The Isostatic Conjecture

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Abstract

We show that a jammed packing of disks with generic radii, in a generic container, is such that the minimal number of contacts occurs and there is only one dimension of equilibrium stresses, which have been observed with numerical Monte Carlo simulations. We also point out some connections to packings with different radii and results in the theory of circle packings whose graph forms a triangulation of a given topological surface.

Keywords Packings · Square torus · Density · Granular materials · Stress distribution · Koebe · Andreev · Thurston

1 Introduction

Granular material, made of small rocks or grains of sand, is often modeled as a packing of circular disks in the plane or round spheres in space. In order to analyze the internal stresses that resolve external loads, there is a lot of interest in the distribution of the

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stresses in that material. See, for example, [18–20,29]. A *self stress* is an assignment of scalars to the edges of the graph of contacts such that at each vertex (the disk centers) there is a vector equilibrium maintained. One property that has come up in this context is that, when a packing is jammed in some sort of container, there is necessarily an internal self stress that appears. It seems to be taken as a matter of (empirical) fact that when the radii of the circles (or spheres) are chosen generically, there is only one such self stress, up to scaling. In that case, one says that the structure is *isostatic*. This statement seems to be borne out in many computer simulations, since it is essentially a geometric property of the jammed configuration of circular disks. See, for example, the work of Roux [32], Atkinson et al. [3,4,9]. When the disks all have the same radius, and thus are non-generic, for example, it quite often turns out that the packing is not isostatic. Here, we refer to the (mathematical) statement that when the radii and lattice are generic, the packing has a single stress up to scaling, as the *isostatic conjecture*. Note that when the radii of the packing disks are chosen generically, this does not imply that the coordinates of the configuration of the centers of the disks are generic. There is a wide literature on the rigidity of frameworks, when the configuration is generic, for example from the basic results in the plane starting with Laman [26], and more generally as described in Asimow and Roth [1,2]. But if the graph of the packing has any cycle of even length, the corresponding edge lengths between the vertex centers will not be generic since the sum of the lengths of half of the edges will be the same as the sum of the lengths of the other half of the edges. The configuration of the centers will not be generic either, since if they were, the edge lengths of a cycle would also be generic.

There are many different instances when the isostatic conjecture could be posed. For example, one could enclose a collection of disks with fixed radii inside a polygon and squeeze the shape of the polygon until the packing inside is jammed. It can happen that there is an occasional disk that is not fixed to the others, which we call a *rattler*, and in that case we ignore it, since it does not contribute to the self stress lying in the packing. See also [3,4] for the effect of rattlers on the density. Another example, and one that we will investigate in our study, is when a packing is periodic with a given lattice determining its overall symmetry. We then increase the radii uniformly, keeping a fixed ratio between every pair of radii, maintaining the genericity. In dimension three and higher, we do not provide any method to prove that jammed packings are isostatic. We rely very heavily on two-dimensional techniques.

Another interesting aspect of the ideas here is that we connect some of the principles of the rigidity theory of jammed packings as in the work of Dickinson et al. [11], Musin and Nikitenko [11,12,30], with another theory of analytic circle packings as in the book of Stephenson [33]. There are essentially two seemingly independent methods of creating a circle packing. One is created by modeling the disks as having fixed ratios and increasing the packing density until they jam, and the other is based on an idea that goes back to at least Koebe, Andreev and Thurston [35], where the graph of the packing is a predetermined triangulation. Here we use an extension by Guo [22], where some of the distances between disks are determined by an inversive distance, defined later, between circles.

Fig. 1 This is a maximally dense packing of seven disks in a square torus, with a rattler in the middle, found by Musin and Nikitenko [30]. The thin horizontal and vertical lines outline fundamental regions of the torus
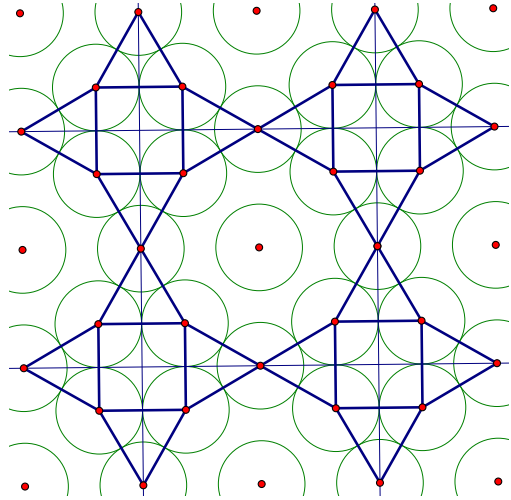
2 Rigidly Jammed Circle Packings

We need a container for our packings. For the sake of simplicity and because of the lack of boundary effects, we will use the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ regarded as the Euclidean plane $\mathbb{R}^2$ modulo the fixed integer lattice Λ as the container. A packing **P** in $\mathbb{T}^2$ is a finite union of labeled circular disks with disjoint interiors. We say that **P** is *locally maximally dense* if there is an $\epsilon > 0$ such that for any other packing **Q** with corresponding radii in the same ratio and $|\mathbf{Q} - \mathbf{P}| < \epsilon$, one has $\rho(\mathbf{Q}) \leq \rho(\mathbf{P})$. The density of **P** is $\rho(\mathbf{P}) = \sum_i A(D_i)/A(\mathbb{T}^2)$, where $A()$ is the usual area function. The distance between packings is regarded as the distance between the vectors of the centers and radii of the packing disks. In other words, near the packing **P**, except for translates, we cannot increase the packing radii uniformly and maintain the packing constraints.

A first process is that we can "inflate" the packing disks uniformly until some subset of the packing disks jam and prevent any other expansion. This process is called a "Monte Carlo" method in Torquato and colleagues [16,17]. Let $\mathbf{r} = (r_1, \ldots, r_n)$ be the radii of the corresponding packing disks $\mathbf{P} = (D_1, \ldots, D_n)$. The idea is to continuously increase the radii to for $t > 1$, and at the same time continuously deform the packing to $\mathbf{P}(t)$ so that the radius of D_i is tr_i, until no further increase in t is possible. Then the resulting packing $\mathbf{P}(t_1)$ will be locally maximally dense. We would like to say that $\mathbf{P}(t_1)$ is rigid or jammed. But there is a problem with rattlers as in Fig. 1. Considering the rigidity of the packing, we just discard the rattlers.

We need some tools to determine the rigidity of packings. Given a packing **P** in a torus, define a packing graph $G(\mathbf{P})$, where vertices are the centers of the packing disks, and edges connect centers whose disks touch, as in Fig. 1. Note that in the torus $G(\mathbf{P})$ may have loops and multiple edges, but in all our calculations, we work in the universal cover where $G(\mathbf{P})$ has no loops or multiple edges. We are effectively working with the equivalence classes of lattices $\mathbf{p}_i + \Lambda$ in the plane. When there is an (oriented)

edge joining two vertices of $G(\mathbf{P})$ this can always be represented as a well-defined vector $\mathbf{p}_i - \mathbf{p}_j$ in the plane $\mathbb{R}^2$.

Let $\mathbf{p}_i$ be the center of disk D_i. Let $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$ be a corresponding sequence of vectors, where $\mathbf{p}'_i \in \mathbb{R}^2$ is in the tangent space of $\mathbb{T}^2$ at $\mathbf{p}_i$. We say that $\mathbf{p}'$ is an *infinitesimal flex* of $G(\mathbf{P})$, if for each edge $\{i, j\}$ of G,

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0. \tag{2.1}$$

We say that $\mathbf{p}'$ is a *trivial infinitesimal flex* of $G(\mathbf{P})$, if all the $\mathbf{p}'_i$ are the same vector, for $i = 1 \ldots, n$. We say that a packing $\mathbf{P}$ is *(locally) rigid* or *collectively jammed* if the only continuous motion of the packing (preserving the radii) is by translations. We say that a packing $\mathbf{P}$ is *infinitesimally rigid* if every infinitesimal flex is trivial.

Theorem 2.1 *A packing $\mathbf{P}$ in a torus $\mathbb{T}^2$ is collectively jammed if and only if it is infinitesimally rigid. Furthermore $\mathbf{P}$ is locally maximally dense if and only if there is a subset of the packing that is collectively jammed.*

We call the packing disks, minus the rattlers, a *spine* of the jammed packing.

