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Universal Rigidity of Complete Bipartite Graphs

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Abstract We describe a very simple condition that is necessary for the universal rigidity of a complete bipartite framework $(K(n, m), \mathbf{p}, \mathbf{q})$. This condition is also sufficient for universal rigidity under a variety of weak assumptions, such as general position. Even without any of these assumptions, in complete generality, we extend these ideas to obtain an efficient algorithm, based on a sequence of linear programs, that determines whether an input framework of a complete bipartite graph is universally rigid or not.

Keywords Rigidity · Prestress stability · Universal rigidity

1 Introduction and Definitions

1.1 Main Results

A bar and joint framework, denoted as $(G, \mathbf{p})$, is a graph G together with a configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ of points in $\mathbb{R}^d$. A bar and joint framework is *universally rigid* if it is rigid in any Euclidean space that contains it. This is equivalent to the property that the framework must be congruent to any other configuration of the vertices of the

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underlying graph, in any dimension, whenever the corresponding edge lengths are the same.

In this paper, we provide a complete characterization of which realizations of a complete bipartite graph, $G = K(n, m)$, are universally rigid and which realizations are not. As a necessary condition, we show (Theorem 2.2) that, except for $K(1, 0)$ (a single vertex) and $K(1, 1)$, if the partitions can be strictly separated by a quadric surface, then the framework is not universally rigid. Conversely, as a sufficient condition, we show (Corollary 4.6) that, if the vertices of the configuration are in general (affine) position in $\mathbb{R}^d$ and there is no quadric surface strictly separating the partitions, then the framework is universally rigid. Alternatively (Corollary 4.8) if there are at least $(d+1)(d+2)/2+1$ vertices and no $(d+1)(d+2)/2$ of them lie in a quadric surface and if the partitions cannot be strictly separated by a quadric surface, then the framework is universally rigid.

Even without any of these general position assumptions, in complete generality, we extend these ideas to obtain an efficient algorithm, based on a sequence of linear programs, that determines whether an input framework of a complete bipartite graph is universally rigid or not.

Surprisingly, our results are closely related to some older statements about extremal correlation matrices due to Tsirelson, which arose in his study of quantum Bell-type inequalities. In particular, our Theorem 2.2 is related to Tsirelson's [19, Thm. 2.21(c)], while our Corollary 4.8 is related to Tsirelson's in [19, Thm. 2.22]. Proofs for these statements were not provided in [19]. This connection was pointed out in recent work [13] which also showed that one can indeed prove Tsirelson's statements using our results and techniques (which were posted in an earlier draft of the current paper).

A closely related concept to universal rigidity is *global rigidity* in $\mathbb{R}^d$, which is similar except that the other configurations $\mathbf{q}$, where corresponding edge lengths are the same, are restricted to be in $\mathbb{R}^d$. Clearly, if $(G, \mathbf{p})$ is universally rigid, then it is automatically globally rigid in $\mathbb{R}^d$. But in most results, for a framework to be globally rigid, it is assumed that the configuration $\mathbf{p}$ is *generic*, which means that there is no non-zero integral polynomial relation among the coordinates of $\mathbf{p}$, and it may be very hard to verify that some specific framework is acting generically. So, when possible, the stronger condition of universal rigidity can be a useful condition that is sufficient to show that particular configuration is globally rigid in $\mathbb{R}^d$.

1.2 Definitions

The basic tool we use in this paper is a stress $\omega = (\ldots, \omega_{ij}, \ldots)$, which is an assignment of a real scalar $\omega_{ij} = \omega_{ji}$ to each edge, $\{i, j\} \in E(G)$. We assume $\omega_{ij} = 0$, when $\{i, j\} \notin E(G)$. We say that a stress ω is an *equilibrium* stress for $(G, \mathbf{p})$ if the vector equation

$$\sum_i \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{1.1}$$

holds for all vertices j of G. We associate an N-by-N *stress matrix* Ω to a stress ω, for N, the total number of vertices, by saying that i, j entry of Ω is $-\omega_{ij}$, for $i \neq j$, and the diagonal entries of Ω are such that the row and column sums of Ω are zero. If the dimension of the affine span of the vertices $\mathbf{p}$ is d, then the rank of an equilibrium stress matrix Ω is at most $N - d - 1$, but it could be less.

We say that $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, a finite collection of non-zero vectors in $\mathbb{R}^d$, lie on a *conic at infinity* of $\mathbb{R}^d$ if, when regarded as points in real projective $(d-1)$ space $\mathbb{RP}^{d-1}$, they lie on a conic. This means that there is a non-zero d-by-d symmetric matrix Q such that for all $i = 1, \ldots, m$, $\mathbf{v}_i^t Q \mathbf{v}_i = 0$, where $()^t$ is the transpose operation.

1.3 Basic Results

Definition 1.1 A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is said to be *universally rigid* if any other corresponding framework with the same edge lengths in $\mathbb{R}^D$, for any D, is congruent to $(G, \mathbf{p})$.

The following fundamental theorem [6] is a basic tool used to establish universal rigidity.

Theorem 1.2 *Let $(G, \mathbf{p})$ be a framework whose affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$, with an equilibrium stress ω and stress matrix Ω. Suppose further*

(i) *Ω is positive semi-definite (PSD).*
(ii) *The rank of Ω is $N - d - 1$.*
(iii) *The edge directions of $(G, \mathbf{p})$ do not lie on a conic at infinity of $\mathbb{R}^d$.*

Then $(G, \mathbf{p})$ is universally rigid.

There are several examples of the universally rigid frameworks in [8], where conditions (i) and (ii) of Theorem 1.2 do not hold, and yet they are still universally rigid.

Definition 1.3 When conditions (i), (ii), (iii) hold for a framework $(G, \mathbf{p})$ with affine span $\mathbb{R}^d$, we say it is *super stable*.

Remark 1.4 If a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ happens to have an affine span of some smaller dimension d', then the framework can be rigidly placed in $\mathbb{R}^{d'}$ and Theorem 1.2 can be applied if appropriate. In this case, we also say that $(G, \mathbf{p})$ is super stable.

When the sign of the stress $\omega_{i,j}$ is positive (respectively negative), then the constraint on the lengths of the edges of the possible alternative configurations $\mathbf{q}$ can be weakened to be not longer (respectively not shorter), and the conclusion of Theorem 1.2 still holds. Those edges with a positive stress are called *cables* and those with a negative stress are called *struts*. When possible, in the following, we will designate cables with dashed line segments and struts with heavy solid line segments. The default is that the edge lengths are constrained to stay the same length.

Definition 1.5 A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ with a d'-dimensional span is said to be *dimensionally rigid* if any other corresponding framework with the same edge lengths in $\mathbb{R}^D$, for any D, has affine span at most d'.

For completeness, we note the following from [2]:

Theorem 1.6 *If* $(G, \mathbf{p})$ *is a dimensionally rigid framework in* $\mathbb{R}^d$ *with* N *vertices whose affine span is* d*-dimensional and condition* (iii) *of Theorem* 1.2 *holds, then* $(G, \mathbf{p})$ *is universally rigid.*

The following result in [1] will be important in this paper. (See also [8] for another point of view for the proof.)

Theorem 1.7 *If* $(G, \mathbf{p})$ *is a dimensionally rigid or universally rigid framework in* $\mathbb{R}^d$ *with* N *vertices whose affine span is* d'*-dimensional,* $d' \leq N-2$*, then it has a non-zero equilibrium stress with a positive semi-definite* (*PSD*) *stress matrix* Ω (*with rank* ≥ 1).

So, in particular, if $(G, \mathbf{p})$ has no non-zero PSD equilibrium stress matrix, and the dimension of the affine span of $\mathbf{p}$ is $\leq N-2$, then it is not universally rigid.

2 Bipartite Frameworks and Quadrics

Let $K(n, m)$ be the complete bipartite graph on n and m vertices, and let $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_n})$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$ be two configurations of points in $\mathbb{R}^d$. Then we denote by $(K(n, m), \mathbf{p}, \mathbf{q})$ the associated complete bipartite framework.

Recall that a quadric surface in $\mathbb{R}^d$, is the solution to a non-zero quadratic function in the coordinates of $\mathbb{R}^d$. For the line $\mathbb{R}$, a quadric surface is two points. For the plane $\mathbb{R}^2$, a quadric surface is a conic, which includes the possibility of two straight lines as well as ellipses and a hyperbola. (We do not need to consider quadrics that consist of just one hyperplane in $\mathbb{R}^d$.) By adjoining the projective space $\mathbb{RP}^{d-1}$, we can complete $\mathbb{R}^d$ to real projective space $\mathbb{RP}^d$, and a quadric will separate $\mathbb{RP}^d$ into two components. For any vector $\mathbf{x} \in \mathbb{R}^d$, define $\hat{\mathbf{x}} \in \mathbb{R}^{d+1}$ by adding a 1 as the last coordinate. A quadric can be written in the form $\{\mathbf{x} \in \mathbb{R}^d \mid \hat{\mathbf{x}}^t A \hat{\mathbf{x}} = 0\}$, where A is a $(d+1)$-by-$(d+1)$ symmetric matrix, $\hat{\mathbf{x}}$ is a column vector, and $\hat{\mathbf{x}}^t$ is its transpose. So the two components determined by the matrix A are given by $\hat{\mathbf{x}}^t A \hat{\mathbf{x}} < 0$ and $0 < \hat{\mathbf{x}}^t A \hat{\mathbf{x}}$.

