GENERICALLY GLOBALLY RIGID GRAPHS HAVE GENERIC UNIVERSALLY RIGID FRAMEWORKS*

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We show that any graph that is generically globally rigid in $\mathbb{R}^d$ has a realization in $\mathbb{R}^d$ that is both generic and universally rigid. This also implies that the graph also must have a realization in $\mathbb{R}^d$ that is both infinitesimally rigid and universally rigid; such a realization serves as a certificate of generic global rigidity.

Our approach involves an algorithm by Lovász, Saks and Schrijver that, for a sufficiently connected graph, constructs a general position orthogonal representation of the vertices, and a result of Alfakih that shows how this representation leads to a stress matrix and a universally rigid framework of the graph.

1. Introduction

In this paper we clarify one central aspect in the relationship between global and universal rigidity of frameworks of a graph.

Given a graph G (with n vertices and m edges) and a configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of its vertices in $\mathbb{R}^d$, we refer to the pair $(G, \mathbf{p})$ as a *framework*, and measure the Euclidean lengths along the edges of G between pairs of vertices in $\mathbb{R}^d$. We call two frameworks, $(G, \mathbf{p})$ and $(G, \mathbf{q})$ *congruent* if there is an isometry of all of $\mathbb{R}^d$ that takes $\mathbf{q}$ to $\mathbf{p}$. This is equivalent to the property that the Euclidean lengths are preserved between *all pairs* of points in a configuration.


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We say that $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{R}^d$ if every framework, $(G, \mathbf{q})$ in $\mathbb{R}^d$, with the same edge lengths as $(G, \mathbf{p})$, is congruent to $(G, \mathbf{p})$.

We say that $(G, \mathbf{p})$ is *universally rigid* if every framework, $(G, \mathbf{q})$ with the same edge lengths as $(G, \mathbf{p})$ in *any* dimension $\mathbb{R}^D$ is congruent to $(G, \mathbf{p})$.

We say that a graph G is *generically globally rigid* (GGR) in $\mathbb{R}^d$ if every "generic framework" of G in $\mathbb{R}^d$ (you can think of this as "almost every" framework in $\mathbb{R}^d$) is globally rigid. It turns out that if a graph is not generically globally rigid, then every "generic framework" of G in $\mathbb{R}^d$ is not globally rigid [20].

The universal rigidity of frameworks in $\mathbb{R}^d$ of G does not have such a simple behavior. There are graphs with Euclidean open (positive measure) sets of frameworks that are universally rigid, and other open sets of frameworks that are not universally rigid (see, e.g., [18, Remark 1.7]). For example, for the line $\mathbb{R}^1$, and when the graph G is a cycle, the only universally rigid configurations are when one edge length is the sum of the others, although all the generic configurations are globally rigid in the line (see [27] for more about universal rigidity in $\mathbb{R}^1$).

If one framework $(G, \mathbf{p})$ is universally rigid, then clearly this framework is globally rigid in $\mathbb{R}^d$, but this does not imply that the graph G itself, is generically globally rigid in $\mathbb{R}^d$. Indeed, $\mathbf{p}$ might be somehow exceptional, and not representative of the generic behavior of frameworks of G in $\mathbb{R}^d$. Figure 1 shows two examples of frameworks on $K_{3,3}$ that are universally rigid in the plane because the vertices are not separable by a quadric [13, Theorem 4.4] (see also [9]). Generically, $K_{3,3}$ is minimally rigid, and thus not globally rigid [22].
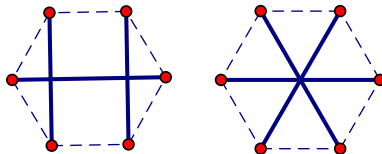
Figure 1. Universally rigid frameworks of a graph that is not generically globally rigid. The edges are drawn to indicate the signs of the entries of a PSD equilibrium stress matrix (see Section 2) for each framework: dashed lines correspond to negative entries and thick ones to positive entries

On the other hand, if we can find a Euclidean open set of configurations of G in $\mathbb{R}^d$ that are universally rigid (and thus globally rigid), then G is generically globally rigid in $\mathbb{R}^d$ [20]. (We may replace "open set of configu-

rations" by either "a single generic framework" or "a single infinitesimally rigid framework" without changing the conclusion.)

In this paper we show the converse. Namely, if G is generically globally rigid in $\mathbb{R}^d$, then it has a Euclidean open set of frameworks in $\mathbb{R}^d$ that are universally rigid. This answers a question posed by Gortler and Thurston [18] for $d \geq 3$. Our method applies for $d \geq 1$, but the cases $d = 1$ and $d = 2$ have already been settled [28,25]. Both [28] and [25] rely, in a fundamental way, on the combinatorial classification of GGR graphs for $d = 2$ [7,24] which does not apply to higher dimensions.

Our approach is to analyze a construction due to Alfakih [1], which builds on work of Lovász, Saks and Schriver [32]. The main result of [32] is that any $(d+1)$-connected graph admits an $(n-d-1)$-dimensional "orthogonal representation" in general position. Alfakih [1] showed how to convert these representations into positive semidefinite (PSD) stress matrices of rank $n-d-1$, which then yield universally rigid frameworks $(G, \mathbf{p})$.

Since $(d+1)$-connectivity is strictly weaker than generic global rigidity [22], there are graphs G for which *all* the universally rigid frameworks $(G, \mathbf{p})$ constructed by Alfakih's method are infinitesimally flexible and thus lie in a proper algebraic subset of configurations. Our main result says that this does not happen when G is GGR.

2. Background

Let G be a graph with n vertices and m edges. Let d be a fixed dimension. Throughout, we will assume that $n \geq d+2$.

A *(bar and joint) framework* in $\mathbb{R}^d$, denoted as $(G, \mathbf{p})$, is a graph G together with a configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of points in $\mathbb{R}^d$.

2.1. Rigidity of frameworks

We say that the framework $(G, \mathbf{p})$ is *locally rigid* in $\mathbb{R}^d$ if, except for congruences, there are no continuous motions in $\mathbb{R}^d$ of the configuration $\mathbf{p}(t)$, for $t \geq 0$, that preserve the edge lengths:

$$(1) \qquad |\mathbf{p}_i(t) - \mathbf{p}_j(t)| = |\mathbf{p}_i - \mathbf{p}_j|$$

for all edges, $\{i, j\}$, of G, where $\mathbf{p}(0) = \mathbf{p}$. If a framework is not locally rigid in $\mathbb{R}^d$, it is called *locally flexible* in $\mathbb{R}^d$ or equivalently just *flexible* or a *finite mechanism*.

The simplest way to confirm that a framework is locally rigid in $\mathbb{R}^d$ is look at the linearization of the problem.

A *first-order flex* or *infinitesimal flex* of $(G, \mathbf{p})$ in $\mathbb{R}^d$ is a corresponding assignment of vectors $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$, $\mathbf{p}'_i \in \mathbb{R}^d$ such that for each $\{i, j\}$, an edge of G, the following holds:

$$(2) \qquad\qquad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0.$$

A first-order flex in $\mathbb{R}^d$ $\mathbf{p}'$ is *trivial* if it is the restriction to the vertices, of the time-zero derivative of a smooth motion of isometries of $\mathbb{R}^d$. The property of being trivial is independent of the graph G.

The *rigidity matrix* $R(\mathbf{p})$ is the nd-by-m matrix, where

$$R(\mathbf{p})\mathbf{p}' = (\ldots, (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j), \ldots)^T,$$

for $\mathbf{p}' \in \mathbb{R}^{nd}$.

A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is called *infinitesimally rigid* in $\mathbb{R}^d$ if it has no infinitesimal flexes in $\mathbb{R}^d$ except for trivial ones. When $n \geq d$ this is the same as saying that the rank of $R(\mathbf{p})$ is $nd - \binom{d+1}{2}$. If a framework is not infinitesimally rigid in $\mathbb{R}^d$, it is called *infinitesimally flexible* in $\mathbb{R}^d$.

A classical theorem states:

Theorem 2.1. *If a framework $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^d$, then it is locally rigid in $\mathbb{R}^d$.*

The converse is not true (but see Theorem 2.4 below).

A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is called *globally rigid in $\mathbb{R}^d$* if, there are no other (even distant) frameworks $(G, \mathbf{q})$ in $\mathbb{R}^d$ having the same edge lengths as $(G, \mathbf{p})$, other than congruent frameworks.

A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is called *universally rigid* if, there are no other (even distant) frameworks $(G, \mathbf{q})$ in $\mathbb{R}^D$, for any D, having the same edge lengths as $(G, \mathbf{p})$, other than congruent frameworks in $\mathbb{R}^D$.

Clearly, universal rigidity implies global rigidity (in any dimension) which implies local rigidity (in any dimension).

Given a graph G, a *stress vector* $\omega = (\ldots, \omega_{ij}, \ldots)$, is an assignment of a real scalar $\omega_{ij} = \omega_{ji}$ to each edge, $\{i, j\}$ in G. (We have $\omega_{ij} = 0$, when $\{i, j\}$ is not an edge of G.)

We say that ω is an *equilibrium stress vector* for $(G, \mathbf{p})$ if the vector equation

$$(3) \qquad\qquad \sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0$$

holds for all vertices i of G. The equilibrium stress vectors of $(G, \mathbf{p})$ form the co-kernel of its rigidity matrix $R(\mathbf{p})$.

We associate an n-by-n *stress matrix* Ω to a stress vector ω, by setting the i,jth entry of Ω to $-\omega_{ij}$, for $i \neq j$, and the diagonal entries of Ω are set such that the row and column sums of Ω are zero. The stress matrices of G are simply the symmetric matrices with zeros associated to non-edge pairs that, additionally, have the all-ones vector in their kernel.

If ω is an equilibrium stress vector for $(G,\mathbf{p})$, then we say that the associated Ω is an *equilibrium stress matrix* for $(G,\mathbf{p})$. For each of the d spatial dimensions, if we define a vector $\mathbf{v}$ in $\mathbb{R}^n$ by collecting the associated coordinate over all of the points in $\mathbf{p}$, we have $\Omega\mathbf{v}=0$. Thus if the dimension of the affine span of the vertices $\mathbf{p}$ is d, then the rank of Ω is at most $n-d-1$, but it could be less.