A proof of this statement and Corollary 3.3, later, can be found in [9,17]. The idea goes back at least to Danzer [13], and it works for all dimensions for all compact surfaces of constant curvature, except, interestingly, the proof of the "only if" part fails for surfaces of positive curvature, such as the sphere. The infinitesimal flex can be used to define a continuous flex for the whole configuration and the higher-order terms work in our favor. The "if" part is standard.

Another closely related property is when the lattice Λ itself is allowed to move. Following [17], we say that a packing in a torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ is *strictly jammed* if it is rigid allowing both the configuration and the lattice Λ to move locally with the constraint that the total area of $\mathbb{T}^2$ not increase. Here we will concentrate mostly on collective jamming.

3 Basic Rigidity of Tensegrities

The rigidity part of Sect. 2 can be rephrased in the language of frameworks and tensegrities. One is given a finite configuration of points $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$, in our case in the 2-torus $\mathbb{T}^2$, and a *tensegrity graph* G, where each edge of G is defined to be a *cable*, which is not allowed to increase in length, or a *strut*, which is not allowed to decrease in length, and a *bar*, which is not allowed to change in length. In our case the packing graph consists entirely of struts.

The next tool we need is the concept of a *stress* for the graph, which is just a scalar $\omega_{ij} = \omega_{ji}$ assigned to each edge $\{i, j\}$ of G. We say that a stress $\omega = (\ldots, \omega_{ij}, \ldots)$ is an *equilibrium stress* if for each vertex i of G the following holds

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0,$$

where $\omega_{ij} = 0$ for non-edges $\{i, j\}$. Furthermore, we say that a stress for a graph G is a *strict proper stress* if $\omega_{ij} > 0$ for a cable $\{i, j\}$, and $\omega_{ij} < 0$ for a strut $\{i, j\}$. There is no condition for a bar. The following is a basic duality result of Roth and Whiteley [31].

Theorem 3.1 *A tensegrity is infinitesimally rigid if and only if the underlying bar framework is infinitesimally rigid and there is strict proper equilibrium stress.*

Since infinitesimal rigidity involves the solution to a system of linear equations and inequalities, we have certain relationships among the number of vertices of a tensegrity, say n, the number of edges, e, and the dimension of the space of equilibrium stresses s. Recall that the dimension of the space of trivial infinitesimal flexes is 2, given by translations in the torus.

Proposition 3.2 *For an infinitesimally rigid tensegrity on a torus $\mathbb{T}^2$ with all struts,*

$$e \geq 2n - 1 \quad and \quad s = e - (2n - 2).$$

Corollary 3.3 *If a packing $\mathbf{P}$ in a torus $\mathbb{T}^2$ is collectively jammed with n disks and k contacts, then $k \geq 2n - 1$, and further when $\mathbf{P}$ is collectively jammed, it has exactly one stress if and only if $k = 2n - 1$.*

The idea is that there are $2n$ variables describing the configuration of the disk centers. One packing disk can be pinned to eliminate trivial translations, and at least one extra constraint must be added to insure a self stress, $2(n-1) + 1 = 2n - 1$ constraints, corresponding to contacts altogether.

With isostatic packings, the stress space is one-dimensional, assuming the packing is rigid, not counting rattlers, if and only if $e = 2n - 1$. In the granular material literature, this situation is called *isostatic*. However, in the mechanical engineering literature a bar tensegrity (framework) is called isostatic if it is infinitesimally rigid and it has no non-zero equilibrium stress, because when the framework is subjected to an external load, it can "resolve" that load with a unique single internal stress. So for packings, the isostatic conjecture is that when the packing disks are sufficiently generic, then there is only a one-dimensional equilibrium stress and $e = 2n - 1$.

Notice that the packing of the six disks in Fig. 1, with the rattler missing, has $12 = 2 \cdot 6$ edges and so is not isostatic as we have defined it above. All the disks have the same radius, and so are not generic. By contrast, the packing graph of two disks in Fig. 2 in a torus with a slanted lattice has $2 \cdot 2 - 1 = 3$ edges and when it is jammed, and so it is isostatic. On the other hand, the packing in Fig. 3 in the torus defined by a rectangular lattice has four contacts and is not isostatic even though the ratio of the radii are generic and the lattice has one free parameter in the space of rectangular lattices. So in this case, the isostatic conjecture is false, if one insists on choosing that subset of the possible lattices. The moral of this story is that all the lattice parameters should be included in the generic condition.

Fig. 2 The slanted torus, with an isostatic packing
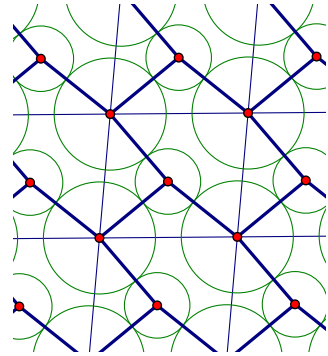
Fig. 3 The rectangular torus, with a non-isostatic packing
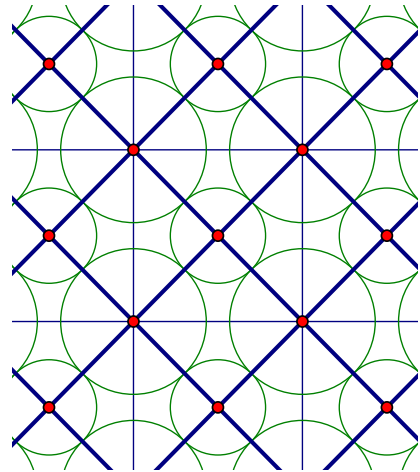
4 Coordinates

In order to do calculations later, we will describe the lattice, radii, and configuration in terms of canonical coordinates.

(a) The lattice $\Lambda = \{z_1\lambda_1 + z_2\lambda_2\}$ where $\lambda_1 = (a, 0)$, $\lambda_2 = (b, c)$, and z_1, z_2 are integers $a, b, c > 0$. The dimension of $\mathcal{L}$ of all such lattices is 3. The dimension of such lattices with a fixed determinant, say 1, is 2.

(b) The configuration $\mathcal{C}$ is the set of $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$, where $\mathbf{p_1} = (0, 0)$, and otherwise $\mathbf{p_2}, \ldots, \mathbf{p_n}$ are free, and not constrained. Note that a point $\mathbf{p}_i$ is defined to be equivalent to the point $\mathbf{q}_i$ if $\mathbf{p}_i + \Lambda = \mathbf{q}_i + \Lambda$ as sets. The dimension of $\mathcal{C}$ is $2(n - 1) = 2n - 2$.

(c) If the radius of the ith disk is $r_i > 0$, we denote the vector of radii as $\mathbf{r} = (r_1, \ldots, r_n)$. Later we will be interested in the relative ratios of these disk radii. So we will denote the set of ratios as $\mathcal{R} = \{(1, r_2/r_1, \ldots, r_n/r_1) \mid r_i > 0\}$. For any $\mathbf{r} = (r_1, \ldots, r_n)$, define $\bar{\mathbf{r}} = (r_2/r_1, \ldots, r_n/r_1)$. So the dimension of $\mathcal{R}$ is $n - 1$.

A packing **P** is described uniquely by all the coordinates above, where $|\mathbf{p}_i - \mathbf{p}_j| = r_i + r_j$, for all i, j. The space of all such packings will be denoted by $\mathcal{P}$. If we fix the lattice Λ, then the corresponding space of such packings will be denoted by $\mathcal{P}(\Lambda)$. Similarly if we additionally fix the radius ratios $\bar{\mathbf{r}}$, we denote that restricted packing space as $\mathcal{P}(\Lambda, \bar{\mathbf{r}})$.

5 Dimension Calculations

When we have a collectively jammed, or a locally maximally dense, packing, we would like to perturb the parameters, the radii and lattice, and still maintain that property. In the following we will assume that the lattice Λ is constant with determinant 1. So the packing is determined completely by the n centers **p** of the packing disks and **r**, the n radii of the packing disks. The pair $(\mathbf{p}, \mathbf{r})$ determine a packing uniquely if and only if for all i, j between 1 and n and $r_i > 0$,

$$|\mathbf{p}_i - \mathbf{p}_j| \geq r_i + r_j. \tag{5.1}$$

The density of $(\mathbf{p}, \mathbf{r})$ is $\rho = \rho(\mathbf{p}, \mathbf{r}) = \pi \sum_i r_i^2 = \rho(\mathbf{r})$, which only depends on **r** and is clearly continuous.