Definition 2.1 If $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_n})$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m)$ are two configurations of points in $\mathbb{R}^d$, we say that they are *strictly separated by a quadric*, given by a matrix A, if for each $i = 1, \dots, n$ and $j = 1, \dots, m$,

$$\hat{\mathbf{q}}_j^t A \hat{\mathbf{q}}_j < 0 < \hat{\mathbf{p}}_i^t A \hat{\mathbf{p}}_i. \tag{2.1}$$

A stress matrix for a complete bipartite framework $(K(n, m), \mathbf{p}, \mathbf{q})$, has the following form, where $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_m$ are the diagonal entries, whereas all the non-diagonal entries in the upper left and lower right blocks are zero

$$\Omega = \begin{pmatrix} \lambda_1 & 0 & 0 & -\omega_{11} & \cdots & -\omega_{1m} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \lambda_n & -\omega_{n1} & \cdots & -\omega_{nm} \\ -\omega_{11} & \cdots & -\omega_{n1} & \mu_1 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ -\omega_{1m} & \dots & -\omega_{nm} & 0 & 0 & \mu_m \end{pmatrix}. \tag{2.2}$$

So the diagonal entries are such that $\sum_{j=1}^{m} \omega_{ij} = \lambda_i$, and $\sum_{i=1}^{n} \omega_{ij} = \mu_j$, from the definition of Ω.

Our first main result is the following necessary condition for the universal rigidity of a complete bipartite framework.

Theorem 2.2 *If $(K(n, m), (\mathbf{p}, \mathbf{q}))$ is a complete bipartite framework in $\mathbb{R}^d$, with an affine span of dimension $\leq m+n-2$, such that the partition vertices $(\mathbf{p}, \mathbf{q})$ are strictly separated by a quadric, then it is not universally rigid.*

Proof Let A be the $(d+1)$-by-$(d+1)$ symmetric matrix for the separating quadric as above, and let ω be any equilibrium stress for $(K(n, m), (\mathbf{p}, \mathbf{q}))$ with stress matrix Ω. For any vertex $\mathbf{q}_j$ in one partition, the equilibrium condition of Eq. (1.1) can be written, for each $j = 1, \ldots, m$ as

$$\sum_{i=1}^{n} \omega_{ij}(\hat{\mathbf{p}}_i - \hat{\mathbf{q}}_j) = 0,$$

or equivalently

$$\sum_{i=1}^{n} \omega_{ij}\hat{\mathbf{p}}_i = \left(\sum_{i=1}^{n} \omega_{ij}\right)\hat{\mathbf{q}}_j = \mu_j \hat{\mathbf{q}}_j.$$

Then taking the transpose of this equation, and multiplying on the right by $A\mathbf{q}_j$, we get

$$\sum_{i=1}^{n} \omega_{ij}\hat{\mathbf{p}}_i^t A\hat{\mathbf{q}}_j = \mu_j \hat{\mathbf{q}}_j^t A\hat{\mathbf{q}}_j.$$

Similarly, for $\mathbf{p}_i$ in the other partition,

$$\sum_{j=1}^{m} \omega_{ij}\hat{\mathbf{q}}_j^t A\hat{\mathbf{p}}_i = \lambda_i \hat{\mathbf{p}}_i^t A\hat{\mathbf{p}}_i.$$

Since the matrix A is symmetric,

$$\sum_{j=1}^{m} \mu_j \hat{\mathbf{q}}_j^t A\hat{\mathbf{q}}_j = \sum_{ij} \omega_{ij}\hat{\mathbf{p}}_i^t A\hat{\mathbf{q}}_j = \sum_{ij} \omega_{ij}\hat{\mathbf{q}}_j^t A\hat{\mathbf{p}}_i = \sum_{i=1}^{n} \lambda_i \hat{\mathbf{p}}_i^t A\hat{\mathbf{p}}_i. \tag{2.3}$$

By Theorem 1.7, if $(K(n, m), (\mathbf{p}, \mathbf{q}))$ were universally rigid, then there would be an equilibrium stress with a stress matrix Ω that would be PSD and non-zero. Then $\mu_j \geq 0$ for all $j = 1, \ldots, m$, $\lambda_i \geq 0$ for all $i = 1, \ldots, n$, and we would have at least one positive diagonal term. But then Eq. (2.3) would contradict the assumed quadric separation condition of Eq. (2.1). □

This result is a generalization of, and inspired by, the main result in [15], which is the result here for the line $d = 1$. We will see that the quadric separation condition is also the critical sufficient condition for complete bipartite graphs to be universally rigid, including in higher dimensions, but we need to use a technique that allows us to find PSD matrices with a given kernel and appropriate rank, which we describe in later sections.

3 The Veronese Map

Vectors in $\mathbb{R}^d$ will be regarded as column vectors, and in general, for any vector or matrix X, we will denote by $\hat{X}$ the same object with a row of 1's added on the bottom. We denote X^t as the transpose of a matrix or vector X.

If two configurations $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ and $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m)$ in $\mathbb{R}^d$ cannot be separated by a quadric, i.e. when the condition of Eq. (2.1) cannot be made to hold for any A, we show here how find a certificate of this non-separability, that can help us to establish universal rigidity.

Definition 3.1 We define $\mathcal{M}_d$ to be the $(d+1)(d+2)/2$-dimensional space of $(d+1)$-by-$(d+1)$ symmetric matrices, which we call the *matrix space*.

Definition 3.2 We define the map $\mathcal{V} : \mathbb{R}^d \to \mathcal{M}_d$ by $\mathcal{V}(\mathbf{v}) = \hat{\mathbf{v}}\hat{\mathbf{v}}^t$, which is a $(d+1)$-by-$(d+1)$ symmetric matrix, with a lower right-hand coordinate of 1.

So $\mathcal{V}(\mathbb{R}^d)$ is a d-dimensional set embedded in a $((d+1)(d+2)/2 - 1)$-dimensional affine subspace of $\mathcal{M}_d$. The function $\mathcal{V}$ is called the *Veronese map*. See [16, p. 244], for very similar properties that are used here.

Proposition 3.3 *In $\mathbb{R}^d$ the vertices of the configurations $\mathbf{p}$ and $\mathbf{q}$ can be strictly separated by a quadric A as in Sect. 2, if and only if the matrix configurations $\mathcal{V}(\mathbf{p})$ and $\mathcal{V}(\mathbf{q})$ can be strictly separated by the the hyperplane given by A in $\mathcal{M}_d$.*

Proof The configurations $\mathbf{p}$ and $\mathbf{q}$ are separated by the quadric given by the matrix A when

$$\langle A, \mathcal{V}(\mathbf{q}_j)\rangle = \mathrm{tr}(A\hat{\mathbf{q}}_j\hat{\mathbf{q}}_j^t) = \hat{\mathbf{q}}_j^t A\hat{\mathbf{q}}_j < 0 < \hat{\mathbf{p}}_i^t A\hat{\mathbf{p}}_i = \mathrm{tr}(A\hat{\mathbf{p}}_i\hat{\mathbf{p}}_i^t) = \langle A, \mathcal{V}(\mathbf{p}_i)\rangle,$$

where the inner product $\langle *, *\rangle$ on symmetric matrices is given by the trace operator tr as above. $\square$

When the configurations $\mathbf{p}$ and $\mathbf{q}$ cannot be separated by a quadric in $\mathbb{R}^d$, then from Proposition 3.3 the convex hull of $\mathcal{V}(\mathbf{p})$ must intersect the convex hull of $\mathcal{V}(\mathbf{q})$ in $\mathcal{M}_d$. This means that there are non-negative coefficients, not all 0, denoted as $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, such that

$$\sum_{i=1}^{n} \lambda_i \hat{\mathbf{p}}_i \hat{\mathbf{p}}_i^t = \sum_{j=1}^{m} \mu_j \hat{\mathbf{q}}_j \hat{\mathbf{q}}_j^t. \tag{3.1}$$

Definition 3.4 The matrix $\hat{P}$ whose columns are $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_n$ is called the *configuration matrix* of $\mathbf{p}$.

In terms of matrices Eq. (3.1) is the same as saying:

$$\hat{P}\Lambda\hat{P}^t = \hat{Q}M\hat{Q}^t, \tag{3.2}$$

where Λ is the n-by-n diagonal matrix whose entries are $\lambda_1, \ldots, \lambda_n$, and M is the m-by-m diagonal matrix whose entries are $\mu_1, \ldots, \mu_m$. This is the starting point for constructing a PSD stress matrix in Sect. 4.

4 The Singular Value Decomposition

We first show that when each of the partition's affine span is the full $\mathbb{R}^d$, we will not need to worry about condition (iii) of Theorem 1.2.

