, H.: Convex drawings of graphs with non-convex boundary constraints. Discrete Appl. Math. **156**, 2368–2380 (2008)
12. Jackson, B., Jordán, T.: Connected rigidity matroids and unique realizations of graphs. J. Comb. Theory Ser. B **94**(1), 1–29 (2005)

13. Jelínek, V., Kratochvíl, J., Rutter, I.: A Kuratowski-type theorem for planarity of partially embedded graphs. Comput. Geom. Theory Appl. **46**(4), 466–492 (2013)
14. Kurowski, M.: Planar straight-line drawing in an $O(n) \times O(n)$ grid with angular resolution $\Omega(1/n)$. In: Proceedings of the 31st Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM). Lecture Notes in Computer Science 3381, pp. 250–258. Springer, New York (2005)
15. Lenhart, W.J., Whitesides, S.H.: Reconfiguring closed polygonal chains in euclidean d-space. Discrete Comput. Geom. **13**, 123–140 (1995)
16. Patrignani, M.: On extending a partial straight-line drawing. Found. Comput. Sci. **17**(5), 1061–1069 (2006)
17. Sauer, N.W.: Distance sets of Urysohn metric spaces. Can. J. Math. **65**, 222–240 (2013)
18. Sauer, N.W.: Edge labelled graphs and metric spaces. In: Abstracts of the Erdős Centennial, p. 78. Budapest (2013)
19. Schnyder, W.: Embedding planar graphs in the grid. In: Proceedings of the 1st ACM-SIAM Symposium on Discrete Algorithms, pp. 138–147. ACM, San Francisco (1990)
20. Tutte, W.T.: How to draw a graph. Proc. Lond. Math. Soc. **3–13**(1), 743–767 (1963)