Proposition 5.1 *Suppose that* $(\mathbf{p}, \mathbf{r})$ *is locally maximally dense in a torus given by the lattice* Λ. *Let* $\epsilon > 0$ *be given. Then there is a* $\delta > 0$ *such that for any packing* $(\mathbf{q}, \mathbf{s})$ *such that* $|\mathbf{q} - \mathbf{p}| < \delta$ *and* $|\mathbf{s} - \mathbf{r}| < \delta$, *there is a locally jammed packing* $(\mathbf{q}(1), \mathbf{s}(1))$ *with* $|\mathbf{q}(1) - \mathbf{p}| < \epsilon$, *and* $\bar{\mathbf{s}} = \mathbf{s}(1)$.

Proof Fix $\bar{\mathbf{r}}(0)$ and any configuration $\mathbf{p}(0)$ such that $(\mathbf{p}(0), \mathbf{r}(0))$ is a packing, i.e., it satisfies (5.1), and such that it is collectively jammed. If there are any rattlers, we can deal with the collectively jammed subset, which we call the *spine*. So we can assume that $(\mathbf{p}(0), \mathbf{r}(0))$ is collectively jammed. Let $\mathcal{P}$ be the space of packings of n disks in the Λ torus given by $(\mathbf{p}, \mathbf{r})$, which corresponds to $(\mathbf{p}, s, \bar{\mathbf{r}})$, where $s = r_1$.

We define a *constraint space* $\mathcal{E} = ((\ldots l_{ij} \ldots), (r_1, r_2, \ldots, r_n))$, where l_{ij} are positive real variables that correspond to edges of the contact graph on n vertices, and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ are also positive real variables that correspond to the disks in the packing. We define a set $E \subset \mathcal{E}$ defined by the following constraints:
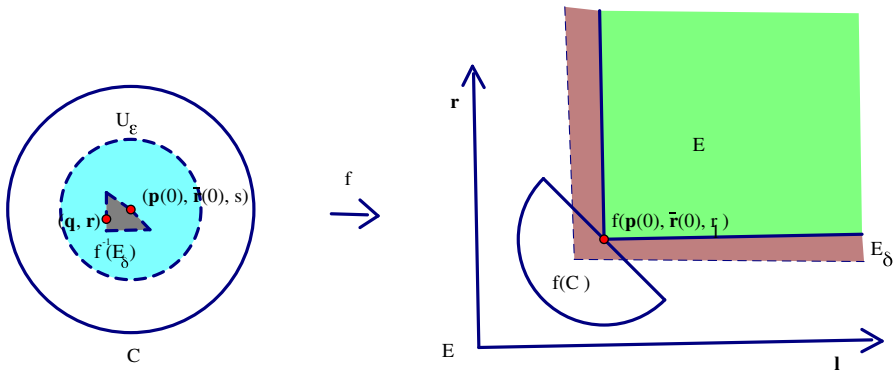
$$l_{ij} \geq s(\bar{r}_i + \bar{r}_j), \tag{5.2}$$
$$\bar{\mathbf{r}} = \bar{\mathbf{r}}(0), \tag{5.3}$$
$$\rho(\mathbf{r}) \geq \rho(\mathbf{r}(0)). \tag{5.4}$$

We then define a continuous map $f : \mathcal{P} \to \mathcal{E}$, by

$$f(\mathbf{p}, \bar{\mathbf{r}}, s) = (|\mathbf{p}_i - \mathbf{p}_j| \ldots, \bar{\mathbf{r}}, r_1).$$

Since $(\mathbf{p}(0), \mathbf{r}(0))$ is collectively jammed, there is compact neighborhood $C \subset \mathcal{P}$, the *rigidity neighborhood* such that $f^{-1}(E) \cap C$ is just $\{(\mathbf{p}(0), r_1, \bar{\mathbf{r}}(0))\}$, that is the packing given by $(\mathbf{p}(0), \mathbf{r}(0))$.

Fig. 4 This shows the argument in the proof of Proposition 5.1. The vertical direction in this figure also represents an increase in density ρ

Define the δ neighborhood of E, E_δ by the conditions

$$l_{ij} > s(\bar{r}_i + \bar{r}_j) - \delta, \tag{5.5}$$

$$|\bar{r} - \bar{r}(0)| < \delta, \tag{5.6}$$

$$\rho(\mathbf{r}) > \rho(\mathbf{r}(0)) - \delta. \tag{5.7}$$

Note that conditions (5.5), (5.6), (5.7) correspond to a slackened versions of (5.2), (5.3), (5.4), respectively. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that $f(\mathbf{q}, \bar{r}, s) \in E_\delta$ implies that $|(\mathbf{p}(0), \mathbf{r}(0)) - (\mathbf{q}, \mathbf{r})| < \epsilon$. That is $f^{-1}(E_\delta) \cap C \subset U_\epsilon$, the ϵ neighborhood of $(\mathbf{p}(0), \mathbf{r}(0))$. This is due to C being compact and E closed. See [8, Thm. 1], for a similar argument.

Next start with any packing $(\mathbf{q}, \mathbf{r}) \in f^{-1}(E_\delta) \cap C \subset U_\epsilon$ that maps to E_δ and continuously increase its density $\rho(\mathbf{r})$ fixing $\bar{r}$ until it reaches a local maximum, where it becomes locally maximally dense. During this process, the packing will always satisfy (5.5)–(5.7), and therefore be a local maximally dense packing $(\mathbf{q}(1), \mathbf{r}(1))$, remaining in U_ϵ. See Fig. 4 for a visualization of this process. $\square$

Corollary 5.2 *Under the same assumptions as Proposition* 5.1, *for any radius ratio in* $\mathcal{R}$ *within* δ *of* $\bar{r}$, *there is a locally maximally dense packing* $(\mathbf{q}, \mathbf{s})$ *such that* $\bar{s}$ *is that radius ratio, and* $\mathbf{q}$ *is within* δ *of* $\mathbf{p}$. *Furthermore, the lattice can be perturbed by* δ *as well, with the same conclusion.*

Proof Apply Proposition 5.1 to the packing $(\mathbf{p}, \mathbf{r}')$, where all the radii of $\mathbf{r}'$ are strictly smaller than those of $\mathbf{r}$ but $\bar{r}'$ still close to $\bar{r}$, which is a feasible packing, but such that the ratios are the given ratio in $\mathcal{R}$. Similarly, one can alter the lattice by a sufficiently small amount and apply the same limiting argument to the altered lattice converging to Λ. $\square$

Let $X(n, CJ)$ be the space of collectively jammed packings with n disks. Note that this space is quantified over all lattices, and dimension of such lattices, by the

definition (a), is two, where packings are identified with the configuration $\mathbf{p}$, and radii $\mathbf{r}$. We assume that there is at least one collectively jammed packing.

Corollary 5.3 *The dimension of $X(n, CJ)$ is $n + 1$.*

Notice that the perturbed packing $(\mathbf{q}, \mathbf{s})$ in the proof of Proposition 5.1 may loose some packing contacts from the original and even possibly create some rattlers as in Figs. 10 and 11 in the tricusp case. In that case, such packings, being not collectively jammed, are not in the set $X(n, CJ)$ and, being of lower dimension than $n + 1$, do not contribute to the dimension of $X(n, CJ)$. Note that the area of the torus corresponding to the lattice Λ is the determinant of the matrix defining Λ, which is just ac from the definition in Sect. 4. With this generality we have the following.