Lemma 4.1 *Suppose that the configurations* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{R}^d$ *each have affine span equal to all of* $\mathbb{R}^d$*, and the bipartite framework* $(K(n, m), (\mathbf{p}, \mathbf{q}))$ *has a stress with stress matrix,* Ω*, satisfying* (i) *and* (ii) *of Theorem* 1.2*. Then* $(K(n, m), (\mathbf{p}, \mathbf{q}))$ *is super stable and universally rigid. Likewise, if, instead of* (i) *and* (ii) *of Theorem* 1.2*,* $(K(n, m), (\mathbf{p}, \mathbf{q}))$ *is just dimensionally rigid, then it is still universally rigid.*

Proof We have only to check (iii) of Theorem 1.2, that the edge directions do not lie on a conic at infinity of $\mathbb{R}^d$. Suppose there is a non-zero symmetric d-by-d matrix Q such that $(\mathbf{p}_1 - \mathbf{q}_j)Q(\mathbf{p}_1 - \mathbf{q}_j)^t = 0$ and $(\mathbf{p}_2 - \mathbf{q}_j)Q(\mathbf{p}_2 - \mathbf{q}_j)^t = 0$, for all $j = 1, \ldots, m$. Expanding these terms and subtracting we get

$$\begin{aligned} \mathbf{p}_1 Q\mathbf{p}_1^t - 2\mathbf{p}_1 Q\mathbf{q}_j^t - \mathbf{p}_2 Q\mathbf{p}_2^t + 2\mathbf{p}_2 Q\mathbf{q}_j^t &= 0, \quad \text{which gives us} \\ \mathbf{p}_1 Q\mathbf{p}_1^t - \mathbf{p}_2 Q\mathbf{p}_2^t &= 2(\mathbf{p}_1 - \mathbf{p}_2)^t Q\mathbf{q}_j^t, \end{aligned}$$

which is a non-trivial affine linear constraint on the vertices of $\mathbf{q}$, unless for all $\mathbf{p}_i$ and $\mathbf{p}_k$,

$$(\mathbf{p}_i - \mathbf{p}_k)^t Q = 0.$$

The first case implies that the vertices of $\mathbf{q}$ lie in a proper affine subspace, while the latter implies that the vertices of $\mathbf{p}$ lie in a proper affine subspace. □

Alfakih and Ye in [3] show that if a configuration of a framework is in general position and satisfies (i) and (ii) of Theorem 1.2, then it is universally rigid. Lemma 4.1 is more precise and general for complete bipartite graphs.

For any diagonal matrix X, with non-negative entries, denote $X^{1/2}$ as another diagonal matrix whose entries are the square roots of the entries of X.

Definition 4.2 For any a-by-b matrix X, $a \leq b$, a *singular value decomposition* (SVD) is a factoring $X = USV^t$, where U is an a-by-a orthogonal matrix, V is a

b-by-b orthogonal matrix, and S is the matrix of $S = [D, 0]$, where D is an a-by-a diagonal matrix of non-negative *singular values*. Such a decomposition always exists (see e.g. [14]).

Our next step is to show that when Eq. (3.2) holds, $\hat{P}\Lambda^{1/2}$ and $\hat{Q}M^{1/2}$ must share their singular values and their left singular structure.

Lemma 4.3 *Suppose Eq.* (3.2) *holds where* Λ *and* M *are non-negative, diagonal matrices as above. Then the SVD factors can be taken such that* $\hat{P}\Lambda^{1/2} = U S_n V_n^t$, *and* $\hat{Q}M^{1/2} = U S_m V_m^t$, *with a common matrix* U *and where* $S_n S_n^t = S_m S_m^t$.

Proof By definition, the squared singular values and the left singular vectors of $\hat{P}\Lambda^{1/2}$ are the eigenvalues and eigenvectors of $\hat{P}\Lambda\hat{P}^t$. Likewise, the squared singular values and the left singular vectors of $\hat{Q}M^{1/2}$ are the eigenvalues and eigenvectors of $\hat{Q}M\hat{Q}^t$.

Since, by assumption, $\hat{P}\Lambda\hat{P}^t = \hat{Q}M\hat{Q}^t$, these singular values and left singular vectors agree. Thus we can pick a single shared $(d+1)$-by-$(d+1)$ matrix U, along with with appropriately sized diagonal matrices S_n and S_m, and appropriate orthogonal matrices V_n and V_m, such that we obtain the singular value decompositions:

$$\hat{P}\Lambda^{1/2} = U S_n V_n^t \quad \text{and} \quad \hat{Q}M^{1/2} = U S_m V_m^t$$

where $S_n S_n^t = S_m S_m^t$. In particular

$$S_n = D\big[I^{d+1}, 0^{n-d-1}\big] \quad \text{and} \quad S_m = D\big[I^{d+1}, 0^{m-d-1}\big]$$

for a single shared diagonal matrix D. $\square$

Our next result is our main sufficient condition for the universal rigidity of a complete bipartite framework. The central idea is to use the conditions of Eq. (3.2) to directly construct Ω, a PSD equilibrium stress matrix for $(K(n, m), \mathbf{p}, \mathbf{q})$ that has rank $n + m - d - 1$. To do this we will use the SVD provided by Lemma 4.3 in order to *transform* the matrix $[\hat{P}, \hat{Q}]$ into a very specific and simple canonical form. It will be easy to see that this canonical form is annihilated by a certain simple PSD matrix Ψ described below. We can then reverse this transformation, thus constructing a Ω with the same signature as Ψ.

Theorem 4.4 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be configurations* (*in any dimension*), *such that Eq.* (3.1) *holds with strictly positive coefficients. Then the framework* $(K(n, m), \mathbf{p}, \mathbf{q})$ *is super stable, and thus universally rigid. Additionally, the affine span of* $\mathbf{p}$ *is the same as the affine span of* $\mathbf{q}$.

Proof Let d be the dimension of the combined span of $(\mathbf{p}, \mathbf{q})$. Without loss of generality, we can rigidly place $(\mathbf{p}, \mathbf{q})$ in $\mathbb{R}^d$ and continue.

By Lemma 4.3 we have the following $(d+1)$-by-$(n+m)$ matrix equality:

$$\big[\hat{P}\Lambda^{1/2}, \hat{Q}M^{1/2}\big] = \big[U S_n V_n^t, U S_m V_m^t\big],$$

where U, V_n, V_m are orthogonal matrices, of the appropriate size,

$$S_n = D\big[I^{d+1}, 0^{n-d-1}\big], \quad \text{and} \quad S_m = D\big[I^{d+1}, 0^{m-d-1}\big],$$

where D is a $(d+1)$-by-$(d+1)$ diagonal matrix, I^{d+1} is the $(d+1)$-by-$(d+1)$ identity matrix, 0^{n-d-1} is a $(d+1)$-by-$(n-d-1)$ zero matrix and 0^{m-d-1} is a $(d+1)$-by-$(m-d-1)$ zero matrix. Then

$$\begin{aligned}\big[\hat{P}, \hat{Q}\big] &= \big[\hat{P}\Lambda^{1/2}, \hat{Q}M^{1/2}\big]\begin{bmatrix}\Lambda^{-1/2} & 0 \\ 0 & M^{-1/2}\end{bmatrix} \\ &= UD\big[I^{d+1}, 0^{n-d-1}, I^{d+1}, 0^{m-d-1}\big]\begin{bmatrix}V_n^t & 0 \\ 0 & V_m^t\end{bmatrix}\begin{bmatrix}\Lambda^{-1/2} & 0 \\ 0 & M^{-1/2}\end{bmatrix}.\end{aligned} \tag{4.1}$$

(The matrices $\Lambda^{-1/2}$ and $M^{-1/2}$ are well defined due to our assumption of strictly positive coefficients.)

Define the following symmetric $(n+m)$-by-$(n+m)$ matrix:

$$\Psi = \begin{bmatrix} I^{d+1} & & -I^{d+1} & \\ & I^{n-d-1} & & \\ -I^{d+1} & & I^{d+1} & \\ & & & I^{m-d-1} \end{bmatrix}, \tag{4.2}$$

where the blank entries are zero matrices of the appropriate dimensions. It is easy to check that

$$\big[I^{d+1}, 0^{n-d-1}, I^{d+1}, 0^{m-d-1}\big]\Psi = 0,$$

and that Ψ is PSD of rank $n+m-d-1$. Then we define a stress matrix

$$\Omega = \begin{bmatrix}\Lambda^{1/2} & 0 \\ 0 & M^{1/2}\end{bmatrix}\begin{bmatrix}V_n & 0 \\ 0 & V_m\end{bmatrix}\Psi\begin{bmatrix}V_n^t & 0 \\ 0 & V_m^t\end{bmatrix}\begin{bmatrix}\Lambda^{1/2} & 0 \\ 0 & M^{1/2}\end{bmatrix}. \tag{4.3}$$

Clearly Ω has zero entries for all of the non edges of the complete bipartite graph. Thus by unraveling Eqs. (4.1), (4.2), (4.3), and using the assumption that the diagonal entries of Λ and M are all positive, we see that Ω is PSD of rank $n+m-d-1$, and $[\hat{P}, \hat{Q}]\Omega = 0$. This is sufficient to obtain conditions (i) and (ii) of Theorem 1.2.

The equilibrium condition of Eq. (1.1) at each vertex, and the non-zero diagonal entries in the stress matrix, imply that each $\mathbf{p}_i$ is in the affine span of $\mathbf{q}$, and similarly each $\mathbf{q}_j$ is in the affine span of $\mathbf{p}$. So the affine span of $\mathbf{p}$ is the same as the affine span of $\mathbf{q}$, which by our assumptions must then be all of $\mathbb{R}^d$. Lemma 4.1 then implies that condition (iii) of Theorem 1.2 holds. □

The next two corollaries describe partial converses to our Theorem 2.2, each requiring some kind of general position for the configuration. Without any such assumptions, the converse of Theorem 2.2 does not hold. In Sect. 8, we use our Theorem 4.4 as the basis of a complete algorithm for determining the universal and dimensional rigidity of any complete bipartite framework.

Definition 4.5 A configuration $\mathbf{p}$ in $\mathbb{R}^d$ is in *general position* if every $k+1$ of the points of $\mathbf{p}$ span a k-dimensional affine subspace for $k = 1, \ldots, d$.