Let $(G,\mathbf{p})$ be a framework (in any dimension) with a d-dimensional affine span, denoted $\langle\mathbf{p}\rangle$. Fixing an affine frame for $\langle\mathbf{p}\rangle$, we can represent $\mathbf{p}$ using coordinates in $\mathbb{R}^d$. We say that the edges directions of $(G,\mathbf{p})$ lie on a conic at infinity of $\langle\mathbf{p}\rangle$ if there exists a non-zero symmetric d-by-d matrix Q such that for all of the edges, $\{i,j\}$ in G, we have $(\mathbf{p}_i-\mathbf{p}_j)^t Q(\mathbf{p}_i-\mathbf{p}_j)=0$.

Following [9] we say a framework $(G,\mathbf{p})$ (in any dimension) with a d-dimensional affine span is *super stable* if there is an equilibrium stress ω for $(G,\mathbf{p})$ such that its associated stress matrix Ω is PSD, the rank of Ω is $n-d-1$, and the edge directions do not lie on a conic at infinity of $\langle\mathbf{p}\rangle$.

The following is a classic theorem by Connelly [9]

Theorem 2.2. *Let $(G,\mathbf{p})$ be a framework (in any dimension). If $(G,\mathbf{p})$ is super stable, then $(G,\mathbf{p})$ is universally rigid.*

Alfakih and Ye [2] showed that one can easily avoid the explicit assumption about conics at infinity in the case of general position.

Theorem 2.3. *Let $(G,\mathbf{p})$ be a framework with a d-dimensional affine span. If $(G,\mathbf{p})$ is in general affine position within $\langle\mathbf{p}\rangle$ and has an (even indefinite) equilibrium stress matrix of rank $n-d-1$, then the edge directions of $(G,\mathbf{p})$ do not lie on a conic at infinity of $\langle\mathbf{p}\rangle$.*

Thus if $(G,\mathbf{p})$ is a framework with a d-dimensional affine span and in general affine position within $\langle\mathbf{p}\rangle$ and it has a PSD equilibrium stress matrix of rank $n-d-1$, then it is super stable and thus universally rigid.

2.2. Rigidity of graphs

We say that a configuration $\mathbf{p}$, or a framework $(G,\mathbf{p})$, in $\mathbb{R}^d$ is *generic*, if there is no non-zero polynomial relation, with coefficients in $\mathbb{Q}$, among the coordinates of $\mathbf{p}$.

We say that a graph G is *generically locally rigid (resp. flexible)* in $\mathbb{R}^d$ if every generic framework of G in $\mathbb{R}^d$ is locally rigid (resp. flexible) in $\mathbb{R}^d$.

We say that a graph G is *generically infinitesimally rigid (resp. flexible)* in $\mathbb{R}^d$ if every generic framework of G in $\mathbb{R}^d$ is infinitesimally rigid (resp. flexible) in $\mathbb{R}^d$.

As described in [4,5], generic local rigidity is determined by generic infinitesimal rigidity

Theorem 2.4. *If some framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^d$ *is infinitesimally rigid in* $\mathbb{R}^d$, *then the graph* G *is generically infinitesimally rigid in* $\mathbb{R}^d$ *and thus generically locally rigid in* $\mathbb{R}^d$.

If a graph, G, *is not generically infinitesimally rigid in* $\mathbb{R}^d$, *then it is generically locally flexible in* $\mathbb{R}^d$.

Thus, if G *is not generically locally rigid in* $\mathbb{R}^d$, *then it is generically locally flexible in* $\mathbb{R}^d$.

We say that a graph G is *generically (resp. not) globally rigid* in $\mathbb{R}^d$ if every generic framework of G in $\mathbb{R}^d$ is (resp. not) globally rigid in $\mathbb{R}^d$.

The following is the easy half of a theorem by Hendrickson [21], which we will need below.

Theorem 2.5. *If* G *is generically globally rigid in* $\mathbb{R}^d$, *then it must be* $(d+1)$-*connected.*

Connelly [11] proved the following sufficient condition for global rigidity.

Theorem 2.6. *If some generic framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^d$ *has an (even indefinite) equilibrium stress matrix of rank* $n - d - 1$, *then the graph* G *is generically globally rigid in* $\mathbb{R}^d$.

This was refined slightly in [20,14] giving the following sufficient certificate for generic global rigidity

Theorem 2.7. *If some framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^d$ *is infinitesimally rigid in* $\mathbb{R}^d$ *and* $(G, \mathbf{p})$ *has an (even indefinite) equilibrium stress matrix* Ω *of rank* $n - d - 1$, *then the graph* G *is generically globally rigid in* $\mathbb{R}^d$.

Thus, the pair Ω and $(G, \mathbf{p})$ serve as a certificate for the generic global rigidity of G in $\mathbb{R}^d$. Note that this does not imply that the specific framework $(G, \mathbf{p})$ in the above certificate is globally rigid in $\mathbb{R}^d$ [14].

Gortler, Healy and Thurston [20] proved the strong converse to Theorem 2.6.

Theorem 2.8. *If some generic framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ does not have equilibrium stress matrix of rank $n-d-1$, then the graph G is not generically globally rigid in $\mathbb{R}^d$. It is, in fact, generically not globally rigid in $\mathbb{R}^d$. Thus, if a graph G is not generically globally rigid in $\mathbb{R}^d$, then it is generically not globally rigid in $\mathbb{R}^d$.*

Remark 2.9. The above theorems tell us that a graph G is either generically (locally/infinitesimally/globally) rigid in $\mathbb{R}^d$, or it is generically not (locally/infinitesimally/globally) rigid in $\mathbb{R}^d$.

Due to the semi-algebraic nature of rigidity, if G is generically (resp. not) (locally/infinitesimally/globally) rigid in $\mathbb{R}^d$, then the only exceptional frameworks must be contained in an strict algebraic subset (defined over $\mathbb{Q}$) of configuration space.

Universal rigidity does not behave so simply. In particular, there are graphs with Euclidean open sets of frameworks in $\mathbb{R}^d$ that are universally rigid, and other open sets of frameworks in $\mathbb{R}^d$ that are not universally rigid.

The examples above indicate that a graph can be generically globally rigid in $\mathbb{R}^d$, while having some generic frameworks in $\mathbb{R}^d$ that are not universally rigid. One open question that has been open in the rigidity community since 2010 (see [18]) asks:

> *If G is generically globally rigid in $\mathbb{R}^d$, must it have* some *generic framework in $\mathbb{R}^d$ that is universally rigid?*

The main result of this paper answers this question in the affirmative:

Theorem 2.10. *If G is generically globally rigid in $\mathbb{R}^d$, then there exists a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ that is infinitesimally rigid in $\mathbb{R}^d$ and super stable. Moreover, every framework in a small enough neighborhood of $(G, \mathbf{p})$ will be infinitesimally rigid in $\mathbb{R}^d$ and super stable, and thus must include some generic framework.*

The first part of this theorem tells us that if G is generically globally rigid in $\mathbb{R}^d$, then it must have a certificate, Ω and $(G, \mathbf{p})$ in the sense of Theorem 2.7, where $(G, \mathbf{p})$ is itself certifiably super stable and thus globally rigid.

Remark 2.11. Theorem 2.10 yields a weak converse to Connelly's Theorem 2.6. Namely, if some generic framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ does not have an equilibrium stress matrix of rank $n-d-1$, then the graph G is not generically globally rigid in $\mathbb{R}^d$. But this does not, alone, prove that G is, in fact, generically not globally rigid in $\mathbb{R}^d$. (This requires showing the existence of an equivalent, but not congurent framework for each generic $(G, \mathbf{p})$.)

This paper will also use some basic facts from (semi-)algebraic geometry, which are summarized in the appendix.

3. Stresses from GORs

In [32], Lovász, Saks and Schrijver define a concept called a (GOR) general position orthogonal representation of a graph G in $\mathbb{R}^{n-d-1}$. Alfakih [1] has shown how these relate to a certain class of equilibrium stresses for d-dimensional frameworks of G. This section reviews and extends these results.

3.1. GORs and connectivity

Definition 3.1. Let G be a graph and let D be a fixed dimension. An (OR) *orthogonal representation* of G in $\mathbb{R}^D$ is a vector configuration $\mathbf{v}$ indexed by the vertices of G in $\mathbb{R}^D$ with the following property: $\mathbf{v}_i$ is orthogonal to the vectors associated with each non-neighbor of vertex i. The set of ORs form an algebraic set (defined over $\mathbb{Q}$).

A (GOR) *general position orthogonal representation* of G in $\mathbb{R}^D$ is an OR in $\mathbb{R}^D$ with the added property that the $\mathbf{v}_i$ are in general linear position. The set of GORs form a semi-algebraic set (defined over $\mathbb{Q}$).

The relevant results from [32] are the following.

Theorem 3.2. *Let G, a graph on n vertices, be $(n-D)$-connected for some D. Then G must have a GOR in dimension $\mathbb{R}^D$ [32, Theorem 1.1]. Moreover, the set of all such GORs of G is irreducible [32, Theorem 2.1].*

In our terminology, we will set $D := n - d - 1$, where d is fixed, and thus we will need $(d+1)$-connectivity to obtain GORs in $\mathbb{R}^{n-d-1}$.

Definition 3.3. Let G be a $(d+1)$-connected graph with n vertices, for some d. Denote by D_G the dimension of the set its GORs in $\mathbb{R}^{n-d-1}$.

We wish to compute D_G which is done in the following corollary proven below.

Corollary 3.4. *Let G be a $(d+1)$-connected graph with n vertices and m edges. Then the dimension D_G is $n(n-d) - \binom{n+1}{2} + m$.*

The idea behind the corollary, which is present in [32], is that we can build a GOR of G by selecting vectors one at a time from a linear space of known dimension (that depends on G and the vertex order). The proof of the corollary relies on several lemmas that formalize this intuition.