Theorem 5.4 *The dimension of the space of packing radii for locally maximally dense packings in a neighborhood of a fixed collectively jammed configuration with a lattice in the neighborhood of that fixed lattice, and radius ratios in the neighborhood of those fixed ratios, with n disks is $n + 1$, modulo rattlers.*

Proof By Proposition 5.1, for each configuration $\mathbf{p}$, radius $\mathbf{r}$ and lattice Λ that is collectively jammed, and therefore locally maximally dense, there is a locally maximally dense packing $(\mathbf{q}, \mathbf{s}, \Lambda)$. Each choice of Λ and $\bar{\mathbf{r}}$ has a distinct locally maximally dense packing modulo rattlers. There are $n - 1$ choices for radius ratios and 2 choices for lattices, $n + 1$ in all. □

Here the rattlers are counted as not contributing to the dimension of the space of packings. It is as if they were stuck to the rest of the packing. If they were counted, then each rattler would add 2 degrees of freedom to their configuration space. If the given packing has rattlers they will contribute the same degrees of freedom to each of the approximations, and they can be disregarded. If the given packing has no rattlers, it can happen that some of the approximation packings could themselves have rattlers. But we will see that this cannot happen when the given packing is isostatic. Next we suppose that the packing is approximated by another with the same graph.

Proposition 5.5 *Let $\mathbf{P} \in \mathcal{P}$ be any collectively jammed packing with configuration $\mathbf{p}$, radii $\mathbf{r}$, and lattice Λ. Then there is an $\epsilon > 0$ such that for any other packing $\mathbf{Q} \in \mathcal{P}$ where $|\mathbf{P} - \mathbf{Q}| < \epsilon$, and the packing graph of $\mathbf{P}$ is the same as the packing graph of $\mathbf{Q}$, then $\mathbf{Q}$ is collectively jammed as well.*

Proof By Theorem 2.1, $\mathbf{P}$ is collectively jammed if and only if it is infinitesimally rigid. As before, if there is no ϵ as in the statement, there is a sequence of $\mathbf{Q}_j$, $j = 1, 2, \ldots$, converging to $\mathbf{P}$, each with its own configuration $\mathbf{q}(j)$ and infinitesimal flex $\mathbf{q}'(j)$ satisfying the infinitesimal rigidity constraint (2.1). By renormalizing we can assume that $|\mathbf{q}'(j)| = 1$. So as the $\mathbf{q}(j)$ converge to $\mathbf{p}$, and by taking a subsequence, the $\mathbf{q}'(j)$ converge to a non-zero (and thus non-trivial by the conventions in Sect. 4) infinitesimal flex of $\mathbf{p}$. Thus there is such an epsilon as in the conclusion. □

Suppose that a packing has contact graph G, and define $X(n, CJ(G)) \subset X(n, CJ)$ as the set of collectively jammed packings with the given contact graph G.

Corollary 5.6 *In a sufficiently small neighborhood of a packing* **P** *with contact graph G, suppose that no contacts are lost in the space of collectively jammed packings. Then the dimension of $X(n, CJ(G))$ is $n + 1$.*

Notice that the statements here do not depend on the two-dimensional analytic theory that we describe in Sect. 6, and indeed there are higher dimensional statements that we will not go into detail here. Notice, also, that the results in this section hold in any higher dimension with appropriate adjustment for the dimension of the space of lattices in Theorem 5.4.

6 Analytic Theory of Circle Packings

We need to first do some bookkeeping as far as the topology of graphs on the surface of $\mathbb{T}^2$. One of the first results of Andreev and Koebe was to start with a triangulation of a surface, say a torus, and then create a circle packing whose graph is that triangulation. This is explained in careful detail in Stephenson's book [33]. Note that this pays no attention to the radii of the packing. So suppose that there are n circles in a triangulated packing, with e_T edges, and T triangles. Since each edge is adjacent to two triangles and each triangle is adjacent to three edges, we have

$$3T = 2e_T,$$

and since the Euler characteristic of $\mathbb{T}^2$ is 0, we get that

$$n - e_T + \frac{2}{3} e_T = 0, \quad \text{and} \quad e_T = 3n.$$

In our case, we usually do not have a complete triangulation of the torus. Indeed, from Sect. 3, we only have $e = 2n - 1$ edges if the packing is isostatic. But our packing graph is embedded in $\mathbb{T}^2$, and it can be completed to a triangulation by adding $e_T - e = n + 1$ additional edges. But this does not help us since we do not want the extra contacts of a triangulation. For example, in Fig. 5 we see an isostatic packing of three disks in a square torus with $2 \cdot 3 - 1 = 5$ edges, where $3 + 1 = 4$ additional edges have been inserted to create a triangulation of the torus. Notice that there are multiple edges between some pairs of vertices, but in the universal cover, as shown, the edges form an actual triangulation of the plane.

There has been a lot of interest in circle arrangements, where some pairs of circles are forced to intersect at certain specified angles, generalizing the Koebe–Andreev–Thurston result, where they are made to be tangent. At the other end of that construction, there is a way to measure the distance between pairs of non-tangent circles with disjoint interiors. The *inversive distance* between two circles is defined as

$$\sigma(D_1, D_2) = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2 - (r_1^2 + r_2^2)}{2r_1 r_2},$$

Fig. 5 This is a maximally dense packing of three equal disks in a square torus [15] with a minimal number of contacts, namely five, and so it is isostatic. We have inserted four additional dashed edges to create a triangulation with nine edges total
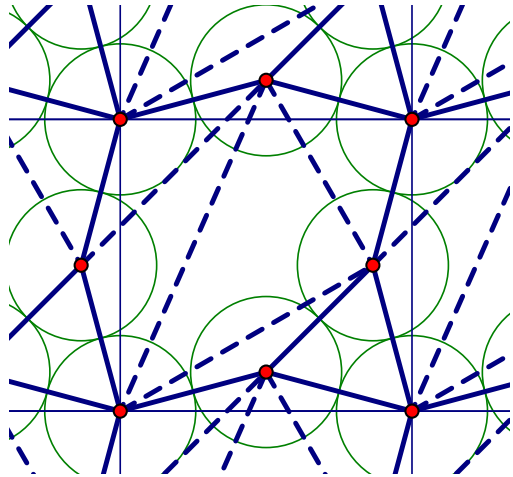
where D_1 and D_2 are disks with corresponding radii r_1 and r_2. It does not seem that there is a proof known, where any set of inversive distances imply the existence of a configuration with those inversive distances, although a result of Luo [27] shows its global uniqueness. But the following local result by Guo [22] is enough for our purposes.

Theorem 6.1 *Let $\mathcal{T}$ be a triangulation of a torus corresponding to a circle packing $\mathbf{P}$ where $\sigma(D_i, D_j) \geq 0$ is the inversive distance between each pair of disks i, j that are an edge in the triangulation $\mathcal{T}$. Then the inversive distance packings are locally determined by the values of $\sigma(D_i, D_j)$.*

Corollary 6.2 *The dimension of $X(n, CJ(G))$ is no greater than $3n - k$, where the number of contacts in the graph G is k.*

Proof By Theorem 6.1 the dimension of all packings with G as the contact graph is no greater than $3n - k$ and $X(n, CJ(G))$ is a subset of those packings. □

By varying the values of the inversive distances, it is possible to show that in the neighborhood of a collectively jammed packing, a configuration that has those particular inversive distances and that the dimension of $X(n, CJ(G))$ is exactly $3n - k$. We expect to show this in a later work. One should keep in mind, though, that the metric of the ambient space, which in our case is the lattice determining the torus, may change with the deformations of the inversive distance data, and this should be taken into account when doing our dimension calculations.

7 The Generic Property

If one has a collection of real numbers X, they are defined to be *generic* if there are no solutions to non-zero polynomials $p(x_1, \ldots, x_n) = 0$ with integer coefficients, where each $x_i \in X$. Each such polynomial defines an algebraic set and being generic

implies that the points of X avoid that set. But this is something of an overkill. For example, in many cases, for some given set X and situation at hand, there are only some finite number of such algebraic sets that have to be avoided, but it may be difficult to explicitly define what the particular polynomials are.

Another way to think of the generic parameters $x_1, \ldots, x_n$ is as independent variables satisfying no polynomial relations over the integers (or equivalently the rationals). In our case, the independent parameters are the ratios of distinct radii of the packing disks.