Corollary 4.6 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be configurations in* $\mathbb{R}^d$*. Suppose there exists subsets of the corresponding configurations* $\mathbf{p}' \subset \mathbf{p}$ *and* $\mathbf{q}' \subset \mathbf{q}$*, such that that the points of* $(\mathbf{p}', \mathbf{q}')$ *are in general position in* $\mathbb{R}^d$*, and such that there is no quadric strictly separating* $\mathbf{p}'$ *and* $\mathbf{q}'$*. Then* $(K(n,m), \mathbf{p}, \mathbf{q})$ *is universally rigid. Additionally the affine span of* $\mathbf{p}$ *and the affine span of* $\mathbf{q}$ *must be all of* $\mathbb{R}^d$*.*

Proof Since there is no quadric strictly separating $\mathbf{p}'$ and $\mathbf{q}'$, the convex hulls $\mathcal{V}(\mathbf{p}')$ and $\mathcal{V}(\mathbf{q}')$ must intersect in matrix space, $\mathcal{M}_d$, and Eq. (3.1) holds with strictly positive coefficients λ_i, and μ_i for some subsets $\mathbf{p}'' \subset \mathbf{p}'$ and $\mathbf{q}'' \subset \mathbf{q}'$. By Theorem 4.4, that subframework is super stable.

Also from Theorem 4.4 each vertex of $\mathbf{p}''$ must be in the affine span of the $\mathbf{q}''$, and so due to general position assumption, the affine span of $\mathbf{q}''$ must then be d-dimensional. Since each of the vertices of $\mathbf{p}$ has at least $d+1$ neighbors in $\mathbf{q}''$, each $\mathbf{p}$ has a fixed distance to all the vertices of $\mathbf{q}''$. The same argument applies to $\mathbf{q}$. This *trilateration* argument shows that all of $(K(n,m), \mathbf{p}, \mathbf{q})$ is universally rigid. □

Note that it may be the case that, even assuming general position, the framework is not super stable, because all the stress coefficients may vanish for some vertex. See the example of Fig. 2 that shows this possibility, and other examples of universally rigid frameworks.

Definition 4.7 We say that a configuration $\mathbf{p}$ in $\mathbb{R}^d$ is in *quadric general position* if every $k+1$ of the points of $\mathcal{V}(\mathbf{p}) \subset \mathcal{M}_d$ span a k-dimensional affine subspace for $k = 1, \ldots, (d+1)(d+2)/2 - 1$. (The vertices of $\mathcal{V}(\mathbf{p})$ are automatically mapped into a co-dimension one subspace of $\mathcal{M}_d$, where the last coordinate is one.) Essentially this means that if there are at least $(d+1)(d+2)/2$ points, then no $(d+1)(d+2)/2$ of them lie on a quadric.

Corollary 4.8 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be configurations in* $\mathbb{R}^d$*. Suppose there exist subsets of the corresponding configurations* $\mathbf{p}' \subset \mathbf{p}$ *and* $\mathbf{q}' \subset \mathbf{q}$*, such that that the points of* $(\mathbf{p}', \mathbf{q}')$ *are in quadric general position in* $\mathbb{R}^d$*, and such that there is no quadric strictly separating* $\mathbf{p}'$ *and* $\mathbf{q}'$*. Then* $(K(n,m), \mathbf{p}, \mathbf{q})$ *is super stable, and* $n + m \geq (d+1)(d+2)/2 + 1$*. Additionally, the affine span of* $\mathbf{p}$ *is the same as the affine span of* $\mathbf{q}$*, which must be all of* $\mathbb{R}^d$*.*

Proof In matrix space $\mathcal{M}_d$, since the points $\mathcal{V}(\mathbf{p}')$ and $\mathcal{V}(\mathbf{q}')$ cannot be separated by a hyperplane, Eq. (3.1) holds with non-negative coefficients, not all 0. But since $(\mathcal{V}(\mathbf{p}'), \mathcal{V}(\mathbf{q}'))$ is in general position, then at least $(d+1)(d+2)/2+1$ of the coefficients are positive, corresponding to subsets $\mathbf{p}'' \subset \mathbf{p}$ and $\mathbf{q}'' \subset \mathbf{q}$.

This gives us $n+m \geq (d+1)(d+2)/2+1$. Additionally, this lower bound together with our quadric general position assumption, forces the combined span of $(\mathbf{p}'', \mathbf{q}'')$ to be all of $\mathbb{R}^d$.

By Theorem 4.4, the bipartite graph restricted to $(\mathbf{p}'', \mathbf{q}'')$ is super stable. Additionally the span of $\mathbf{p}''$ and the span of $\mathbf{q}''$ must be all of $\mathbb{R}^d$.

Additionally, due to Eq. (3.1) and the quadric general position assumption, the affine span of this $(\mathcal{V}(\mathbf{p}''), \mathcal{V}(\mathbf{q}''))$ in matrix space must be the full $(d+1)(d+2)/2 - 1$ dimensions. Thus, for any additional point, $\mathbf{p}_i \in \mathbf{p}$, there is an affine relation non-zero on $\mathcal{V}(\mathbf{p}_i)$ and involving the $\mathcal{V}(\mathbf{p}'')$ and $\mathcal{V}(\mathbf{q}'')$. When that relation is added to both sides of Eq. (3.1), choosing the coefficient of $\mathcal{V}(\mathbf{p}_i)$ to be positive, and the whole relation small enough, we enlarge the number of indices, where $\lambda_i > 0$, and $\mu_j > 0$, until all the coefficients are positive, applying this argument to any $\mathbf{q}_j \in \mathbf{q}$ as well. Then again Theorem 4.4 implies that all of $(K(n, m), \mathbf{p}, \mathbf{q})$ is super stable. □

Remark 4.9 The smallest example of Corollary 4.8 in the line is $K(2, 2)$. In the plane, the smallest example is $K(4, 3)$. In $\mathbb{R}^3$, the smallest examples are $K(7, 4)$ and $K(6, 5)$. Section 5 shows some examples of these. For the first three examples, any equilibrium stress matrix with positive diagonals will be PSD, which implies directly that they are super stable. However, for $K(6, 5)$ in $\mathbb{R}^3$, there are always equilibrium stress matrices with all positive diagonals but with negative eigenvalues. At first it is a little surprising that Theorem 4.4 guarantees that there will always be some such PSD stress matrix.

Remark 4.10 We note that Theorem 4.4 is also gives us an alternative proof for [5, Thm. 6], under the restriction that the coefficients $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$ of Eq. (3.1) are positive. This is because the proof of our Theorem 4.4 provides a construction of an equilibrium stress matrix for $(K(n, m), \mathbf{p}, \mathbf{q})$ with these coefficients on its diagonal.

Indeed, by slightly generalizing this construction, we can produce *all* of the equilibrium stress matrices for $(K(n, m), \mathbf{p}, \mathbf{q})$ with all positive diagonals. To do this, all we need to do is replace Eq. (4.2) with

$$\Psi = \begin{bmatrix} I^{d+1} & & -I^{d+1} & \\ & I^{n-d-1} & & C \\ -I^{d+1} & & I^{d+1} & \\ & C^t & & I^{m-d-1} \end{bmatrix},$$

where C is an arbitrary diagonal $(n-d-1)$-by-$(m-d-1)$ matrix, and also we need to allow for any U, V_m and V_n such that

$$\hat{P}\Lambda^{1/2} = U S_n V_n^t \quad \text{and} \quad \hat{Q}M^{1/2} = U S_m V_m^t.$$

Additionally, whenever any of the diagonal entries in C have a magnitude equal to 1 the rank of Ψ will drop, and whenever all of diagonal entries of C have magnitudes less than or equal to 1, then Ψ will be PSD.

It is less clear if we can use the ideas in this paper to prove [5, Thm. 6], when the coefficients $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$ include negative values.

It is easy to see how our necessity result of Theorem 2.2 fits in with the ideas of this section. In particular we have the following Proposition, which is essentially [5, Lem. 5].

Proposition 4.11 *Suppose that* Ω *is an equilibrium stress matrix for* $(K(n, m), \mathbf{p}, \mathbf{q})$*, where* Ω *is of the form*

$$\begin{bmatrix} \Lambda & B \\ B^t & M \end{bmatrix},$$

where Λ *and* M *are diagonal matrices of size* n *and* m *respectively. Then Eq.* (3.2) *holds with this* Λ *and* M.

Proof Since Ω is an equilibrium stress matrix we have

$$[\hat{P}, \hat{Q}] \begin{bmatrix} \Lambda & B \\ B^t & M \end{bmatrix} = 0$$

and so we have $\hat{P}\Lambda = -\hat{Q}B^t$ and $\hat{Q}M = -\hat{P}B$. This gives us $\hat{P}\Lambda\hat{P}^t = -\hat{Q}B^t\hat{P}^t$ and $\hat{Q}M\hat{Q}^t = -\hat{P}B\hat{Q}^t$. Since these are symmetric matrices, this gives us $\hat{P}\Lambda\hat{P}^t = \hat{Q}M\hat{Q}^t$, which is Eq. (3.2). □

Thus, when $(K(n, m), \mathbf{p}, \mathbf{q})$ is universally rigid, from Theorem 1.7 it must have a non-zero equilibrium stress matrix with non-negative (and not all zero) Λ and M. Then Propositions 4.11 and 3.3 imply that $\mathbf{p}$ and $\mathbf{q}$ cannot be strictly separated by a quadric, which gives us the result of our Theorem 2.2.