Definition 3.5. Let G be a graph with n ordered vertices $\{1, 2, \ldots, n\}$. Fix d. Let G_i, for $2 \leq i \leq n$ be the subgraph of G induced by vertices j such that $j \leq i$.

Let GOR_{i-1} be the set of GORs of G_{i-1} in $\mathbb{R}^{n-d-1}$, where n is the number of vertices on the full graph G.

Fix $\mathbf{v}^{i-1}$, some configuration in GOR_{i-1}. Let $A \subset \mathbb{R}^{n-d-1}$ be the linear span of the $\mathbf{v}_j$ in $\mathbf{v}^{i-1}$ corresponding to non-neighbors of vertex i in G_{i-1}. We say that $\mathbf{v}^{i-1}$ is *inextendable* if there is a set of $\mathbf{v}_j$ in $\mathbf{v}^{i-1}$ of cardinality at most $n - d - 2$ such that $A^\perp$ is in the span of these $\mathbf{v}_j$. Otherwise we say that $\mathbf{v}^{i-1}$ is *extendable*. Every extendable configuration in GOR_{i-1} can be extended to a configuration in GOR_i by some appropriate placement of vertex i in $A^\perp$.

Let $\bar{e}_i$ denote the number of vertices in G_i that are not neighbors of vertex i and e_i be the number of its neighbors in G_i.

Lemma 3.6. *Let G be a $(d+1)$-connected graph with n vertices, for some d. Then for any $1 \leq i \leq n$, we have GOR_i non-empty and irreducible.*

Proof. The graph G_i has i vertices and must be at least $(d+1-(n-i))$-connected. (Negative connectivity is the same as having no connectivity conditions at all). Meanwhile, in order to apply Theorem 3.2 directly to G_i, we only need it to be $(i-(n-d-1))$-connected. ∎

Lemma 3.7. *The subspace $A^\perp$ has dimension*

$$(n - d - 1) - \bar{e}_i = (n - d - 1) - (i - 1 - e_i) = n - d - i + e_i.$$

Proof. By assumption, the $\bar{e}_i$ non-neighbors are in general position, giving us the first expression. The rest is obvious. ∎

Lemma 3.8. *Let $2 \leq i \leq n$. The subset of GOR_{i-1} that is inextendable is semi-algebraic and of strictly lower dimension than GOR_{i-1}. Thus the extendable subset has full dimension.*

Proof. The conditions describing inextendibility can be described with algebraic equations (using determinants). Also, from Lemma 3.6, GOR_{i-1} is irreducible. Thus, the inextendable set is either all of GOR_{i-1} or it is of lower dimension.

Meanwhile, from Lemma 3.6 GOR_i is not empty, thus it contains some configuration $\mathbf{v}^i$. By forgetting the last vertex, we obtain a configuration $\mathbf{v}^{i-1}$ in GOR_{i-1} that must be extendable. ∎

Lemma 3.9. *Let $2 \leq i \leq n$. Suppose that $\mathbf{v}^{i-1}$ is an extendable configuration in GOR_{i-1}. Then there is a Zariski open subset of $A^\perp$ such that placing $\mathbf{v}_i$ in this subset produces an element in GOR_i.*

Proof. The set of disallowed placements for $\mathbf{v}_i$ (violating general position) is the intersection of the irreducible $A^\perp$ with a subspace arrangement arising from the linear spans of subsets of $\mathbf{v}^{i-1}$. Either $A^\perp$ is contained in this arrangement, or the disallowed subset is algebraic and has lower dimension. ∎

Proof of Corollary 3.4. Let $2 \leq i \leq n$. Let π be the map $\pi\colon \mathrm{GOR}_i \to \mathrm{GOR}_{i-1}$ that forgets the last vertex. From Lemma 3.8, we have $\dim(\pi(\mathrm{GOR}_i)) = \dim(\mathrm{GOR}_{i-1})$.

Due to irreducibility of GOR_i (Lemma 3.6), we can apply the fiber dimension Theorem A.12 in the appendix to see that

$$\dim(\mathrm{GOR}_i) = \dim(\pi(\mathrm{GOR}_i)) + \dim(\pi^{-1}(\pi(x)) \cap N(x)),$$

where x is generic in GOR_i and $N(x)$ is a neighborhood around x.

Meanwhile, from Lemmas 3.9 and 3.7 any fiber $\pi^{-1}(\pi(x))$ is a Zariski open subset of a linear space of dimension $n-d-i+e_i$. Thus, the dimension of GOR_i is $n-d-i+e_i$ more than the dimension of GOR_{i-1}. Also the dimension of GOR_i for $i=1$ is $n-d-1=n-d-i+e_i$.

Summing over all i gives

$$D_G = \sum_{i=1}^{n} n - d - i + e_i = n(n-d) - \binom{n+1}{2} + m$$

as claimed. ∎

3.2. Alfakih's construction

Because of the orthogonality property of a GOR, its Gram matrix has the right zero/non-zero pattern to be a stress matrix. Alfakih [1] builds on this. First we set some notation.

Definition 3.10. Let G be a $(d+1)$-connected graph and $\mathbf{v}$ a GOR of G in dimension $n-d-1$. The $n \times (n-d-1)$ matrix X with the $\mathbf{v}_i$ as its rows is the *configuration matrix* of $\mathbf{v}$. We denote the Gram matrix XX^t of $\mathbf{v}$ by Ψ. Note that Ψ is, by construction, PSD and has rank $n-d-1$, (as $\mathbf{v}$ is in general position).

A GOR $\mathbf{v}$ is called *centered* if its barycenter is the origin. We define GOR^0 to be the semi-algebraic set of centered GORs.

The Gram matrix Ψ is a stress matrix (which we will call Ω) if and only if $\mathbf{v}$ is centered. (Recall that the extra condition is that the all-ones vector is in the kernel.) Such an Ω is PSD and of rank $n-d-1$.

We define the set LSS of *Lovász-Saks-Schrijver stresses* to be the collection of stress matrices Ω arising as the Gram matrices of centered GORs. Denote its dimension by D_L.

We wish to compute D_L. Heuristically, we expect the relationship

$$(4) \qquad D_L + \binom{n-d-1}{2} = D_G - n + d + 1$$

to hold because both sides correspond to the dimension of the set GOR^0 of centered GORs. In particular, given a stress matrix $\Omega \in \text{LSS}$, we can change the underlying GOR $\mathbf{v}$ by an orthogonal transformation on $\mathbb{R}^{n-d-1}$ without changing Ω. This corresponds to the left-hand side of (4). The right-hand-side comes from noting that the centering condition imposes a linear constraint on each column of the matrix X.

This does not constitute a proof because we need to show that the linear constraints from the centering condition change the dimension as predicted by the r.h.s. of (4). Instead of checking this directly, we use the construction from [1].

Definition 3.11. Let $\mathbf{v}$ be a vector configuration. A *centering map* φ is a map $\mathbf{v}_i \mapsto \alpha_i \mathbf{v}_i$ so that $\varphi(\mathbf{v})$ has its barycenter at the origin. A centering map is *full rank* if none of the α_i are zero. The coefficients α_i defining φ correspond to a row vector in the co-kernel of the configuration matrix, X, of $\mathbf{v}$.

We define $\varphi(X)$ to be DX, where D is the $n \times n$ diagonal matrix with the α_i on its diagonal.

Specialized to the case where $\mathbf{v}$ is a GOR for a graph G in $\mathbb{R}^{n-d-1}$, $\varphi(\mathbf{v})$ is a centered GOR if and only if φ is full rank, since general position must be maintained.

Thus, the set of full rank centering maps is the semi-algebraic set arising by removing vectors with any zero coordinates from the co-kernel of X.

Lemma 3.12. *Let G be a $(d+1)$-connected graph and $\mathbf{v}$ a GOR of G in $\mathbb{R}^{n-d-1}$. If there is at least one full-rank centering map φ of $\mathbf{v}$, then the set of all full rank centering maps is $(d+1)$-dimensional.*

Proof. The co-kernel K of X is a linear space, and so is irreducible. The subset of vectors in K with any zero coordinate is an algebraic subset of it, and so is either all of K or of lower dimension. Since K is assumed to contain a vector with no zeros, the set of full rank centering maps is then

the (semi-algebraic) complement of a proper algebraic subset, so it has the same dimension as K.

Because $\mathbf{v}$ is in general position, the dimension of K is $d+1$.   ∎

Alfakih's main result in [1] is the following.

Theorem 3.13 ([1])**.** *Let G be $(d+1)$-connected. Then any GOR in $\mathbb{R}^{n-d-1}$ for G has a full rank centering map. This gives rise to stress matrix Ω in LSS. Moreover, any framework $(G,\mathbf{p})$ with d-dimensional affine span, that has Ω as an equilibrium stress matrix, must be in general position. (And, consequently, super stable.)*

Remark 3.14. In Theorem 3.13, the existence of the full rank centering map relies crucially on the general position property of the input GOR. The existence of a general position kernel framework relies on the general position of the centered GOR obtained by scaling. A centered OR that is not in general position but still has rank $n-d-1$ has only non-general position frameworks in its kernel.

Corollary 3.15. *If G is $(d+1)$-connected, then $D_L = m - \binom{d+1}{2}$.*

Proof. If we can establish (4), then we have

$$D_L = D_G - \binom{n-d-1}{2} - n + d + 1$$

$$= m + n(n-d) - \binom{n+1}{2} - \binom{n-d-1}{2} - n + d + 1$$

$$= m - \binom{d+1}{2}.$$

Now we show (4) by computing the dimension of GOR^0 two ways.

For the first way, we build a semi-algebraic bundle B of points $(\mathbf{v}, \varphi)$, where $\mathbf{v}$ is a GOR and φ is full rank centering map for $\mathbf{v}$. Combining Lemma 3.12 and Theorem 3.13, for each fixed $\mathbf{v}$ the set of φ is $(d+1)$-dimensional. Thinking of B as a bundle (X^t, φ), we may apply Lemma B.1, to get that B is irreducible and of dimension $D_G + d + 1$.