8 The Isostatic Theorem

Theorem 8.1 *If a collectively jammed packing* $\mathbf{P}$ *with n vertices in a torus* $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ *is chosen so that the ratio of packing disks* $\bar{\mathbf{r}}$, *and torus lattice* Λ, *is generic*, *then the number of contacts in* $\mathbf{P}$ *is* $2n - 1$, *and the packing graph is isostatic.*

Proof Restrict to a sufficiently small neighborhood of a collectively jammed packing $\mathbf{P}$. Any restriction on the number of contacts in the space $X(n, CJ)$ of collectively jammed packings constitutes an algebraic or semi-algebraic subset, and corresponds to a constraint in the $\bar{\mathbf{r}}$ and lattice variables unless it defines (an open subset of) the whole space $X(n, CJ)$. This is because there is a natural projection from collectively jammed packings $(\mathbf{p}, \mathbf{r}, \Lambda)$ to $(\bar{\mathbf{r}}, \Lambda)$, where Λ has determinant 1, the dimension of $X(n, CJ)$ in $n+1$, and the dimension of the $(\bar{\mathbf{r}}, \Lambda)$ is also $n+1$. Therefore for generic $(\bar{\mathbf{r}}, \Lambda)$, where Λ has determinant 1, we may assume the number of contacts in the neighborhood of $\mathbf{P}$ is constant. By Corollaries 6.2, 5.6, 5.3, for packings with n vertices and k contacts that locally have a constant number of contacts among collectively jammed packings and thus have constant graph G, the following holds:

$$3n - k \geq \dim(X(n, CJ(G))) = \dim(X(n, CJ)) = n + 1.$$

Thus $3n - (n+1) = 2n - 1 \geq k$. By Corollary 3.3 such packings are isostatic. If the radii ratios $\bar{\mathbf{r}}$ and the lattice variables are generic, and the packings in the neighborhood of $\mathbf{P}$ are collectively jammed with a constant number of contacts, then those packings are isostatic, as was to be shown. $\square$

One could do a similar calculation for the case when each packing is assumed to be strictly jammed, allowing the lattice to vary. Then the number of contacts in this case is $2n + 1$ instead of $2n - 1$.

It is also possible to extend Theorem 8.1 to cases when the number of free variables used to define a generic $\bar{\mathbf{r}}$ and lattice are less than $n + 1$, where n is the number of disks. For example, suppose the number of free lattice variables is one instead of two. We can show, if the $\bar{\mathbf{r}}$ and lattice variables are otherwise generic, then the number of contacts is at most $2n$ instead of $2n - 1$. An example of this is shown in Fig. 3, where $n = 2$, the number of free lattice variables is 1 instead of 2 and there is just one radius ratio. The number of contacts in a generic case is 4 instead of 3. This is the only example we know so far though.

Fig. 6 The most dense packing of five disks in the square torus
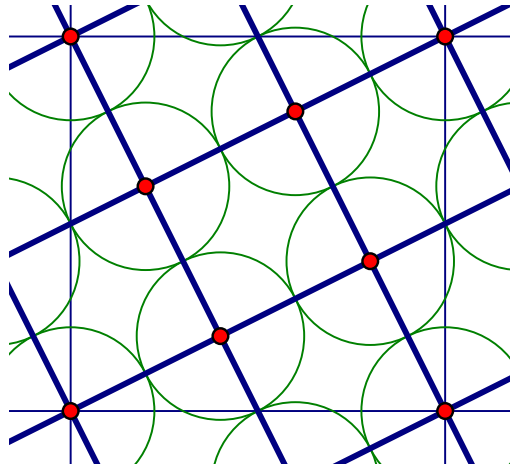
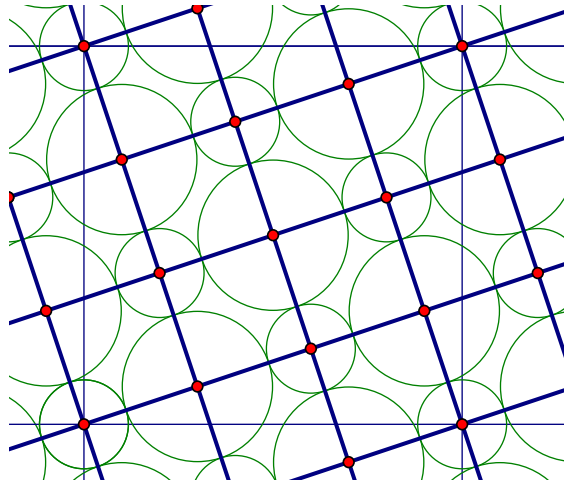**Fig. 7** An isostatic packing of five disks with generic radii in the square torus
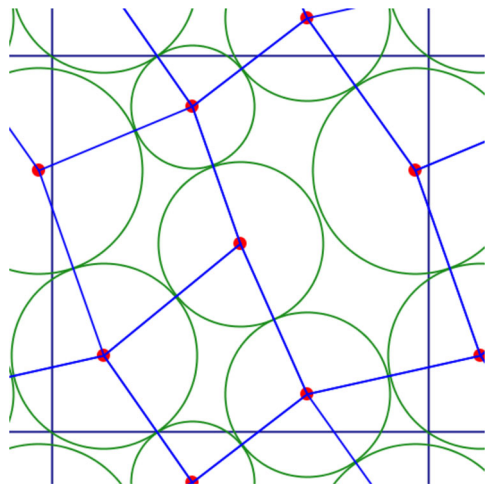

9 Computations

Will Dickinson et al. [15] showed that the most dense packing of five equal circles in a square torus is when they form a grid as in Fig. 6. This packing is not isostatic since it clearly does not have just a one-dimensional equilibrium stress, but two. It has $2 \cdot 5 = 10$ contacts, one more than needed for rigidity. When the disk radii are perturbed, we get the packing in Fig. 7, which is isostatic with nine contacts even though the defining lattice is still the square lattice.

Similarly, for 10 disks in a square torus, even when there are two different radii, a rigid jammed packing may not be isostatic as in Fig. 8. When the radii are sufficiently varied, Fig. 9 shows how one of the contacts breaks, obtaining another isostatic packing. Note that in these packings, we still get an isostatic packing without having to perturb the underlying square lattice. Indeed, it is tempting to propose that if the

**Fig. 8** A packing of 10 disks in the square torus

**Fig. 9** An isostatic packing of 10 disks with generic radii in the square torus
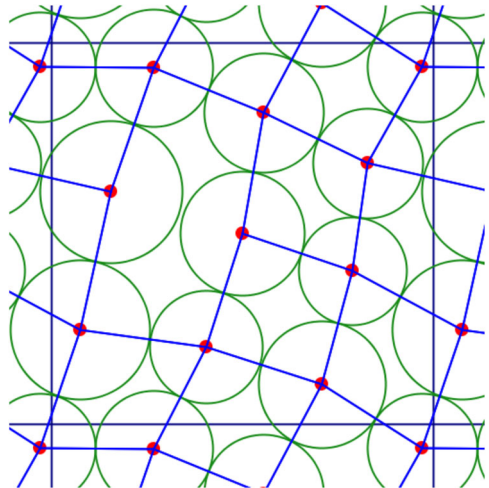
packing has a sufficiently large number of packing elements, then it will be isostatic or at least have the minimum number of contacts for the collectively jammed case (or the strictly jammed case) even if all the disks have the same radius. This does not seem to be the case with some of our calculations, and for calculations done by Atkinson et al. [4].

The packings of Figs. 7 and 9 were obtained with a "Monte Carlo" algorithm similar to the one described in [16,17], where a seed packing with generic radii with no contacts allows the radii to grow until the packing jams. This generally works when there is enough random motion to force the packing to be rigid. If the packing seems not to be converging to an infinitesimally rigid configuration, it is always possible to apply a linear programming algorithm to break up any configuration that is not converging sufficiently rapidly, as was described in [17].