5 Examples

Figure 1 shows examples of bipartite frameworks that are super stable in quadric general position with the minimal number of vertices. Dashed edges have a positive equilibrium stress, and for solid edges the equilibrium stress is negative. These represent cables and struts, respectively, where cables cannot increase in length, and struts cannot decrease in length. These examples have symmetry, and for the calculation of the separating quadric or conic, this allows us to only consider symmetric quadrics or conics, since we can average those that separate the two partitions to get one that is symmetric. Note that the $K(6, 5)$ example is such that it is in quadric general position, but since there are several sets of three vertices that collinear, it is not in general position.

In Fig. 2 the top examples are frameworks of the graph $K(3, 3)$ in the plane. The top left example is super stable. It lies on a conic, which corresponds to a co-dimension two subspace of $\mathcal{M}_3$. It has equilibrium stress which is PSD since it cannot be separated by a conic. See also [6]. The top right example is not universally rigid, even though the vertices lie on a conic, since the partitions can be separated by a conic consisting of two lines, as shown. The bottom example is the same as the top left example, except a red vertex is inserted and attached to the blue vertices forming a $K(4, 3)$. The stress on the edges on the central vertex is zero, but the entire configuration is in general position in the plane. So Corollary 4.6 applies and it is universally rigid, but not super stable.

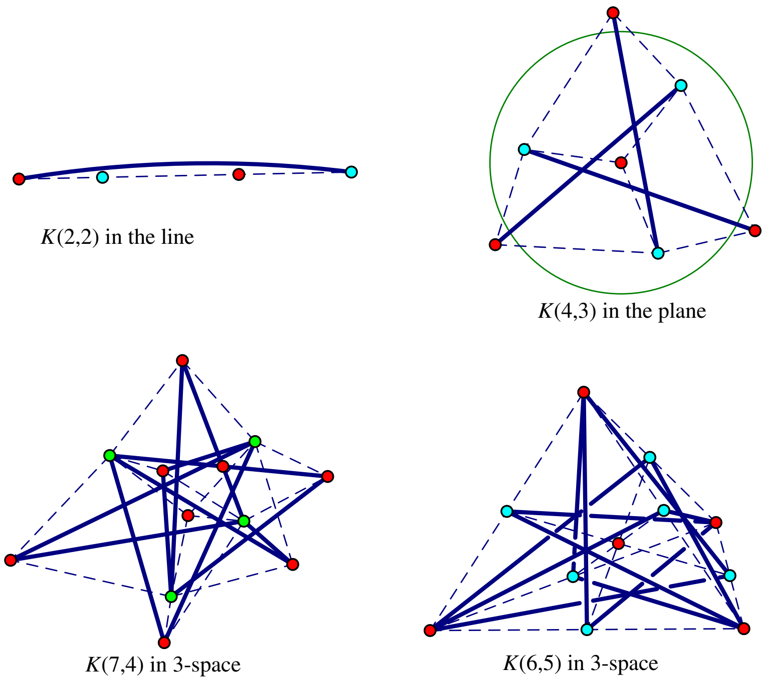


Fig. 1 Super stable bipartite tensegrities in quadric general position

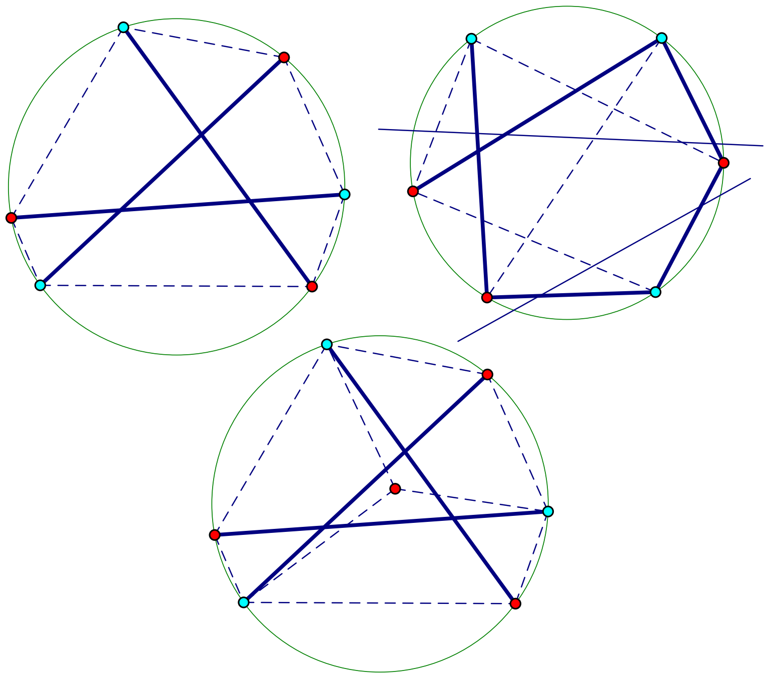


Fig. 2 Tensegrities in the plane lying on conics. See the text

6 Primitive Cores

Definition 6.1 Following [11, Thm. 9.1], we say that a partition $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ in $\mathcal{M}^d$ is *primitive* if the convex hull of $\mathcal{V}(\mathbf{p})$ intersects the convex hull of $\mathcal{V}(\mathbf{q})$ and no proper subset of $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ has this property.

From our discussion above and [11] it is clear that if the convex hull of $\mathcal{V}(\mathbf{p})$ intersects the convex hull of $\mathcal{V}(\mathbf{q})$, there are subsets $\mathbf{p}' \subset \mathbf{p}$ and $\mathbf{q}' \subset \mathbf{q}$ such that the convex hull of $\mathcal{V}(\mathbf{p}')$ intersects the convex hull of $\mathcal{V}(\mathbf{q}')$ in their relative interiors with a minimal number of vertices. We call the subframework $(K(n', m'), \mathbf{p}', \mathbf{q}')$ a *primitive core* of $(K(n, m), \mathbf{p}, \mathbf{q})$. Here we list all the primitive cores of complete bipartite graphs for dimensions one, two, three. It is easy to see how to extend this higher dimensions.

Note that when $K(n, m)$ is a primitive core with affine span of dimension d, then $n \geq d + 1$, $m \geq d + 1$, and $n + m - 2$ is the dimension of the affine span of $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ in $\mathcal{M}^d$. Since $(d + 1)(d + 2)/2 - 1$ is the dimension of the affine span of the image of $\mathcal{V}(\mathbb{R}^d)$ in $\mathcal{M}_d$, the vertices of $(\mathbf{p}, \mathbf{q})$ lie in the intersection of $(d + 1)(d + 2)/2 + 1 - (n + m)$ quadrics, corresponding to hyperplanes in $\mathcal{M}_d$. Furthermore, for a primitive core, $(K(n, m), \mathbf{p}, \mathbf{q})$ is super stable.

6.1 Dimension One

There is only one primitive core given by $K(2, 2)$, where the partitions alternate along the line, as in Fig. 1. This is the main result of [15].

6.2 Dimension Two

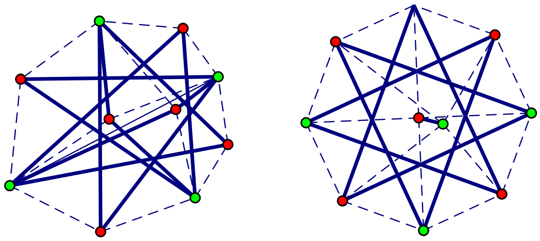
When the core vertices are in quadric general position there is only $K(4, 3)$ as in Fig. 1. Here the dimension of the affine span of $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ is 5-dimensional.

When the Veronese images of the core vertices $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ have a 4-dimensional affine span, then there is one more example, $K(3, 3)$ as in Fig. 2. The vertices of this $K(3, 3)$ lie on a single conic in the plane. This particular example was described in [6] as well, for example.

6.3 Dimension Three

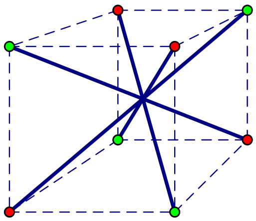
When the core vertices are in quadric general position, there are two examples, $K(6, 5)$ and $K(7, 4)$ as in Fig. 1. Here the the core vertices $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ have a 9-dimensional affine span in $\mathcal{M}_3$.

When the core vertices $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ have an 8-dimensional affine span in $\mathcal{M}_3$, then there are two examples, $K(5, 5)$ and $K(6, 4)$. Figure 3 shows an example for $K(6, 4)$ and $K(5, 5)$ lying on a sphere. The configuration for $K(6, 4)$ is obtained by taking the green vertices as the vertices of a regular tetrahedron, and the red vertices as the midpoints of the 6 edges rescaled out to be on the circumsphere of the tetrahedron. The configuration for $K(5, 5)$ is obtained by taking the red and green vertices as a regular



$K(6,4)$ $K(5,5)$

Fig. 3 Super stable bipartite tensegrities in 3-space



$K(4,4)$

Fig. 4 A super stable bipartite tensegrity cube, with struts as long diagonals

octahedron, but with the red vertices translated up and the green vertices translated down. Then another red and green vertex is added down and up, respectively, to avoid separating the two partitions.

When the core vertices $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ have a 7-dimensional affine span in $\mathcal{M}_3$, then there is an example, $K(5, 4)$. (One can use the analysis of Theorem 4.4 to construct examples in this range.) This configuration lies on the intersection of two quadrics.

When the core vertices $(\mathcal{V}(\mathbf{p}), \mathcal{V}(\mathbf{q}))$ have a 6-dimensional affine span in $\mathcal{M}_3$, then there is an example, $K(4, 4)$, which is the intersection of three quadrics. One example is a cube with its long diagonal as in Fig. 4. This was also shown in [6].