Any two GORs $\mathbf{v}$ and $\mathbf{v}'$, can be scaled to the same element of GOR^0 iff all of their corresponding vectors $\mathbf{v}_i$ and $\mathbf{v}'_i$ share the same direction. Thus, the natural map $B \to \mathrm{GOR}^0$ given by $(\mathbf{v}, \varphi) \mapsto \varphi(\mathbf{v})$ has n-dimensional fibers. This map is also surjective since $\mathrm{GOR}^0 \subset \mathrm{GOR}$, so $\mathbf{v} \in \mathrm{GOR}^0$ is in the image by taking φ to be the identity. We may now apply Theorem A.12 again to conclude that, GOR^0 has dimension $D_G - n + d + 1$. We also see, that as the image of this polynomial map, GOR^0 is irreducible.

For the second way, as discussed above, the map from $\mathrm{GOR}^0 \to \mathrm{LSS}$, given by $X \mapsto XX^t$ is invariant under the orthogonal group, so its fibers are $\binom{n-d-1}{2}$-dimensional. This map is, by definition, surjective, so by Theorem A.12 the dimension of GOR^0 is $D_L + \binom{n-d-1}{2}$. ∎

Remark 3.16. In Alfakih's construction, if one starts with a fixed GOR and varies the full rank centering maps φ, the resulting Ω matrices will differ only through scaling. Thus, all of the d-dimensional frameworks in the kernels of these Ω must only differ through d-dimensional projective transforms [14].

Remark 3.17. Since all the configurations produced by Alfakih's construction are in affine general position, and because it only applies to $(d+1)$-connected graphs, there are many frameworks with a maximal rank PSD equilibrium stress matrix that it does not construct. For an example, see Figure 2.
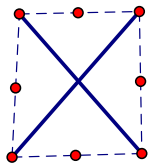
Figure 2. A framework with a maximal rank PSD equilibrium stress matrix that does not arise from Alfakih's construction. The edge styles have the same meaning as in Figure 1

Remark 3.18. Alfakih's construction is an example of what the engineering literature calls "form finding" [42]. It is particularly similar to [12], in that it produces a configuration with specific properties by searching among configurations with a specific equilibrium stress.

4. Example: $K_{2,2}$ in $\mathbb{R}^1$

As a concrete example of Alfakih's construction, we consider the case of $K_{2,2}$ in $\mathbb{R}^1$. For convenience label the vertices u_1, u_2, v_1, v_2, so that the edges are $\{u_i, v_j\}$ for $i, j \in \{1, 2\}$. The universally rigid configurations are those with one "long edge" and three "short" ones [28,25]. We can explore these from the perspective of GORs for $K_{2,2}$ in dimension $\mathbb{R}^2$ (since $n = 4$ and $d = 1$). Denote a GOR of $K_{2,2}$ by $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2)$.

The space of GORs is easy to describe: we have $\mathbf{u}_1 \perp \mathbf{u}_2$, $\mathbf{v}_1 \perp \mathbf{v}_2$, and the angle, θ, between $\mathbf{u}_1$ and $\mathbf{v}_1$ is not a multiple of $\pi/2$. This is clearly a 6-dimensional set, in accordance with Theorem 3.2.

To explore the example more directly, we define a curve of reference GORs parameterized by the angle θ.

$$\mathbf{u}_1^\theta := \mathbf{i}e^{-\mathbf{i}\theta/2}, \quad \mathbf{u}_2^\theta := e^{-\mathbf{i}\theta/2}, \quad \mathbf{v}_1^\theta := \mathbf{i}e^{\mathbf{i}\theta/2}, \quad \mathbf{v}_2^\theta := -e^{\mathbf{i}\theta/2},$$

where we have identified $\mathbb{R}^2$ with $\mathbb{C}$ to keep the formulas compact, and $\mathbf{i}$ is the imaginary unit.

Following Alfakih's construction, we first scale our reference curve of GORs to centered ones. Noting the reflection symmetry of this parameterization, we see that, for $\theta \in [0, \pi/2]$

$$(5) \qquad |\mathbf{u}_2^\theta + \mathbf{v}_2^\theta|(\mathbf{u}_1^\theta + \mathbf{v}_1^\theta) + |\mathbf{u}_1^\theta + \mathbf{v}_1^\theta|(\mathbf{u}_2^\theta + \mathbf{v}_2^\theta) = 0$$

and

$$(6) \qquad |\mathbf{u}_2^\theta + \mathbf{v}_1^\theta|(\mathbf{v}_2^\theta - \mathbf{u}_1^\theta) + |\mathbf{v}_2^\theta - \mathbf{u}_1^\theta|(\mathbf{u}_2^\theta + \mathbf{v}_1^\theta) = 0.$$

This gives us basis for scalings of a reference GOR to GOR^0. These two scalings, along with θ, parameterize LSS for this interval, since there are three independent parameters. Thus, $K_{2,2}$ has a 3-dimensional space for LSS, as expected from Corollary 3.15.
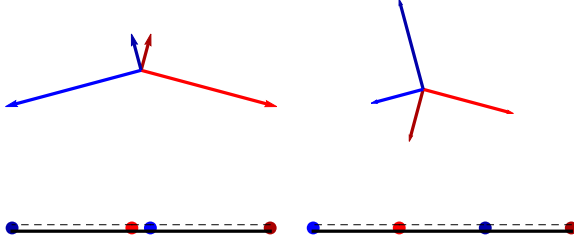
Figure 3. Two centered GORs with $\theta = \pi/6$. Their kernel frameworks are related by a projective transformation

For a fixed θ, applying a different scaling leads to a different point of GOR^0. The effect on the kernel framework, by Remark 3.16, is to apply a projective transformation. This is illustrated in Figure 3, where the left column uses the scaling (5) and the right column uses (6). The color coding is: u_1 in dark red; u_2 in bright red; v_1 in dark blue; v_2 in bright blue. The top row shows two centered GORs at $\theta = \pi/6$ and the bottom row the associated kernel frameworks. As is expected, in each case there is one red-blue vector pair with positive dot product, corresponding to a negative ω value on the long edge of that framework (represented with a thick line). Edges with
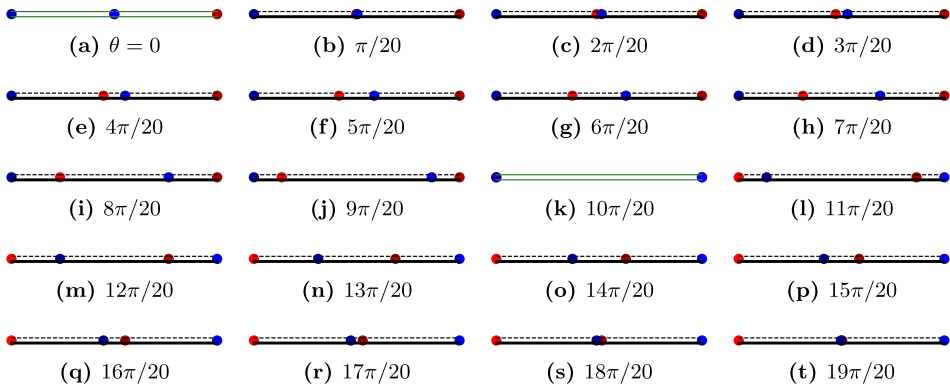
Figure 4. Universally rigid configurations of $K_{2,2}$ in $\mathbb{R}^1$

positive ω are shown with a dotted line and edges with zero ω are shown in thin green.

Figure 4 illustrates the sequence of kernel frameworks associated with the scaling (5), which holds on $[0, \pi]$, as θ increases from 0 to π, using a consistent affine normalization. At the endpoints of the interval, the "GOR" is no longer in general position. Moreover, the "centered GOR" generated by this scaling degenerates, yielding a rank 1 stress matrix; in the figure, we show the kernel framework that is the limit as $\theta \to 0$ in part (a). The limit stress at $\theta = 0$ has $\omega = 1$ for the edge between bright red and bright blue and $\omega = 0$ for the other edges. In part (k), where $\theta = \pi/2$, the "GOR" is also no longer in general position, but under the scaling, we still obtain a stress matrix of rank 2. This stress has $\omega = 1$ for the edges between dark blue and bright red, and between bright blue and dark red, and $\omega = 0$ for the other edges.

5. Stratification of stresses

We want to look at all equilibrium stresses for all d-dimensional frameworks of G, and see which ones correspond to those in LSS. We will do this by slicing the configuration space into subsets that are easy to analyze on their own. We will find that, when G is generically globally rigid in $\mathbb{R}^d$, the equilibrium stresses arising from infinitesimally flexible frameworks can account for only a low dimensional subset of LSS. This will then lead immediately to a proof of our Theorem 2.10.

5.1. The subsets

We now chop up the configuration space, $\mathbb{R}^{nd}$ into nicely behaved subsets. It will be helpful for each of these subsets, S, to be invariant with respect to invertible affine transforms. That means that if $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2.., \mathbf{p}_n) \in S$ and A is an invertible affine transform, then $A(\mathbf{p}) := (A(\mathbf{p}_1), A(\mathbf{p}_2).., A(\mathbf{p}_n)) \in S$.

Lemma 5.1. *Let S be a semi-algebraic set of configurations that is invariant with respect to invertible affine transforms. Let r be the maximal rank of the rigidity matrices over all $\mathbf{p} \in S$. Let S' be the semi-algebraic subset of S, where the rigidity matrix has rank less than r. Then S' is invariant with respect to invertible affine transforms.*

Proof. The rank of the rigidity matrix of $(G, \mathbf{p})$ is invariant with respect to invertible affine transforms acting on $\mathbf{p}$. ∎

Lemma 5.2. *Let S be a semi-algebraic set of configurations that is invariant with respect to invertible affine transforms. Then its singular set is invariant with respect to invertible affine transforms.*

Proof. Each such affine transform gives us a diffeomorphism on $\mathbb{R}^n$ and thus retains smoothness of points in subsets. ∎

We thank Dylan P. Thurston for the proof of the following lemma.