Fig. 10 The tricusp with four
jammed disks, not isostatic
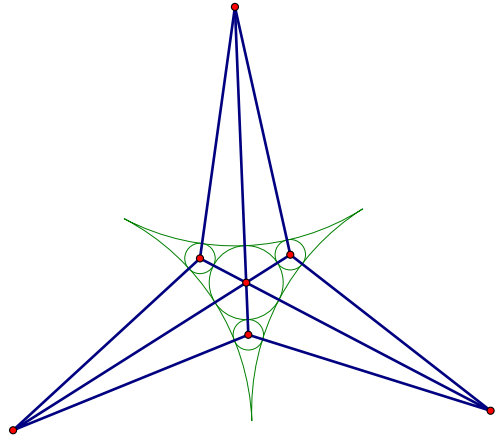
Fig. 11 The tricusp with two
rattlers, isostatic
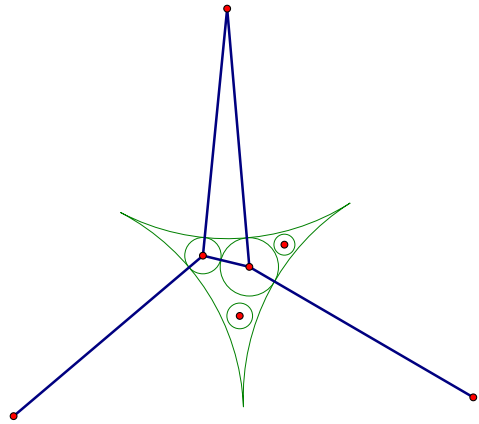
10 The Tricusp Case

There are many circumstances where there is a jammed packing in a bounded container
with an appropriate condition on the boundary of the container, and the infinitesimal
rigidity condition holds. (The condition is that the boundary of the container must
consist of concave up curves like the tricusp in Fig. 10.) It seems reasonable that if the
shape of the container is generic, including the ratio of the radii, that the packing is
isostatic. If the container consists of three mutually tangent circles, then the isostatic
conjecture will hold fixing the boundary, since the packing is determined up to linear
fractional conformal transformations, and the three boundary circles can be fixed. We
call the region between the three mutually tangent circles a *tricusp* following [16], as
in Figs. 11 and 10. If we have a jammed packing of n disks in the tricusp, we regard
the boundary as fixed and since there are no trivial motions, there are $2n$ degrees of
freedom for the centers of the disks. If the packing is isostatic, there is one other
constraint due to the stress condition as before. Thus there are exactly $e = 2n + 1$
contacts, or equivalently edges in the packing graph, when the packing is isostatic.

On the other hand, if e_T is the number of edges in a triangulation of a triangle, then $e_T = 3(n + 3) - 6 = 3n + 3$. This is because adding three edges connecting the fixed edges, we get a triangulation of sphere. Such a triangulation is well-known to have $3m - 6$ edges when there are m vertices, assuming that the graph is 3-connected. So there are $e_T - (e + 3) = 3n + 3 - (2n + 1 + 3) = n - 1$ diagonal edges which can serve as free parameters for the inversive distance as we did for the case of the torus. So we get the following.

Theorem 10.1 *If a jammed packing* $\mathbf{P}_0$ *with n disks in a tricusp is chosen so that the ratio of the radii of the packing disks is generic, then the number of contacts in* $\mathbf{P}_0$ *is* $2n + 1$, *and the packing graph is isostatic.*

The packing in Fig. 10 is jammed, but when the ratios of the generic radii are perturbed so that the smaller three disks are smaller than they are in Fig. 10, while the larger disk, in ratio to the smaller disks, is larger than in Fig. 10, we get packing with two rattlers as in Fig. 11. In this process, there can be no extra edges created in the packing graph. However, Fig. 10 has 12 edges with four vertices in the tricusp, which is three more edges than is needed for being isostatic. Due to the way the radii are perturbed, two of the smaller disks must become rattlers, so n, the number of disks, is decreased by 2, and the number of contacts is decreased by 7, bringing the edge count to the isostatic case.

## 11 Varying the Radii and Lattice

Instead of fixing the lattice that defines the torus, one can allow the lattice to vary as well as the individual packing disks. In this process, if the radii of the disks (or the ratio of the radii) is fixed there is a result analogous to Theorem 2.1. If there is an infinitesimal motion of the lattice and disk centers, that satisfies the packing requirement (i.e. the strut requirements on the edge lengths) and that satisfies the constraints on the lattice (i.e. there is no increase in the area of the whole torus), then there is an actual motion that increases the overall density. This is the following result from [9].

Theorem 11.1 *If* Λ' *and* $\mathbf{p}'$ *represent an infinitesimal motion of a lattice* Λ *and its configuration* $\mathbf{p}$ *that determines a non-positive area change, then there is a smooth motion of the lattice with its configuration that strictly increases adjacent distances and decreases the volume unless* $\mathbf{p}'_i = \mathbf{p}'_j$ *for adjacent disks i and j, and* Λ' *is trivial.*

The effect of this is to increase the density of the configuration while varying both the lattice and the configuration, but keeping the radii (ratios) constant. When the packing is locally maximally dense with these kinds of deformations, in [17] the packing is called *strictly jammed*. Notice, also, that the minimum number of contacts for a strictly jammed packing is at least two more than the minimum when the configuration has its lattice fixed. Namely for n disks, there should be at least $n + 1$ contacts. As an example of the this kind of deformation, if we start with square lattice configuration of two equal disks, perform this deformation and end up with the most dense configuration of equal disks in the plane, where each disk is surrounded by six others as in Fig. 12 deforming to Fig. 13. One can see how the fundamental region has changed shape

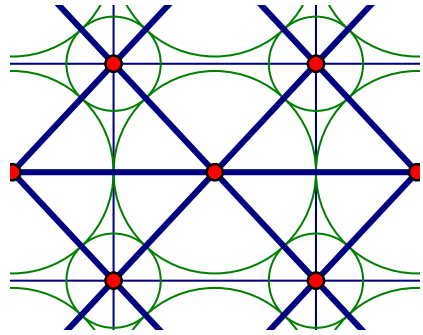**Fig. 12** The most dense packing of two disks in the square torus
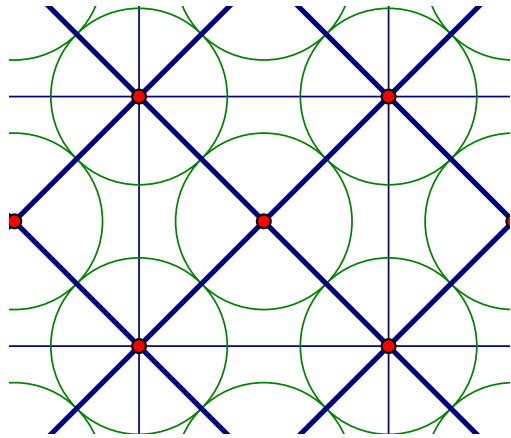
**Fig. 13** The most dense packing of equal disks in the plane
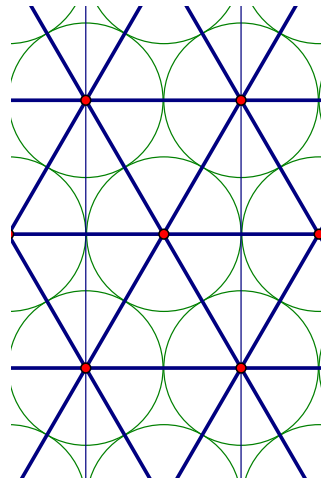
from a square with four contacts and density $\pi/4 = 0.78539\ldots$ to a rectangle with six contacts with density $\pi/\sqrt{12} = 0.90689\ldots$

Another process one can use is to deform a packing, varying not only the configuration of centers, but varying the radii as well, while maintaining the condition that the disk centers with the determined radii form a packing. There is similar second-order calculation for such deformations. For example, for a packing with two disks in a torus having two different radii, one starts again with the packing of Fig. 12 and deforms the lattice, the configuration of centers and radii as in Fig. 14 until eventually there is another contact as in Fig. 15. Indeed, for any infinite packing of disks whose radii are 1 and $\sqrt{2} - 1 = 0.41421\ldots$, in [23] Alidár Heppes proved that the maximum density for such a packing is $\pi(2 - \sqrt{2})/2 = 0.92015\ldots$, which is achieved by the packing in Fig. 15. In the classic book by Tóth [21], it is essentially conjectured that the packing in Fig. 14 has the maximum density for disks with two radii whose ratio is slightly greater than $\sqrt{2} - 1$. The density of the packings during the deformation from Figs. 12, 13 and 14 is $\rho = \pi(1 + r^2)/(4\sqrt{r^2 + 2r})$ as shown in Fig. 16. Note that the

Fig. 14 The conjectured most dense packing of two disks of fixed ratio close to $\sqrt{2}-1$ in the plane

Fig. 15 A most dense packing of disks with radius ratio $\sqrt{2}-1$ in the plane
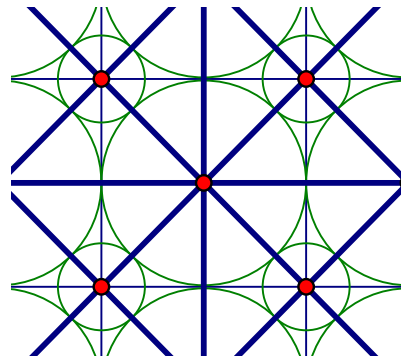

graph is concave up as predicted. It known by Blind [6] that for $0.742 < r < 1$, the most dense packing can do no better than $\pi/\sqrt{12}$ which is achieved by the ordinary triangular lattice packing, with just one disk size.