7 Coning and Projection-Section

Here we describe some general tools that are interesting in their own right and that we will use below in Sect. 8. See also [10] for similar results in the context of generic global rigidity.

7.1 Coning

Definition 7.1 A *coned graph* is one where one of the vertices is connected to all the others.

If a configuration for a complete bipartite graph has coincident vertices from different partitions, we can identify those two vertices as one, and we effectively have a coned graph. Here we first consider a general graph, not just a bipartite graph, that has a distinguished vertex $\mathbf{p}_0$ that is connected to all the vertices of a graph G. We denote this framework as $\mathbf{p}_0 * (G, \mathbf{p})$. We also assume that all the vertices of $\mathbf{p}$ are distinct

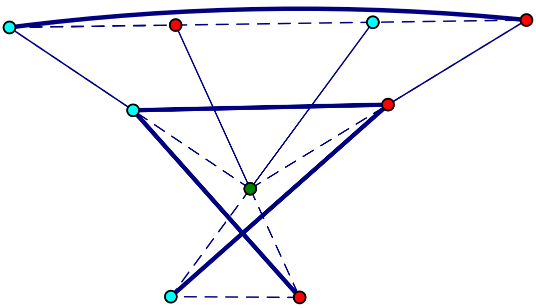


Fig. 5 A bipartite cone that is universally rigid

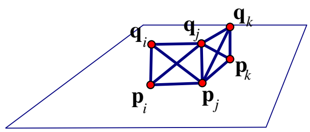


Fig. 6 A construction joining parallel frameworks

from $\mathbf{p}_0$. The following is immediate since one can slide the vertices of G on the lines from $\mathbf{p}_0$ while preserving universal and dimensional rigidity.

Lemma 7.2 *Suppose that* $\mathbf{p}_0 * (G, \mathbf{p})$ *and* $\mathbf{p}_0 * (G, \mathbf{q})$ *are two coned frameworks. For simplicity, we assume* $\mathbf{p}_0 = 0$*, the origin. Suppose that for each edge* $\{i, j\}$ *of* G*,*

$$\frac{|\mathbf{p}_i \cdot \mathbf{p}_j|}{|\mathbf{p}_i||\mathbf{p}_j|} = \frac{|\mathbf{q}_i \cdot \mathbf{q}_j|}{|\mathbf{q}_i||\mathbf{q}_j|}.$$

Then $\mathbf{p}_0 * (G, \mathbf{p})$ *is universally rigid if and only if* $\mathbf{p}_0 * (G, \mathbf{q})$ *is universally rigid, and* $\mathbf{p}_0 * (G, \mathbf{p})$ *is dimensionally rigid if and only if* $\mathbf{p}_0 * (G, \mathbf{q})$ *is dimensionally rigid.*

Figure 5 shows this for a quadrilateral in the plane that is a cone over a $K(2, 2)$ graph on a line. The cable-strut designation is shown as well. The stress values on the "cone edges" over the collinear $K(2, 2)$ are zero.

The following is a general result relating universal and dimensional rigidity to their coned frameworks.

Proposition 7.3 *Suppose that* $\mathbf{p}$ *is in* $\mathbb{R}^d$ *and the cone point* $\mathbf{p}_0 \in \mathbb{R}^{d+1} - \mathbb{R}^d$*. Then the framework* $\mathbf{p}_0 * (G, \mathbf{p})$ *is dimensionally rigid if and only if* $(G, \mathbf{p})$ *is dimensionally rigid, and if* $(G, \mathbf{p})$ *is universally rigid, then* $\mathbf{p}_0 * (G, \mathbf{p})$ *is universally rigid.*

Proof The "if" statements are obvious.

For the other direction, suppose $(G, \mathbf{p})$ is not dimensionally rigid. For now, we will assume that G is connected. Without loss of generality, we can choose d so that the span of $\mathbf{p}$ is full within $\mathbb{R}^d$.

In $\mathbb{R}^{d+1}$ construct a parallel framework $(G, \mathbf{q})$ by translating each vertex $\mathbf{p}_i$ by one unit perpendicular to the $\mathbb{R}^d$ hyperplane to get $\mathbf{q}_i$. Then for each edge $\{i, j\}$ of G, construct the bars connecting all the pairs of vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{q}_i, \mathbf{q}_j$ constructing a new framework in $\mathbb{R}^{d+1}$, $(H, (\mathbf{p}, \mathbf{q}))$, as in Fig. 6.

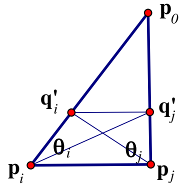


Fig. 7 A coned framework related to the construction in Fig. 6

It is clear that if $(G, \mathbf{p})$ is not dimensionally rigid then $(H, (\mathbf{p}, \mathbf{q}))$ is not dimensionally rigid. Then by [8, Sect. 13], any non-singular projective image of $(H, (\mathbf{p}, \mathbf{q}))$, is not dimensionally rigid as well. But the lines through $\mathbf{p}_i$ and $\mathbf{q}_i$ are all parallel and so in the projective image all these lines intersect at a "meeting" point in $\mathbb{R}^{d+1}$.

For any chosen point $\mathbf{p}_0$, we can find a projective transformation that leaves $\mathbb{R}^d$ fixed and such that the image of $(H, (\mathbf{p}, \mathbf{q}))$ has its meeting point at $\mathbf{p}_0$. Let us denote this framework as $(H, (\mathbf{p}, \mathbf{q}'))$. The point $\mathbf{p}_0$ is on each of the lines through $\mathbf{p}_i, \mathbf{q}'_i$ for all i.

Each edge $\{i, j\}$ of G corresponds to a 4-vertex universally rigid planar framework on the vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{q}'_i, \mathbf{q}'_j$ in the graph H. Each such 4-vertex framework determines a unique "apex point", say, using the angle-side-angle theorem in elementary geometry. This also determines the distance from the apex point to $\mathbf{p}_i$ and to $\mathbf{q}'_i$ along the line spanned by $\mathbf{p}_i$ and $\mathbf{q}'_i$. See Fig. 7. In the framework, $(H, (\mathbf{p}, \mathbf{q}'))$ all of these apicies coincide at $\mathbf{p}_0$.

Suppose there is a second framework of H with the same edge lengths as $(H, (\mathbf{p}, \mathbf{q}'))$ but with an affine span of dimension greater than $d + 1$. Then for each 4-vertex set in this second framework, we can compute its apex point. Since we have assumed that G is connected, these all must agree at a common meeting point. This means that we can find a second framework that has the same edge lengths as the coned framework $\mathbf{p}_0 * (H, (\mathbf{p}, \mathbf{q}'))$, but with an affine span greater than $d + 1$.

From Lemma 7.2, this means that we can find a second framework that has the same edge lengths as the coned framework $\mathbf{p}_0 * (G, \mathbf{p})$ but with an affine span greater than $d + 1$, making it not dimensionally rigid.

Finally, we can look at the case that $(G, \mathbf{p})$ in $\mathbb{R}^d$ has multiple connected components. Suppose one of the components is not dimensionally rigid. We have just shown that the coned framework over that component is not dimensionally rigid as well. This can be used to certify that $\mathbf{p}_0 * (G, \mathbf{p})$ is not dimensionally rigid. Suppose, instead that each of the components is dimensionally rigid but $(G, \mathbf{p})$ is not. This means that we can increase the dimension span by simply rigidly moving one of the components into a larger space. The same will be true for $\mathbf{p}_0 * (G, \mathbf{p})$ using an appropriate rotation of that component about $\mathbf{p}_0$ into some larger space. □

In the setting of the above proposition, it is not true that if $\mathbf{p}_0 * (G, \mathbf{p})$ is universally rigid, then $(G, \mathbf{p})$ is universally rigid. In particular, the proof of the proposition above relies on the invariance of dimensional rigidity with respect to projective transformations. This invariance does not hold for universal rigidity as having edge directions on a conic at infinity is not invariant with respect to projective transformations! Following the example of [8, Fig. 8], the ladder in $\mathbb{R}^2$ as in Fig. 8 is not universally rigid, since it

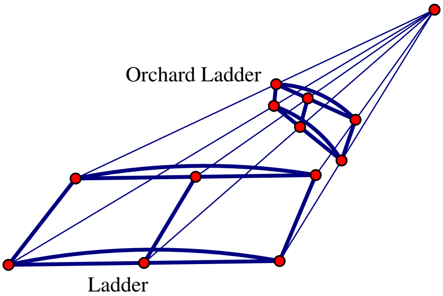


Fig. 8 Two frameworks that are projective images of each other, one universally rigid, the other not

has an affine flex, but the cone over the ladder in $\mathbb{R}^3$ is universally rigid, since it has a section, the orchard ladder which is universally rigid.

But when we specialize to complete bipartite graphs, examples such as with Fig. 8 can be ruled out, and we note the following corollary (which we will not need elsewhere in this paper).