Lemma 5.3. *Let S be a reducible semi-algebraic set of configurations that is invariant with respect to invertible affine transforms. Then each irreducible component S_i of S is invariant with respect to invertible affine transforms.*

Proof. Let $\mathcal{A}$ be the set of invertible affine transforms; this is an irreducible semi-algebraic set. Let g_i be the map $S_i \times \mathcal{A} \to S$, which sends $(\mathbf{p}; A) := (\mathbf{p}_1, \mathbf{p}_2.., \mathbf{p}_n; A)$ to $A(\mathbf{p}) := (A(\mathbf{p}_1), A(\mathbf{p}_2).., A(\mathbf{p}_n))$. Since its domain is irreducible, the image of the polynomial map, g_i, must be an irreducible semi-algebraic set. Moreover, since the identity map is in $\mathcal{A}$, the image of g_i must contain S_i. But since a component, by definition, must be maximal, S_i cannot be a strict subset of this irreducible image. Thus it agrees with it. Thus S_i is invariant with respect to invertible affine transforms. ∎

Now we describe how we slice up the configuration space.

Definition 5.4. Let G be a graph. We will consider a framework $(G, \mathbf{p})$ as a single point in the configuration space, $\mathbb{R}^{nd}$. Let IR be the set of infinitesimally rigid frameworks of G in $\mathbb{R}^d$.

Let IF be the infinitesimally flexible frameworks, with a d-dimensional affine span.

Lemma 5.5. *The set IR is a smooth irreducible semi-algebraic set, invariant with respect to invertible affine transforms.*

Proof. If G is generically infinitesimal rigid in $\mathbb{R}^d$, then IR is a Zariski open subset of configuration space. Otherwise it is empty. Thus, IR is a smooth semi-algebraic set. From Lemma 5.1 we see that IR is invariant with respect to invertible affine transforms. ∎

Next we split up IF into nicely behaved smaller sets.

Splitting by rank. We take IF and we partition it using the rigidity matrix rank of each $\mathbf{p}$ in IF.

In particular we start with IF, which is a semi-algebraic set. Let r be the maximum rank of all of the rigidity matrices in IF. Let IF$'$ be the set of configurations in IF with rigidity matrices of rank $< r$. We partition IF = IF$' \cup W$, where $W := (\text{IF} - \text{IF}')$. As rank dropping can be expressed as an algebraic equality condition, both IF$'$ and W are semi-algebraic sets. All of the frameworks in W have rigidity matrices of the fixed rank r. From Lemma 5.1 we see that IF$'$ and thus also W are invariant with respect to invertible affine transforms.

We can then apply the above splitting recursively on IF$'$. When this terminates (by descent on r), collecting all of the resulting W, we have partitioned IF a finite set of semi-algebraic subsets $\{A_1..A_k\}$ for some k. Each A_i is invariant with respect to invertible affine transforms.

Singularity splitting step. Given a semi-algebraic set A of dimension s, the set can be partitioned, semi-algebraically, as $A = W \cup A'$, where $A' := \text{Sing}(A)$ and $W := A - A'$.

By construction, W must be smooth and of dimension s. From Lemma A.4 the dimension of A' must be strictly less than s. From Lemma 5.2, if A is invariant to affine transforms, then so too is A' and thus also W.

We can recursively apply this procedure to A'. By descent on s, this process must terminate, giving us a collection of W-sets.

Applying this recursive splitting over all of the A_i from the singularity splitting step, we collect all of the W-sets to obtain a set of semi-algebraic sets we call $\{B_1..B_l\}$ for some $l \in \mathbb{N}$. Each B in the resulting collection must be smooth and affine invariant.

Component splitting step. Given a semi-algebraic set B, the set can be written uniquely as the finite union of a set of semi-algebraic irreducible components. From Lemma 5.3, if B is invariant to affine transforms, then

so too is each of its components. If B is smooth, then from Lemma A.6, so too are each of its components.

We collect all of the components over all of the B_i from the singularity splitting step to obtain a set of semi-algebraic sets we call $\{\mathrm{IF}_1..\mathrm{IF}_m\}$ for some m.

We summarize the conclusion of this discussion as follows.

Lemma 5.6. IF *can be written as the union of a finite set of semi-algebraic sets* $\{\mathrm{IF}_1..\mathrm{IF}_m\}$ *for some* m, *where each* IF_i *is smooth, irreducible and affine-invariant. Each* IF_i *is defined over a finite extension of* $\mathbb{Q}$. *All configurations in one* IF_i *share their rigidity matrix rank.*

Definition 5.7. Let C_i be the codimension of IF_i within the nd-dimensional set of configurations.

Let F_i' be the dimension, for each framework in IF_i, of its space of infinitesimal flexes. Let $F_i := F_i' - \binom{d+1}{2}$, which discounts the dimension of the trivial infinitesimal flexes.

Definition 5.8. Let $\mathrm{Str}(\mathrm{IR})$ and $\mathrm{Str}(\mathrm{IF}_i)$ be the union of the equilibrium stress matrices over its framework set.

Lemma 5.9. *The set* $\mathrm{Str}(\mathrm{IR})$ *and each* $\mathrm{Str}(\mathrm{IF}_i)$ *is semi-algebraic.*

Proof. This follows immediately using quantifier elimination.   ∎

Remark 5.10. In the above decomposition, the property of constant rank rigidity matrices will be used throughout our reasoning.

The invariance to invertible affine transforms will be needed in Lemmas 5.11 and 5.13, where we need to carefully count the dimension of $\mathrm{Str}(\mathrm{IR})$ and the $\mathrm{Str}(\mathrm{IF}_i)$.

The irreducibility of each IF_i will be important in Section 5.3, where we want there to be a well defined notion of a generic point, and thus a well defined generic dimension of tangential flexes.

The smoothness of each IF_i will be convenient throughout, but will be especially needed in Lemma 5.18, where we will want the generic frameworks of IF_i to be dense in IF_i. (A non smooth real algebraic set, such as the Whitney umbrella can have a locus of singular points, such as the handle of the umbrella, that have no nearby smooth points.)

The following Lemma is not needed for the proof of our theorem, but is useful in setting up the proof of Lemma 5.13 below.

Lemma 5.11. *Let G be generically globally rigid in $\mathbb{R}^d$. Then*

$$\dim(\mathrm{Str}(\mathrm{IR})) = m - \binom{d+1}{2} = D_L.$$

Proof. We want to count the total dimension of equilibrium stresses over all frameworks in IR, but we need to be careful not to double count.

From Theorem 2.4 the dimension of IR is nd.

The rank of the rigidity matrix of any framework $(G,\mathbf{p})$ in IR is $nd-\binom{d+1}{2}$, and so the dimension of equilibrium stresses for this single framework is $m-nd+\binom{d+1}{2}$.

Let the equilibrium stress bundle of IR, a subset of $\mathbb{R}^{nd} \times \mathbb{R}^m$, consist of pairs $(\mathbf{p},\omega)$, where $(G,\mathbf{p}) \in \mathrm{IR}$ and ω is an equilibrium stress of $(G,\mathbf{p})$. This is a vector bundle over IR. The projection, π_2, of this bundle onto its second factor gives us $\mathrm{Str}(\mathrm{IR})$.

From Lemma B.2 the bundle is irreducible and has dimension $nd+[m-nd+\binom{d+1}{2}]=m+\binom{d+1}{2}$.

Let us now look at one $(\mathbf{p},\Omega)$, some generic point of the bundle. From Lemma A.11, the configuration $\mathbf{p}$ must be a generic configuration. Since G is generically globally rigid, then from Theorem 2.8, the generic framework $(G,\mathbf{p})$ must have an equilibrium stress matrix of rank $n-d-1$. Thus, from genericity, Ω must achieve this rank.

Now we look at π_2, in the neighborhood around $(\mathbf{p},\Omega)$. Since Ω has rank $n-d-1$, the fiber of π_2 consists of affine transforms of a $\mathbf{p}$ and must have dimension $d(d+1)$. So from Theorem A.12, the dimension of the image must be $m+\binom{d+1}{2}-d(d+1)=m-\binom{d+1}{2}$.

From Theorem 2.5, G must be $(d+1)$-connected and so from Corollary 3.15 this dimension agrees with D_L. ∎

Remark 5.12. When G is not generically globally rigid in $\mathbb{R}^d$, then all of the stresses in $\mathrm{Str}(\mathrm{IR})$ must be of rank less than $n-d-1$, and thus the fibers under π_2 are larger and thus $\dim(\mathrm{Str}(\mathrm{IR})) < m - \binom{d+1}{2}$.

5.2. The IF$_i$ with few infinitesimal flexes can only account for low dimensional subsets of LSS

Lemma 5.13. *Let G be any graph. Then for any i such that $C_i > F_i$, we have $\dim(\mathrm{Str}(\mathrm{IF}_i)) < m - \binom{d+1}{2}$. Thus, when G is $(d+1)$-connected, then $\dim(\mathrm{Str}(\mathrm{IF}_i)) < D_L$.*

Proof. We proceed exactly as in the proof of Lemma 5.11.

By assumption the dimension of IF_i is $nd - C_i$ and the dimension of equilibrium stresses for any single framework in IF_i is $m - nd + \binom{d+1}{2} + F_i$.

Let the equilibrium stress bundle of IF_i, a subset of $\mathbb{R}^{nd} \times \mathbb{R}^m$, consist of pairs $(\mathbf{p}, \omega)$, where $(G, \mathbf{p}) \in \mathrm{IF}_i$ and ω is an equilibrium stress of $(G, \mathbf{p})$. The projection, π_2, of this bundle onto its second factor gives us $\mathrm{Str}(\mathrm{IF}_i)$.

From Lemma B.3, this bundle is irreducible and has dimension $[nd - C_i] + [m - nd + \binom{d+1}{2} + F_i]$ which by assumption is strictly less than $m + \binom{d+1}{2}$.