From this discussion it seems that certain particular singular cases, with the most number of contacts, represent the most dense packings for two radii. Away from those cases, the most dense packings may be when the singular cases are perturbed in a particular way. In [21] those singular packings were called *compact* packings, which was defined to be when each packing disk is adjacent to and is surrounded by a cycle of packing disks, each touching the next. But this is the same as saying that the graph of the packing is a triangulation.

From the analytic packing point of view, one is given an abstract triangulation of a particular compact 2-manifold. Then the basic Koebe–Andreev–Thurston algorithm finds a circle packing with that given contact graph in a manifold of constant curvature, and this packing is unique up to the circle preserving linear fractional transformations of the manifold. On the one hand this algorithm has no constraint that preserves the sizes of the radii. On the other hand, if there are few enough of the disks as in Figs. 13 and 15, or they are symmetric enough as in Fig. 17, then the Koebe–Andreev–Thurston algorithm will automatically have just two disk sizes.

The triangulation of Fig. 17 has a 6-fold rotational symmetry about the center of the central small circle, so the three large circles have the same radius. Similarly the six small circles adjacent to the central small circle have the same radius. Since the central circle is adjacent to the six other small circles, it must have the same radius as

```
plot(Pi*(1+r^2)/(sqrt(r^2+2*r)*4),r=sqrt(2)-1.. 1, delta=.84..0.92)
;
```
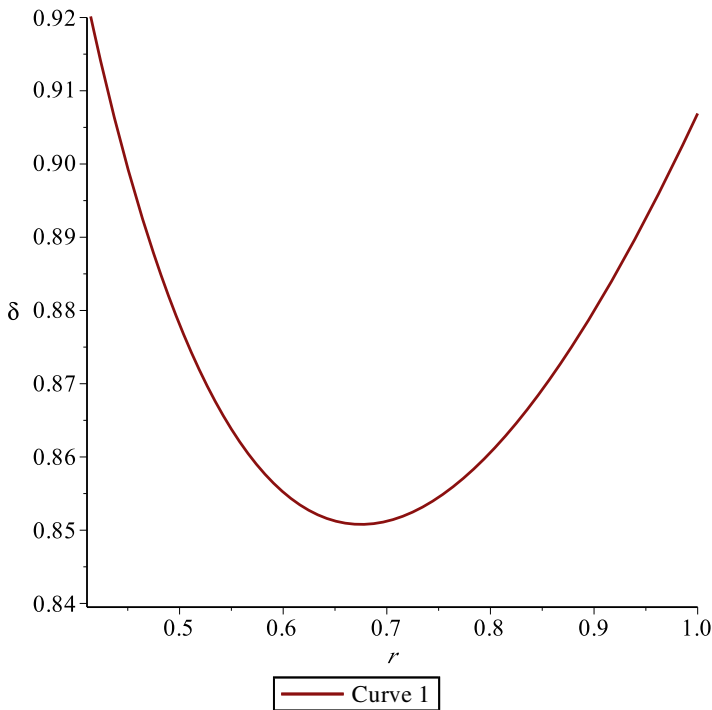

Fig. 16 This plots the density of the packings from Figs. 12, 13, 14 and 15

the other small circles. This is a compact/triangulated packing, one of nine possible classes described by Kennedy [25].

A similar analysis can be done in the tricusp case where some packings are conjectured to be the most dense by Uche, Stillinger and Torquato [16] using the 3-fold symmetry.

There are three types of motions that increase the packing density. Each can be implemented with a Monte Carlo-type process, or a linear programming algorithm.

(1) (Danzer) The lattice defining the torus metric is fixed while the configuration is perturbed so that the radii can be increased uniformly [13].
(2) (Swinnerton-Dyer) The lattice is deformed decreasing its determinant (and therefore the area of the torus) adjusting the configuration while fixing the radii [34].
(3) (Thurston) The radii are adjusted fixing the configuration and the lattice so that the packing condition is preserved while increasing the sum of the squares of the radii. This is essentially maximizing a positive definite quadratic function subject to linear constraints [35].

The idea is that one can perform each of these motions, separately or together depending on what is desired. Each process is named after a person who promoted that process in one form or another.
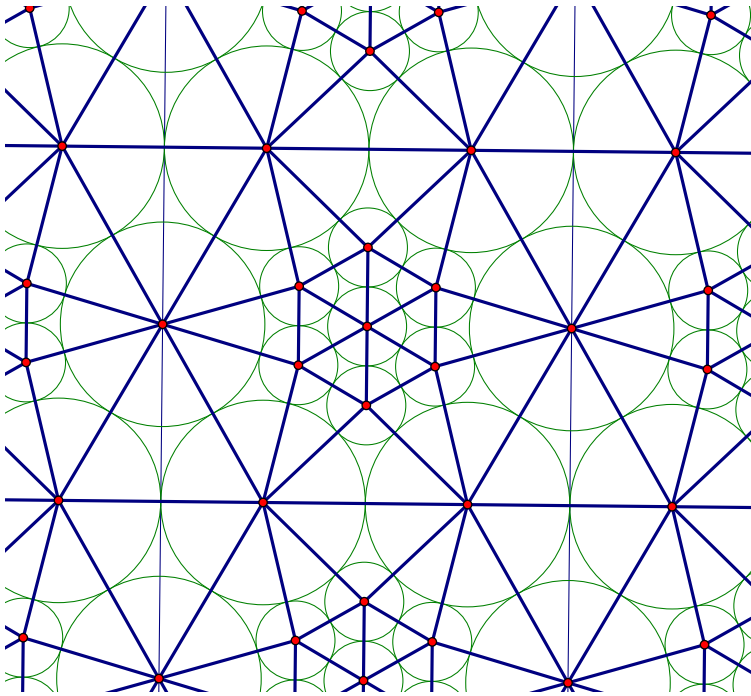
Fig. 17 This is a packing of three large disks and seven smaller ones for the given triangulation

12 Conjecture

Kennedy [25] points out that there are triangulated packings of the plane (and effectively for a flat torus) that are not the most dense for given ratio of radii, which was $\sqrt{2} - 1$ in the case being considered. The idea is to take a square and equilateral tiling of the plane, use the vertices of that tiling for the centers of the larger disks, and the centers of the squares for the centers of the smaller disks. So the final density of the packing is a weighted average of $\pi/\sqrt{12} = 0.906899\ldots$, the density of the triangular close packing, and $\pi(2 - \sqrt{2})/2 = 0.920151\ldots$, the density of the packing in Fig. 15. Figure 18 shows such a periodic triangulated packing with density less than the maximal density. The point is that even if we have a triangulated packing by disks of various sizes, that does not insure that it necessarily represents the maximum density for those sizes.

We say that a packing of disks is *saturated* if there is no place to insert one of the disks in another part of the packing.