Corollary 7.4 *In Proposition 7.3, if we assume in addition that the graph $G = K(n, m)$ is a complete bipartite graph, each of $\mathbf{p}$ and $\mathbf{q}$ span $\mathbb{R}^d$, and $\mathbf{p}_0 * (K(n, m), (\mathbf{p}, \mathbf{q}))$ is universally rigid, then $(K(n, m), (\mathbf{p}, \mathbf{q}))$ is universally rigid.*

Proof By Proposition 7.3 since $\mathbf{p}_0 * (K(n, m), (\mathbf{p}, \mathbf{q}))$ is universally rigid, $(K(n, m), (\mathbf{p}, \mathbf{q}))$ is dimensionally rigid, and Lemma 4.1 implies that $(K(n, m), (\mathbf{p}, \mathbf{q}))$ is universally rigid. $\square$

7.2 Projections and Cross-Sections

Suppose that $\tilde{V} \subset V$ is a subset of the vertices of a graph G, which induces a subgraph $\tilde{G}$ where the edges $E(\tilde{G}) \subset E(G)$. $\tilde{V}$ thus induces $\tilde{\mathbf{p}}$, a subconfiguration of the points $\mathbf{p}$. This gives us $(\tilde{G}, \tilde{\mathbf{p}})$, a subframework of $(G, \mathbf{p})$.

Lemma 7.5 *Suppose that $(\tilde{G}, \tilde{\mathbf{p}})$ is a universally rigid subframework of $(G, \mathbf{p})$ in $\mathbb{R}^d$, where the dimension of the affine span of $\mathbf{p}$ is d, and the dimension of the affine span of $\tilde{\mathbf{p}}$ is $\tilde{d} < d$. Suppose further that for each vertex not in $\tilde{V}$, the dimension of the affine span of its neighbors in $\tilde{\mathbf{p}}$ is $\tilde{d}$-dimensional. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-\tilde{d}}$ be the orthogonal projection that projects all the points of $\tilde{\mathbf{p}}$ to a single point, say $\mathbf{p}_0$.*

Then $(G, \mathbf{p})$ is universally rigid (respectively dimensionally rigid) if and only if $\mathbf{p}_0 * (G, \pi(\mathbf{p}))$ is universally rigid (respectively dimensionally rigid).

Proof We are regarding $\mathbb{R}^d = \mathbb{R}^{\tilde{d}} \times \mathbb{R}^{d-\tilde{d}}$ such that $\tilde{\mathbf{p}} \subset \mathbb{R}^{\tilde{d}}$. Since, for $\mathbf{p}_i$ corresponding to a vertex of $V - \tilde{V}$, the dimension of the affine span of its neighbors in $\tilde{\mathbf{p}}$ is $\tilde{d}$-dimensional, then the distance from such $\mathbf{p}_i$ to $\mathbb{R}^{\tilde{d}}$ is constant for any equivalent realization of $(G, \mathbf{p})$ fixing $\tilde{\mathbf{p}}$ and is equal to $|\mathbf{p}_0 - \pi(\mathbf{p}_i)|$. Additionally with $\tilde{\mathbf{p}}$ fixed, $\tilde{\pi}(\mathbf{p}_i)$ is fixed as well, where $\tilde{\pi} : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ is the orthogonal projection onto $\mathbb{R}^{\tilde{d}}$. Similarly, for $\{i, j\}$ an edge of $G - \tilde{G}$, $|\pi(\mathbf{p}_i) - \pi(\mathbf{p}_j)|^2 + |\tilde{\pi}(\mathbf{p}_i) - \tilde{\pi}(\mathbf{p}_j)|^2 = |\mathbf{p}_i - \mathbf{p}_j|^2$. Then the conclusion follows. $\square$

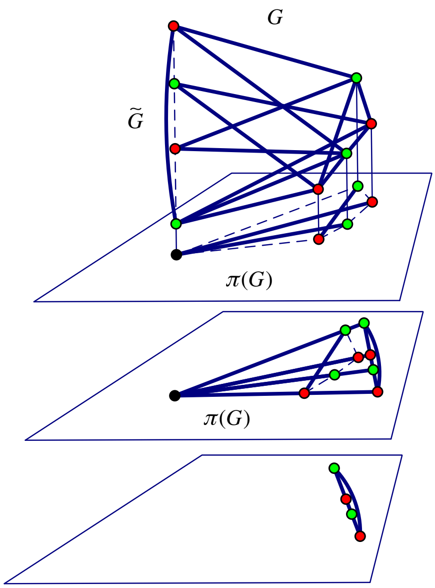


Fig. 9 A universally rigid $K(4,4)$ in 3-space and some of its projections

Figure 9 shows an example of a universally rigid $K(4,4)$ in $\mathbb{R}^3$, using Lemma 7.5 applied to $K(2,2)$ on a line in $\mathbb{R}^3$, then Lemma 7.2 and Corollary 7.4 applied to another $K(2,2)$ on a line, this time in $\mathbb{R}^2$.

8 Algorithm

We can completely test dimensional and universal rigidity of any complete bipartite framework with an efficient algorithm which we describe now. We will assume that the input coordinates $(\mathbf{p}, \mathbf{q})$ in $\mathbb{R}^d$ are given as rational numbers that can be described with L bits. Without loss of generality, we will assume that the affine span of $(\mathbf{p}, \mathbf{q})$ is d-dimensional.

Though it is true that one can also attempt to numerically gain evidence to answer this question using semidefinite programming [18], the lack of complexity results for SDP feasibility [17] makes that approach theoretically less satisfying.

At the heart of our algorithm is a routine that looks for a solution to the following set of conditions over the variables λ_i, μ_j:

$$\sum_{i=1}^{n} \lambda_i \hat{\mathbf{p}}_i \hat{\mathbf{p}}_i^t = \sum_{i=1}^{m} \mu_j \hat{\mathbf{q}}_j \hat{\mathbf{q}}_j^t,$$
$$\sum_{i=1}^{n} \lambda_i = 1,$$
$$\lambda_i \geq 0,$$
$$\mu_j \geq 0.$$

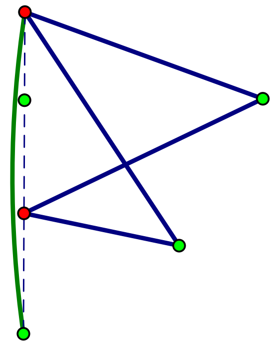


Fig. 10 A complete bipartite framework in the plane that is not universally rigid containing a universally rigid subframework

The second condition rules out the all-zero solution. This is a linear programming feasibility problem that can be exactly solved in worst case time that is polynomial in $(n+m, L)$.

Let us, for now, assume that $m+n \geq \dim \mathrm{Span}(\mathbf{p}, \mathbf{q}) + 2$. If there is no feasible solution, then from Theorem 2.2, the graph must be not dimensionally rigid. On the other hand if we find a feasible solution and *all* of the λ_i and μ_j have positive values, then we know that there is a maximum rank PSD equilibrium stress matrix on the complete bipartite framework and from Theorem 4.4, our framework must be super stable, and thus universally rigid.

Suppose though, we find a feasible solution, where some, but not all of the λ_i and μ_j have positive values. We can easily determine which λ_i and μ_j have positive values, and we will know that there is a maximum rank PSD equilibrium stress matrix on the complete bipartite subframework over the associated $\mathbf{p}_i$ and $\mathbf{q}_j$. From Theorem 4.4, this subframework must be super stable and universally rigid. But what can we say about the complete input framework?

For example, Fig. 10 shows a 2-dimensional framework, where our linear program will find a certifying PSD stress for the collinear $K(2,2)$ subframework. But the vertices that are not on this common line are free to flex continuously in three dimensions.

The idea of our algorithm is to proceed by recording the indices of our "already known to be universally rigid subframework", and then to apply ideas from Sect. 7, to reduce our problem down to a smaller problem. Roughly speaking, we will project the known universally subframework down to a cone point, slide the remaining points into a common hyperplane, and finally remove the cone point (see e.g. Fig. 9). We can we can then apply our linear programming approach to this smaller problem. Infeasibility of the smaller problem will imply that the smaller problem is not dimensionally rigid and so too is the original framework, and we can exit. A feasible solution for the smaller problem will allow us to add even more vertices to the known universally rigid set. We iterate this process until we either exit due to infeasiblity, or we account for all of the vertices and conclude that our input framework is universally rigid, or we end up with a smaller problem where the number of vertices is exactly one more than the dimension of their affine span, in which case the smaller problem, as well as the original framework is dimensionally but not universally rigid.

We note that after the first stage, due to the geometric projection and sliding operations (both which can be determined using a linear system) the input to our linear program may require polynomially more bits than our original input size, L. But in each stage of the iteration, we perform these geometric operations anew starting with the input $(\mathbf{p}, \mathbf{q})$ data, so this avoids any cascading blowup in bit complexity.

8.1 Details

In this algorithm, let $\tilde{V} \subset V$ be an index set recording the vertices of some complete bipartite subframework of $(K(n, m), \mathbf{p}, \mathbf{q})$ in $\mathbb{R}^d$, which has already been determined to be universally rigid. We will refer to this subframework as the "known-UR set". The known-UR set begins as empty. During the algorithm, the known-UR set will also maintain the "invariant" property that the affine span of its $\mathbf{p}$-subset agrees with that of its $\mathbf{q}$-subset.

The complement of the known-UR set is denoted as V', describing a complete bipartite subgraph $K(n', m')$. Suppose the complement is empty, then the known-UR set is the entire $(\mathbf{p}, \mathbf{q})$, and thus $(K(n, m), \mathbf{p}, \mathbf{q})$ is universally rigid. The same is true (due to the invariant) if the complement consists of a single $\mathbf{p}$, or a single $\mathbf{q}$, or one $\mathbf{p}$ and one $\mathbf{q}$ (which must be connected by an edge).