The fiber of π_2 around some generic $(\mathbf{p}, \Omega)$ includes at least the invertible affine images of $\mathbf{p}$. Since, by assumption, $\mathbf{p}$ has a full dimensional affine span, this fiber has dimension at least $d(d+1)$. Thus, the dimension of the image of π_2 is strictly less than $m - \binom{d+1}{2}$.

From Corollary 3.15, when G is $(d+1)$-connected, $D_L = m - \binom{d+1}{2}$. ∎

5.3. Any IF_i with many tangential infinitesimal flexes cannot account for any stresses in LSS

Definition 5.14. Let T_i' be the dimension, at any generic point $\mathbf{p}$ in IF_i, of the space of infinitesimal flexes in $\mathbb{R}^d$ of $(G, \mathbf{p})$ (each thought of a single vector in $\mathbb{R}^{nd}$) that are tangential to the manifold IF_i at $\mathbf{p}$. Define $T_i := T_i' - \binom{d+1}{2}$, to discount the infinitesimal flexes arising from the group $SE(d)$ (these must all be tangential, since IF_i is invariant with respect to invertible affine transforms). Non-generically within IF_i, the dimension of tangential infinitesimal flexes can rise.

Let $X_i := F_i - T_i$. This quantity represents the dimension, at any generic point $\mathbf{p}$ in IF_i, of a linear space of (necessarily non-trivial) infinitesimal flexes that is linearly independent from the tangent space of IF_i at $\mathbf{p}$.

Remark 5.15. As mentioned in the previous definition, non-generically within IF_i, the dimension of tangential infinitesimal flexes can rise. One might be tempted to simply refine our stratification based on this property, cutting out such loci into their own IF_i. The problem with this approach is the resulting subdivided IF_i might not be invariant to invertible affine transformations.

To see the difficulty, suppose $\mathbf{p}$ and $\mathbf{q}$ in some (unsubdivided) IF_i are related by a d-dimensional invertible affine transform with $\mathbf{M}$ as its linear factor. Then the tangent of IF_i at $\mathbf{p}$ will map under $\mathbf{M}$ to the tangent of IF_i at $\mathbf{q}$. On the other hand, some flex $\mathbf{p}'$ of $(G, \mathbf{p})$ will map to a flex of $(G, \mathbf{q})$ through the adjoint map $\mathbf{M}^{-t}$.

Lemma 5.16. *Let G be any graph. For any IF_i such that $T_i \geq 1$, every generic framework $(G, \mathbf{q})$ in IF_i must be locally flexible in $\mathbb{R}^d$.*

Proof. Consider the smooth map from IF_i to $\mathbb{R}^m$ that measures squared edge lengths. The kernel of the linearization of the map at any configuration $\mathbf{p} \in \mathrm{IF}_i$ consists of the infinitesimal flexes for $(G, \mathbf{p})$ that are tangential to IF_i. A regular point of this map is a configuration where the dimension of the kernel of the linearization of the map is at its minimum (and generic) value, $T_i + \binom{d+1}{2}$. Every generic point in IF_i is a regular point of this map.

At a regular point, $\mathbf{q}$, using the constant rank theorem, we see that locally, there is a $T_i + \binom{d+1}{2}$-dimensional submanifold of IF_i that maintains the edge lengths of $\mathbf{q}$. This fiber gives us our desired non-trivial, finite flex. ∎

Lemma 5.17. *If $(G, \mathbf{p})$ is a framework in IF_i (resp. IR) and is super stable, then so too is any other nearby-enough framework $(G, \mathbf{q})$ in IF_i (resp. IR).*

Proof. By assumption, $(G, \mathbf{p})$ has a PSD equilibrium stress matrix Ω of rank $n-d-1$. From Lemma B.4, any nearby $(G, \mathbf{q})$ in IF_i (resp. IR) must have some equilibrium stress matrix close to Ω. Since eigenvalues vary continuously with the matrix, this nearby equilibrium stress matrix must retain its $n-d-1$ positive eigenvalues. As an equilibrium stress matrix, it cannot gain any more non-zero eigenvalues, and is thus PSD.

By assumption, $(G, \mathbf{p})$ does not have its edges on a conic at infinity. Since frameworks with their edges on a conic at infinity are a proper algebraic subset of configuration space, if $(G, \mathbf{p})$ does not have its edges on a conic at infinity, then neither do nearby frameworks in $\mathbb{R}^{nd}$. ∎

Lemma 5.18. *Let G be any graph. For any IF_i such that $T_i \geq 1$, it must be that $\mathrm{Str}(\mathrm{IF}_i)$ is disjoint from LSS.*

Proof. If any framework $(G, \mathbf{p})$ in IF_i were super stable, then from Lemma 5.17, so too would be any nearby enough framework in IF_i. Since IF_i is smooth, the generic frameworks of IF_i are (Euclidean) dense in IF_i (Lemma A.10). Thus, there would be a nearby $\mathbf{q}$ which is generic in IF_i and such that $(G, \mathbf{q})$ is super stable. But from Lemma 5.16, since $T_i \geq 1$, there can be no such $(G, \mathbf{q})$. So $(G, \mathbf{p})$ cannot be super stable.

On the other hand, from Theorem 3.13, all frameworks (with a d-dimensional affine span) arising as the kernels of the stresses from LSS, must be super stable. ∎

5.4. When G is generically globally rigid, there can be no IF_i with many transverse infinitesimal flexes

Lemma 5.19. *Let G be generically globally rigid in $\mathbb{R}^d$. Then for all i, we have $C_i > X_i$.*

Proof. Otherwise we could apply Connelly's global flexibility argument from [10] and obtain a contradiction with the assumed generic global rigidity. For completeness, we will spell out this argument in detail.

We first record the following principle [10, Theorem 6.1].

Lemma 5.20. *Suppose that $\mathbf{p}'$ is an infinitesimal flex for a framework $(G,\mathbf{p})$ in $\mathbb{R}^d$, where the points of $\mathbf{p}$ do not all lie in a hyperplane. Then $(G,\mathbf{p}+\mathbf{p}')$ has the same edge lengths as $(G,\mathbf{p}-\mathbf{p}')$. Moreover, $\mathbf{p}+\mathbf{p}'$ is congruent to $\mathbf{p}-\mathbf{p}'$ iff $\mathbf{p}'$ is a trivial infinitesimal flex.*

From Lemma B.5, we can define a rational map (f stands for flex), $f(\mathbf{p},\mathbf{x})\colon \mathrm{IF}_i\times\mathbb{R}^{F_i}\to\mathbb{R}^{nd}$ that (over its domain/where there is no division by zero) maps to a non-trivial infinitesimal flex $\mathbf{p}'\in\mathbb{R}^{nd}$ of $\mathbf{p}$, and for a fixed $\mathbf{p}$, the map is a linear injective map over $\mathbf{x}$. Let $\mathbf{p}^0$ denote some configuration that is generic in IF_i, around which f is well defined.

Given f, define the rational map (o stands for offset), $o(\mathbf{p},\mathbf{x}):=\mathbf{p}+f(\mathbf{p},\mathbf{x})$, that offsets $\mathbf{p}$ by an infinitesimal flex. Now we look at the image of the linearization, o^* at $(\mathbf{p}^0,\mathbf{0})$. By varying just the $\mathbf{p}$-variables, we see that this image contains the tangent space of IF_i at $\mathbf{p}^0$ (of dimension $nd-C_i$). By varying just the $\mathbf{x}$-variables, we see that this image contains a space of non-trivial flexes of dimension F_i. Since $\mathbf{p}^0$ is generic in IF_i, this space contains a linear space of dimension X_i that is linearly independent from the tangent space of IF_i at $\mathbf{p}^0$. Thus, the image of the linearization is of dimension at least $nd-C_i+X_i$ (and no greater than nd). When $C_i\le X_i$, this rank is nd and so we have a local submersion. As a result, the image of o has dimension nd. Thus, we have shown that a full dimensional subset of configurations can be reached by starting with an infinitesimally flexible framework in IF_i and adding to it some non-trivial infinitesimal flex.

From Lemma 5.20, when $\mathbf{p}$ has a full d-dimensional affine span (as all frameworks in IF do by assumption) and $\mathbf{p}'$ is a non-trivial infinitesimal flex, then $\mathbf{p}+\mathbf{p}'$ must be not globally rigid in $\mathbb{R}^d$. But our construction has found a full dimensional set of such configurations, which contradicts the assumed generic global rigidity. ∎

5.5. Putting the cases together

Having dealt with all the possibilities, we arrive at our main proposition:

Proposition 5.21. *If G is generically globally rigid in $\mathbb{R}^d$, then a full dimensional subset of* LSS *is contained in* Str(IR).

Proof. The sets, IR and IF_i cover all configurations with a full affine span, and thus the union Str(IR) and the Str(IF_i) must contain all of equilibrium stresses of all configurations with full affine spans. Meanwhile, from Theorem 3.13, all of the stresses in LSS arise as equilibrium stress matrices of configurations in general affine position and thus with full affine spans. Thus LSS must be contained in the union Str(IR) and Str(IF_i).

First we show that LSS must be disjoint from the Str(IF_i) where $C_i \leq F_i = X_i + T_i$. When $C_i \leq F_i$, then either (a) $C_i \leq X_i$ or (b) $T_i \geq 1$. The case (a) cannot occur at all due to Lemma 5.19. In case (b), LSS must be disjoint from Str(IF_i) due to Lemma 5.18.

Next we look at the Str(IF_i) where $C_i > F_i$. From Theorem 2.5, G is $(d+1)$-connected. But then from Lemma 5.13, these Str(IF_i) are of lower dimension than LSS. Thus, only a low dimensional subset of LSS can be contained in these Str(IF_i).

Thus, a full dimensional subset of LSS must not be contained in the union of the Str(IF_i) and thus must be be contained in Str(IR). ∎

Remark 5.22. When G is $(d+1)$-connected but not generically globally rigid in $\mathbb{R}^d$, then from the above discussion we see that almost all of the stresses in LSS must come from IF_i, where $C_i \leq X_i$.