Taking a big leap, nevertheless, we conjecture the following:

Conjecture 12.1 *Suppose that* **P** *is a saturated packing, with a triangulated graph, of a finite number of packing disks in a torus with* $n_1, n_2, \ldots, n_k$ *disks of radius* $r_1 > r_2 > \cdots > r_k$ *respectively with density* ρ_0. *Then for all integers* $m \geq 1$, *and a packing of a torus with* $mn_1, mn_2, \ldots, mn_k$ *disks of radius* $r_1 > r_2 > \cdots > r_k$ *respectively, the density is* $\rho \leq \rho_0$.
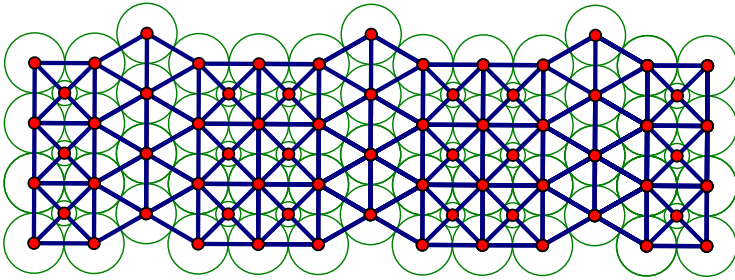
Fig. 18 This is a periodic binary packing of disks with radius ratio $\sqrt{2} - 1$ with density less than the maximum possible $\pi(2 - \sqrt{2})/2 = 0.920151\ldots$
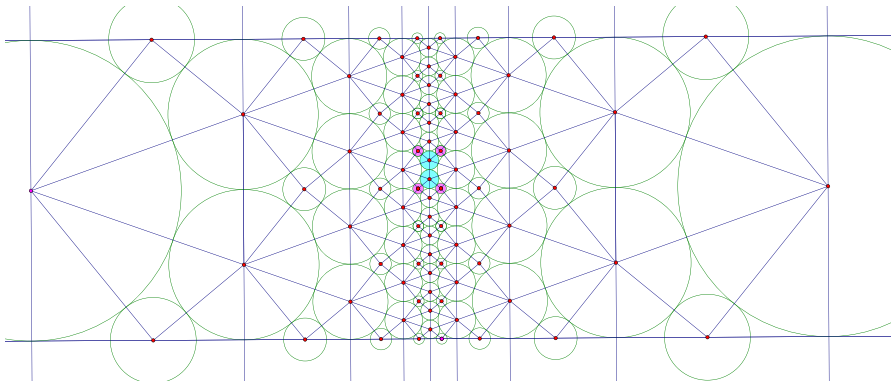
Fig. 19 This is a fundamental region of a triangulated packing of a torus, where the six small colored packing disks can be removed and reinserted in the triangular regions to the right and left without overlap. Then the packing becomes not collectively jammed. Indeed it is not even locally maximally dense, since the packing disks can be displaced one-by-one to another nearby packing with no contacts at all. Then all the disks can grow a small amount to increase the density

Originally the condition that the packing was saturated was omitted, and Fedja Nazarov found the following counterexample, shown in Fig. 19.

In [25], Kennedy shows a list of nine classes of all the triangulated packings of disks in the plane with just two disk sizes. Seven of those nine packing have been shown, by Heppes and Kennedy [23,24] to be the most dense using just those two sizes, which is a bit stronger than the statement of Conjecture 12.1. This is support for Conjecture 12.1.

An interesting special case is a packing of a torus with n_1 disks of radius 1 and n_2 disks of radius $\sqrt{2} - 1$, $n_1 \geq n_2$, then Conjecture 12.1 implies that the maximum density is

$$\rho = \pi \, \frac{n_1 + n_2(\sqrt{2} - 1)^2}{2\sqrt{3}(n_1 - n_2) + 4n_2}.$$

Notice in the case when $k = 2$, and $r_2/r_1 = \sqrt{2} - 1$, and $n_1 = n_2$ the statement of Conjecture 12.1 is weaker that Heppes's Theorem [23], since it assumes $n_1 =$

Fig. 20 This a stellar subdivision of the graph of Fig. 15, and has density $7\pi/24 = 0.91629\ldots$, which is somewhat less than $\pi(2-\sqrt{2})/2 = 0.92015\ldots$, the density of Fig. 15. The radii are in ratio 1:2:3
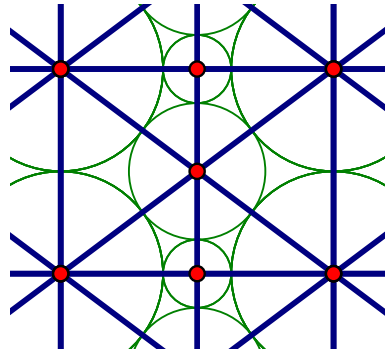
n_2. On the other hand, as far as we know, for other proportions of sizes of disks, Conjecture 12.1 is not known. In particular, continuing with the $r_2/r_1 = \sqrt{2}-1$, $n_2 > n_1$ case, it seems that a triangulated packing of a torus does not exist, and perhaps the most dense packings segregate into the triangular lattice and square lattice pieces.

In another direction, it would be interesting to see how the nature of the triangulation influences the density of the corresponding triangulated packing. Given a triangulation of the plane, an *elementary stellar subdivision* is where a triangle in the triangulation is removed, and it is replaced by the cone over its boundary, or an edge is removed and replaced by the cone over the resulting quadrilateral. For a stellar subdivision of a triangle, it is clear that the density of the corresponding must increase, since one simply places an additional disk in the given triangular region. In many cases, for the stellar subdivision of an edge the density increases. However, if one starts with the Heppes packing graph of Fig. 15 and does a stellar subdivision as indicated in Fig. 20, the density decreases. Indeed, after another stellar subdivision one gets back to a 2-fold covering of the original Fig. 15 with the same density.

13 Remarks and Related Work

There appear to be roughly four groups who work on packing problems, each from their particular point of view.

One group deals with simply finding dense packings of circles in the plane and proving certain packings are the most dense when possible. This group is epitomized by the work of László Fejes Tóth. His book [21] is an early attempt to show what was known and conjectured along with many other related problems and conjectures. Indeed, there are places in [21] where it seems that Conjecture 12.1 is essentially in the background, at least for some particular packings. See Melissen's Ph.D. Thesis [28] for quite a few conjectures for the most dense packings in various containers. Note that the most dense packings conjectured in [11] agree with those in [28]. For provably most dense packings for fixed square and triangular tori, for small numbers of disks there are the results in [10,11,30].

A second group uses linear programing techniques to find upper bounds for sphere packings, particularly in higher dimensions, where things are generally much harder. A good outline of this point of view in the survey by de Laat et al. [14], where their techniques work in dimension two as well. However, their bounds are often not sharp. For example, in dimension three for binary sphere packings where the ratio of the radii is $\sqrt{2} - 1$, the most dense packing is conjectured to be 0.793 which is achieved when the large spheres are centered at the face centered cubic lattice. These centers form the vertices of a tiling of space by regular octahedra and regular tetrahedra. The centers of small spheres are placed at the centers of the octahedra, so that the graph of the packing forms the one-skeleton of a triangulation of space. This is the structure of NaCl ordinary table salt. The techniques of [14] provide an upper bound of 0.813, which is reasonably close to the salty lower bound of 0.793. The salt packing is a three-dimensional extension of the configuration of Fig. 15.

A third group has to do with the Koebe–Andreev–Thurston algorithms that creates packings from the graph of the packings. However, these techniques do not initially specify the radii or density of the resulting packing. A good overview is in the book by Stephenson [33], where many examples are shown as well as connections to conformal mappings, etc. It is interesting to note that one of proofs of the Koebe–Andreev–Thurston packings comes from a minimization argument by Colin de Verdière [7], similar to the process described in Sect. 11.

A fourth group is motivated from the physics of granular material or colloidal clusters is in the work of Torquato and others in [3,17].

There is a lot of room for generalization and possibly improvement of the results here.

(1) Is the example Fig. 3 the only case for $n \geq 2$ disks on a fixed rectangular torus, where the isostatic condition does not hold? It should be kept in mind that the results in [3] show that non-isostatic (strictly) jammed packings seem to be quite frequent for larger numbers when there is one size of radius.

(2) It seems very reasonable that an analysis similar to the argument here shows that a corresponding isostatic conjecture holds, where the parameters of a compact hyperbolic two-dimensional surface are generic, as well as the ratio of the radii.

(3) The proof that an infinitesimal flex implies a finite motion for packings of disks on the two-dimensional sphere is not known, yet it seems reasonable that it is true, and if so, there should be a corresponding isostatic condition for generic radii.

(4) There are many circumstances where there is a jammed packing in a bounded container with an appropriate condition on the boundary of the container, and the infinitesimal flex implies finite motion. It is reasonable that if the shape of the container is generic including the ratio of the radii, that the packing is isostatic.

(5) Is there a way to prove the isostatic conjecture for packings in higher dimensions. Presumably this would involve a different argument, since the analytic packing theory would not be available. In the two-dimensional case, the process described here, with radii, configuration centers, and lattice moving, eventually converges to the case when the faces are all triangles. In higher dimensions, it might be the case that the triangles are replaced by rigid polytopes as in the discussion in [5].

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