Our algorithm is the following:

RigidityTest($\mathbf{p}, \mathbf{q}$)
    $\tilde{V} := \{\}$
    repeat
        $V' := V - \tilde{V}$
        if ($\#(V'_{\mathbf{p}}) \leq 1$ and $\#(V'_{\mathbf{q}}) \leq 1$)   output "universally rigid"
        $(\mathbf{p}_0, \mathbf{p}', \mathbf{q}') := \pi_{\tilde{V}}(\mathbf{p}, \mathbf{q})$
        $(\mathbf{p}'', \mathbf{q}'') := \sigma_{\mathbf{p}_0}(\mathbf{p}', \mathbf{q}')$
        if (dimspan($\mathbf{p}'', \mathbf{q}''$) = $(\#(V') - 1)$)   output "dimensionally rigid"
        $S :=$ findSuperStableSubframework($\mathbf{p}'', \mathbf{q}'', V'$)
        if ($\#(S) = 0$)   output "not dimensionally rigid"
        $\tilde{V} := \tilde{V} \cup S$
        $\tilde{V} :=$ affineClosure$_{\mathbf{p},\mathbf{q}}(\tilde{V})$

Let us denote the dimension of the known-UR set as $\tilde{d}$. The function $\pi_{\tilde{V}}(\mathbf{p}, \mathbf{q})$ performs the orthogonal projection on the points $(\mathbf{p}, \mathbf{q})$ such that the vertices of the known-UR set project to a single point in $\mathbb{R}^{d-\tilde{d}}$. We denote this single point as $\mathbf{p}_0$, and the projection of the complementary vertices as $(\mathbf{p}', \mathbf{q}')$. We can think of this result as describing a framework $\mathbf{p}_0 * (K(n', m'), \mathbf{p}', \mathbf{q}')$ of a cone over the complementary complete bipartite graph in $\mathbb{R}^{d-\tilde{d}}$. (See Fig. 9, top.)

The function $\sigma_{\mathbf{p}_0}(\mathbf{p}', \mathbf{q}')$ slides the points in $\mathbf{p}'$ and $\mathbf{q}'$ along their rays from the cone point $\mathbf{p}_0$ such that they all lie in a hyperplane that does not include the cone vertex (see Fig. 9, middle). We denote the resulting points as $(\mathbf{p}'', \mathbf{q}'')$. By discarding the cone point, we can think of this result as describing a framework $(K(n', m'), \mathbf{p}'', \mathbf{q}'')$ of the complementary complete bipartite graph in $\mathbb{R}^{d-\tilde{d}-1}$. (See Fig. 9, bottom.)

Suppose that $(\mathbf{p}'', \mathbf{q}'')$ has an affine span of maximal dimension, one less then the total number of its vertices. This, and the fact that the complementary graph is not a simplex, makes $(K(n', m'), \mathbf{p}'', \mathbf{q}'')$ dimensionally but not universally rigid. (This follows from simply counting the number of degrees of freedom vs. constraints, the fact that the framework cannot have any non-zero equilibrium stress, and an application of the main results of [4].)

Likewise $\mathbf{p}_0 * (K(n', m'), \mathbf{p}'', \mathbf{q}'')$, being coned in one higher dimension, is also of maximal dimension and not a simplex, making it dimensionally but not universally rigid. Since $(\mathbf{p}', \mathbf{q}')$ is obtained from $(\mathbf{p}'', \mathbf{q}'')$ using sliding through $\mathbf{p}_0$, then by Lemma 7.2, $\mathbf{p}_0 * (K(n', m'), \mathbf{p}', \mathbf{q}')$ too is dimensionally but not universally rigid. By Lemma 7.5, $(K(n, m), \mathbf{p}, \mathbf{q})$ is dimensionally but not universally rigid. Thus we output "dimensionally rigid".

The next step, findSuperStableSubframework, is the heart of the algorithm. Here we find a subframework of $(K(n', m'), \mathbf{p}'', \mathbf{q}'')$ such that Eq. (3.1) holds with strictly positive coefficients. As described above, this can be found by setting up an a linear programming feasibility problem. The output of this step is simply the indices of the vertices, $S \subset V'$ comprising this super stable subframework.

If S is empty, then from Theorem 1.7 $(K(n', m'), \mathbf{p}'', \mathbf{q}'')$ is not dimensionally rigid. Then by Proposition 7.3, so too is $\mathbf{p}_0 * (K(n', m'), \mathbf{p}'', \mathbf{q}'')$, then by Lemma 7.2, so too is $\mathbf{p}_0 * (K(n', m'), \mathbf{p}', \mathbf{q}')$, then by Lemma 7.5, so too is $(K(n, m), \mathbf{p}, \mathbf{q})$. Thus we output "not dimensionally rigid".

If S is not empty, then from Theorem 4.4, the subframwork of $(K(n', m'), \mathbf{p}'', \mathbf{q}'')$ induced by the vertices in S is universally rigid. As before, so is the induced subframeworks of $\mathbf{p}_0 * (K(n', m'), \mathbf{p}'', \mathbf{q}'')$ and $\mathbf{p}_0 * (K(n', m'), \mathbf{p}', \mathbf{q}')$. Finally, by Lemma 7.5, and our invariant, so is the subframework of $(K(n, m), \mathbf{p}, \mathbf{q})$ induced by $\tilde{V} \cup S$.

Likewise, from Theorem 4.4, the affine span of the S-induced subset of $\mathbf{p}''$ agrees with that of the S-induced subset of $\mathbf{q}''$. Thus the invariant is also true for the subset of $(\mathbf{p}, \mathbf{q})$ induced by $\tilde{V} \cup S$. Thus we now include these vertices in our updated known-UR set.

Finally using a trilateration argument, we can also add to $\tilde{V}$ any other the vertices that are in its affine span.

Each iteration in this algorithm always makes progress so it must terminate after at most $n + m$ steps.

In summary:

Theorem 8.1 *Given a complete bipartite framework with rational coordinates. There is a* (*weakly*) *polynomial time algorithm that determines whether or not the framework is dimensionally rigid, and whether or not it is universally rigid.*

Running this algorithm on the example of Fig. 9, will conclude in two iterations, that the framework is universally rigid. For the example of Fig. 10, in the second iteration, V' will consist of two green vertices in $\mathbb{R}^0$, making S empty, and $(\mathbf{p}, \mathbf{q})$ not dimensionally rigid.

9 Tensegrities and Further Work

An important consequence of our approach in this paper is that quite often we can replace the distance equality constraints with inequality constraints as described in Sect. 1. Each edge of the underlying graph G is designated as a cable, strut or bar depending on whether it is constrained not to increase, not to decrease or not to change length, respectively. In [8] we have shown that, in many cases, even though the given framework may not support an equilibrium stress that is non-zero for a given edge, it may still be possible to declare a given edge a cable or strut and maintain universal rigidity. Even in the case when the graph is not bipartite, Proposition 7.3 can apply as in Fig. 5, where due attention should be applied to the signs of the stresses. We do not pursue that extension of the results here, though.

Another application of our approach here is in the local rigidity theory of prestress stability as shown in [9]. There, even if the stress matrix is not PSD, it can still be useful determine local rigidity, especially when the whole framework is not infinitesimally rigid.

Another point is that the stress-energy function determined by the stress and stress matrix provides a measure of how far a given configuration is from an ideal configuration, globally. So if a configuration has some determined edge measurements, the stress-energy function gives an upper bound on how close any configuration is with those edge lengths. Indeed, with the tensegrity constraints it can be possible to eliminate certain edge lengths as feasible. For example, for six points $\mathbf{p}_1, \ldots, \mathbf{p}_6$, there is no configuration where $|\mathbf{p}_i - \mathbf{p}_{i+1}| \leq 1$, and $|\mathbf{p}_i - \mathbf{p}_{i+3}| > 2$, all taken modulo 6. This is shown using the configuration $K(3, 3)$ on a circle as in Fig. 2.

Acknowledgements The impetus for this paper is the result in [15] for $K(n, m)$ on a line. It was a desire to generalize that result, which was the starting point for this paper. The elephant in the room is the paper by E. Bolker and B. Roth [5]. This paper was constantly in the background leading us to what was true and what was not. It gives a reasonably complete picture of which configurations of complete bipartite graphs are infinitesimally rigid. Also, one can see stress matrices there quite naturally. Their basic tool was the tensor product of a vector with itself, where instead we think of it as using the Veronese map. Other work we did not formally use, but is still lurking in the background, is the very insightful paper [20] by W. Whiteley. The idea there is that an infinitesimal flex $\mathbf{p}'$ of a bipartite framework with corresponding configuration $\mathbf{p}$ on a quadric can be easily described. Furthermore, the two configurations $\mathbf{p} + \mathbf{p}'$ and $\mathbf{p} - \mathbf{p}'$ describe equivalent frameworks. Thus they are not even globally rigid, and they are separated by a quadric surface. This is the basis in [7] to show that $K(5, 5)$ is not globally rigid (thus not universally rigid) in $\mathbb{R}^3$. But on the other hand, there are many examples of complete bipartite graphs in any $\mathbb{R}^d$ that are globally rigid, but not universally, as we have shown here. The main result of [12] applied to complete bipartite graphs, shows that when the configuration is generic, the rank and positive semi-definiteness of the stress matrix determines when the configuration is universally rigid. What we have done here, for complete bipartite graphs, is to replace the condition of being generic, which is problematic to determine in general, with the more precise condition of being in quadric general position in Corollary 4.8. We would like to thank Deborah Alves whose experiments kept on suggesting the correctness of Theorem 4.4, long before we knew how to prove it. This work was partially supported by NSF Grants DMS-1564493 and DMS-1564473.

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