And now we can prove our main theorem.

Proof of Theorem 2.10. From Proposition 5.21, there must be an Ω that is both in LSS and in Str(IR). Thus, there must be a framework $(G, \mathbf{p})$ which is infinitesimally rigid and has Ω as one of its equilibrium stress matrix. From Theorem 3.13, Ω must be PSD of rank $n-d-1$, and so $(G, \mathbf{p})$ must be super stable.

For the second part, we use Lemma 5.17 to conclude that any nearby framework in IR must be super stable. As IR is full dimensional and open, any nearby framework in configuration space must be infinitesimally rigid and super stable. Such a neighborhood contains a generic configuration. ∎

6. The stress variety

The main theorem in this paper relates to a deeper question about the algebraic set of stress matrices. As above, let G be a graph with n vertices and m edges, and d a fixed dimension.

Definition 6.1. Let Str be the real algebraic set of n-by-n d-*dimensional stress matrices* for G. Specifically, this is the set of real symmetric matrices that have 0 entries corresponding to non-edges of G, with the all-ones vector in its kernel, and with rank $n-d-1$ or less.

The set Str is the union of $\mathrm{Str}(\mathrm{IR})$ and all of the $\mathrm{Str}(\mathrm{IF}_i)$ described above. In particular since the kernel of an $\Omega \in \mathrm{Str}$ is of dimension at least $d+1$, we can always pick a framework $\mathbf{p}$ with a d-dimensional affine span with spatial coordinates in this kernel. Clearly, Ω is an equilibrium stress matrix for $\mathbf{p}$.

Question 6.2. Suppose that G is generically globally rigid in $\mathbb{R}^d$. Is its associated d-dimensional stress variety, Str, irreducible?

There are some results in the literature about the irreducibility of certain linear sections of determinantal varieties [17,16], but these do not appear to be strong enough to answer the present question.

The irreducibility of Str would be useful, since any strict algebraic subset W of an irreducible (semi-)algebraic set must be of strictly lower dimension! In particular, an affirmative answer to this question would then lead to an alternative direct proof of Theorem 2.10, which we now sketch:

As described in Section B.3, we can select a rational map that maps from a matrix $\Omega \in \mathrm{Str}$ to a framework in its kernel with a d-dimensional affine span. In the image of this map, $d+1$ chosen vertices will always lie in some pinned positions. The map will be undefined over some subvariety V of Str. (The subvariety V consists of all of the Ω of rank strictly less than $n-d-1$ and any Ω of rank $n-d-1$ which is an equilibrium stress matrix of a d-dimensional framework where the chosen $d+1$ vertices lie in a single hyperplane.)

The preimage of the algebraic set, IF, must lie in some algebraic subset W of Str. By Theorem 2.8, this subset of Str is strict.

Suppose that there is an $\Omega \in \mathrm{Str}(\mathrm{IF})$ that has rank $n-d-1$ and is the equilibrium stress of an infinitesimally flexible framework $(G,\mathbf{p})$ with the chosen $d+1$ vertices in general affine position. Then our rational map must map Ω to a configuration $\mathbf{q}$ which is an affine transform of $\mathbf{p}$. The framework $(G,\mathbf{q})$ must be in IF, and thus $\Omega \in W$.

Thus $\text{Str}(\text{IF})$ must lie in the union of V and W. The rest of the matrices, $\text{Str} - (V \cup W)$, must be in $\text{Str}(\text{IR})$. If Str is irreducible, the subset $V \cup W$ must be of strictly lower dimension than Str itself.

Meanwhile, Lemma 5.11 tells us that the dimension of $\text{Str}(\text{IR})$ is $m - \binom{d+1}{2}$. Thus the dimension of $\text{Str}(\text{IF})$ is strictly less than $m - \binom{d+1}{2}$ and thus less than D_L. And we are done. ∎

Our question is also related to one posed by Lovász et al. [32]. Recalling the definition of an OR from Definition 3.1 they ask: under what conditions is the set of ORs irreducible? Some progress on this question is reported in [23].

7. Graph realization SDP

Our results relate to the problem of finding a framework $(G, \mathbf{p})$ with a specific set of desired edge lengths.

Definition 7.1. Let $(G, \mathbf{p})$ be a d-dimensional framework, and let

$$\ell = (\ell_{ij})_{\{i,j\} \in E(G)} := (|\mathbf{p}_i - \mathbf{p}_j|^2)_{\{i,j\} \in E(G)}$$

be the vector of squared edge length measurements. The *graph realization problem* is to find $(G, \mathbf{p})$ given G, ℓ and d. This problem is NP-hard [36]. An instance is well-posed if and only if $(G, \mathbf{p})$ is globally rigid.

Due to its wide applicability, graph realization, and related "distance geometry problems", have received a lot of attention. See the survey [29] for an overview. Given the problem's hardness, practical algorithms will be heuristic[1] in nature, or involve restricting G to some class that is smaller than being generically globally rigid. An important practical approach is based on semidefinite programming (see [41] for a general overview of SDP).

Definition 7.2. Let S_n be the cone of symmetric $n \times n$ real matrices. Let S_n^+ be the cone of symmetric $n \times n$ positive semidefinite matrices, and define an inner product on $n \times n$ matrices by $\langle X, Y \rangle := \text{Tr}\, X^t Y$. A *semidefinite program* (SDP) is an optimization problem of the form

$$\inf_X \{\langle X, \beta \rangle : X \in S_n^+ \cap (L + b)\},$$

---

[1] When G is K_n, one rigorous notion of an approximate solution is a low-distortion embedding (see, e.g., [34, Chapter 15] or [30]). This is a bit different in flavor from distance geometry where the dimension constraint and being exact on the given distances are most important.

where b and β are in S_n and L is a linear subspace of S_n. Semidefinite programming is a convex problem that can be approximated in polynomial time.

A semidefinite program for graph realization has been studied for some time (see [26,3,31]; [29, Section 4] and the references there).

Definition 7.3. Given a framework $(G,\mathbf{p})$ and its edge measurement vector ℓ, define A to be the space of $n{\times}n$ matrices X such that $X_{ii}+X_{jj}-2X_{ij}=\ell_{ij}$ for all edges $\{i,j\}\in E(G)$. (Notice that A is affine.)

The *graph realization semidefinite program* is

$$\inf_X\{\langle X, 0\rangle : X \in S_n^+ \cap A\}.$$

By treating X as the Gram matrix of $\mathbf{p}$ we can recover $\mathbf{p}$ from X.

We say that the graph realization SDP *succeeds on* $(G,\mathbf{p})$ if the only feasible points of the SDP for the associated problem correspond to configurations congruent to $\mathbf{p}$; otherwise we say that it *fails*. (Remember that we will only get a numerical approximation to $\mathbf{p}$ from an SDP solver.)

Remark 7.4. The presentation above follows that in [18].

The graph realization SDP is a convex relaxation of the rank constraint [37] on a Gram matrix for a d-dimensional point set. As the description suggests, it is not difficult to implement, and, when it succeeds, will "guess" the correct dimension d. When it fails, solvers based interior point methods [35] will return a higher dimensional solution. Thus, it is interesting to know, from G only, whether the SDP can succeed on any positive measure set of $\mathbf{p}$.

A connection to universal rigidity was made by Zhu, So, and Ye [44] (building on work of So and Ye [38]).

Theorem 7.5. *Let (G,p) be a generic d-dimensional framework with edge measurement vector ℓ. The graph realization SDP succeeds on the graph realization instance given by G, ℓ and d if and only if $(G,\mathbf{p})$ is universally rigid.*

Combining Theorem 7.5 with our Theorem 2.10, we obtain:

Corollary 7.6. *Let G be a graph and fix a dimension d. Then there is a Euclidean open set of frameworks $(G,\mathbf{p})$ for which the graph realization semidefinite program succeeds if and only if G is generically globally rigid.*

Since we know that universal rigidity is not a generic property, this result is, in a sense, a tight description of which combinatorial types of framework the semidefinite programming algorithm succeeds on. (For example, if we draw **p** from a continuous density, Corollary 7.6 implies that the semidefinite program has a positive probability of success if and only if G is generically globally rigid.)

Characterizing the graphs for which every generic $(G, \mathbf{p})$ is universally rigid, and thus the semidefinite relaxation is tight with probability one, is an open problem.

Appendix

A. Algebraic geometry background

Throughout this paper, we use some basic facts about real algebraic and semi algebraic sets. Here we summarize some preliminaries from real algebraic geometry, somewhat specialized to our particular case. For a general reference, see, for instance, the books [8,6]. Much of this is adapted from [20]. We will spend a bit of time dealing explicitly with some issues of defining fields and genericity as these issues are not fully covered in any single elementary text.

Definition A.1. Let **k** be a subfield of $\mathbb{R}$. An (embedded) affine, real *algebraic set* or *variety* V defined over **k** is a subset of $\mathbb{R}^n$ that can be defined by a finite set of algebraic equations with coefficients in **k**.

A *Zariski open set* is a subset of $\mathbb{R}^n$ defined by removing an algebraic subset.

A real algebraic set has a real *dimension* $\dim(V)$, which we will define as the largest t for which there is an open subset of V, in the Euclidean topology, that is a smooth t dimensional smooth sub-manifold of $\mathbb{R}^n$.

Any nested sequence of strict algebraic subsets must terminate in a finite number of steps. (This is called the Noetherian property). This means that if we continue to take strict algebraic subsets, we must eventually be left with the empty set.

An algebraic set is *irreducible* if it is not the union of two proper algebraic subsets defined over $\mathbb{R}$.

Any reducible algebraic set V can be uniquely described as the union of a finite number of maximal irreducible algebraic subsets called the *components* of V.

Any algebraic subset of an irreducible algebraic set must be of strictly lower dimension.

Lemma A.2. *If a real algebraic set V is defined over* **k**, *a subfield of* $\mathbb{R}$, *then any of its components can be defined over a finite extension of* **k**, *also a subfield of* $\mathbb{R}$.

Proof. Let us define V^*, the *complex Zariski closure* of V, to be the smallest algebraic subset of $\mathbb{C}^n$, defined by polynomials with complex coefficients, that contains V.

If V is defined over **k**, so too is V^* [43, Lemma 6]. Each component of V^* is defined over a subfield of the reals [43, Lemma 7]. Each component of a real algebraic set V is simply the real locus of a corresponding component of V^* [43, Lemma 7]. Thus, our Lemma reduces to understanding the defining field of the components of V^*.

Meanwhile it is standard fact from scheme theory, that given a complex variety V^* defined over **k**, some subfield of $\mathbb{C}$, its irreducible components are themselves defined over some finite extension of **k**. In particular, from [40, Tag 038I], it suffices to just look at the components that are irreducible when working over an algebraic closure of **k**. Then [40, Tag 04KZ], tells us that each of these components is defined over some finite extension of **k**. ∎

Definition A.3. A *semi-algebraic set S* defined over **k** is a subset of $\mathbb{R}^n$ that can be defined by a finite set of algebraic equalities and inequalities with coefficients in **k**, as well as a finite number of Boolean operations. A semi-algebraic set has a well defined (maximal) dimension t which we will define as the largest t for which there is an open subset of S, in the Euclidean topology, that is a smooth t dimensional sub-manifold of $\mathbb{R}^n$.

Any algebraic set is also a t-dimensional semi-algebraic set.

A semi-algebraic set is comprised of a finite number of connected components [8, Theorem 2.4.4].

The real *Zariski closure* of S is the smallest real algebraic set defined over $\mathbb{R}$ containing S.

We call S *irreducible* if its real Zariski closure is irreducible.

A semi-algebraic set S has the same real dimension as its real Zariski closure (see [8, Prop 2.8.2] or [39, Lemma 2]). Thus, if two irreducible semi-algebraic sets of the same dimension have an intersection of that same dimension, then their union must be irreducible.

Any reducible semi-algebraic set S can be uniquely described as the union of a finite number of maximal irreducible semi-algebraic sets called the *components* of S. Each component of S is the intersection of S with a component of the real Zariski closure of S. Thus, if S is defined over **k**, then any of its components can be defined over a finite extension of **k**.

The image of a real semi-algebraic (or algebraic) set under a polynomial or rational map, all defined over **k** is semi-algebraic and defined over **k** [6,

Theorem 2.76]. As a corollary to this, quantifiers can always be eliminated from any first-order formula over the reals, involving polynomial equalities and inequalities, rendering its feasible set semi-algebraic.

The image of a real semi-algebraic set under an injective polynomial or rational map has the same dimension as its domain (see [8, Prop 2.8.8]). The image of an irreducible real semi-algebraic set under a polynomial or rational map is irreducible (see the proof of [8, Prop 2.8.6]).

We call a point on S *smooth* if it has a neighborhood in S that is a smooth sub-manifold of $\mathbb{R}^n$ of dimension $\dim(S)$. (In the semi-algebraic setting, any such smooth sub-manifold will also be a real analytic sub-manifold of $\mathbb{R}^n$ (see [8, Prop 8.1.8])).

If all points of S are smooth, then S is called smooth.

Any semi-algebraic set S of dimension t, defined over **k** can be stratified into the finite disjoint union of smooth semi-algebraic sets of various dimensions defined over **k**. In the stratification, the Euclidean closure in S of one stratum consists of itself and some lower dimensional strata. (See [6, Theorem 5.38] for a detailed description.)

The smooth and non-smooth loci of points of S form semi-algebraic sets, also defined over **k** (the proof of semi-algebraicity in [39], shows how these loci can be defined using quantifier elimination, which establishes that **k** is a defining field).

Lemma A.4. *Let S be a semi-algebraic set of dimension t. Then its singular locus has dimension $< t$.*

Proof. In any smooth stratification of S, points that are not smooth cannot lie in a top-dimensional stratum. (See also [6, Prop 5.53]). ∎

Lemma A.5. *Let S be a smooth and connected semi-algebraic set. Then its real Zariski closure is irreducible.*

(See [8, Prop 8.4.1]).

Lemma A.6. *Let S be a smooth semi-algebraic set. Then each of its irreducible components is smooth.*

Proof. From Lemma A.5, if B is smooth and connected, then it must, itself be irreducible.

In general, S might consist of some finite number of disjoint connected components. In this case, each of the irreducible components of S consists exactly of those connected components that have a common real Zariski closure.

Thus, an irreducible component is the union of disjoint smooth semi-algebraic sets and is thus smooth. ∎

Lemma A.7. *Let S be a semi-algebraic set of dimension t defined over* **k**. *Then the real Zariski closure of S is defined (as a variety) over a finite extension of* **k**.

Proof. Due to the finite stratification, we can assume that S is smooth and connected, of some dimension s. (Then we can just take the finite union over these strata, as the closure of their union is the union of their closures.)

From Lemma A.5, the real Zariski closure V, of S is irreducible and of dimension s. Meanwhile S must be contained in some algebraic set W, that has dimension s and is defined over **k** (see [39, Lemma 2]). As the real Zariski closure of S must be contained in any algebraic set containing S, we must have $V \subset W$. But since V and W have the same dimension, V must be a (maximal) component of W (any algebraic set that is a strict subset of an irreducible component of W must be of lower dimension). From Lemma A.2 components of algebraic sets can always be defined using a finite extensions, thus we are done. ∎

Definition A.8. Let **k** be a countable subfield of $\mathbb{R}$. A point in an irreducible (semi-)algebraic set V defined over **k** is *generic* if its coordinates do not satisfy any algebraic equation with coefficients in **k** besides those that are satisfied by every point on V (such equations are called trivial).

A point that satisfies some non-trivial algebraic equation with coefficients in **k**′, some finite extension of **k**, will always also satisfy some non-trivial algebraic equation with coefficients in **k** (see e.g. [19, Lemma 23]). Thus, a point will remain generic when a finite field extension is applied to the defining field, which might occur, say, when passing from a semi-algebraic set to its real Zariski closure or when splitting an algebraic set into its components.

Almost every point in an irreducible (semi-)algebraic set V defined over **k** is generic.

Lemma A.9. *Every generic point of an irreducible (semi-)algebraic set, defined over a countable field* **k**, *is smooth.*

Proof. From Lemma A.4, any non-smooth point lies in a lower dimensional semi-algebraic set defined over **k**, which remains so after a Zariski closure. Thus, these points must satisfy some extra equation, defined over a finite extension of **k**, that is non-trivial over S. ∎

Lemma A.10. *Let V be an irreducible smooth (semi-)algebraic set, defined over a countable field* **k**. *Then its generic points are (Euclidean) dense in V.*

Proof. Let ϕ be any non-zero algebraic function on V. Its zero set is closed and of dimension lower than that of V and thus is stratified as a union of finite number of smooth manifolds, each with dimension less than that of V. Since V is a smooth manifold, V_ϕ, the complement of this zero set is open and dense (in the subspace topology) in V. The generic points are the intersection of V_ϕ as ϕ ranges over the countable set of all possible ϕ defined over $\mathbf{k}$. Since V is a Baire space, such a countable intersection of open and dense subsets must itself be a dense subset. $\blacksquare$

Lemma A.11. *Let V and W be irreducible semi-algebraic sets and $f: V \to W$ be a surjective polynomial or rational map, all defined over $\mathbf{k}$, a countable subfield of $\mathbb{R}$. Then if $x \in V$ is generic, $f(x)$ is generic inside W.*

Proof. Consider any non-zero algebraic function ϕ on W defined over $\mathbf{k}$. Then $\phi(f(\cdot))$ is a function on V that is not identically zero. Thus, if x is a generic point in V, $\phi(f(x)) \neq 0$. Since this is true for all ϕ, it follows that $f(x)$ is generic. $\blacksquare$

Theorem A.12. *Let S be an irreducible semi-algebraic set. Let π be a polynomial or rational map from S into $\mathbb{R}^n$, for some n, all defined over $\mathbf{k}$, a countable subfield of $\mathbb{R}$. Let s be a generic point in S, and $N(S)$ a sufficiently small Euclidean neighborhood of s in S. Then $\dim(S) = \dim(\pi(S)) + \dim(\pi^{-1}(\pi(s)) \cap N(s))$.*

Proof. From genericity and Lemma A.9, $N(s)$ is smooth, and the rank of the linearization, π^* is of constant rank. Thus by the constant rank theorem, we have $\dim(N(s)) = \dim(\pi(N(s))) + \dim(\pi^{-1}(\pi(s)) \cap N(s))$.

Since $N(s)$ is smooth, we have $\dim(N(s)) = \dim(S)$.

Next we argue that $\dim(\pi(N(s))) = \dim(\pi(S))$, as follows. If $\dim(\pi(N(s)))$ were smaller than $\dim(\pi(S))$, then the semi-algebraic set, $\pi(N(s))$ could be cut out from $\pi(S)$ by a non-trivial algebraic equation (as the real Zariski closure of $\pi(N(s))$ would be of lower dimension than that of the real Zariski closure of $\pi(S)$). This then means that $N(s)$ could be cut out of S by a non-trivial algebraic equation. But a full dimensional subset cannot be cut out from an irreducible semi-algebraic set by an algebraic equation that doesn't identically vanish. $\blacksquare$

Remark A.13. Indeed it also can be shown (say using algebraic Sard's theorem on a smooth stratification of S) that at generic s, $\dim(S) = \dim(\pi(S)) + \dim(\pi^{-1}(\pi(s)))$, which also means that $\dim(\pi^{-1}(\pi(s))) = \dim(\pi^{-1}(\pi(s)) \cap N(s))$. But we will not need this.

B. Rational maps to kernels of matrices

A number of times in this paper, we will have some algebraic set S of n_1-by-n_2 matrices, and we will want to construct a map takes an $\mathbf{M} \in S$ to some vector in the kernel of $\mathbf{M}$. Let r be the maximal rank over the matrices in S. Here we will outline the general procedure and then work out the specific maps that are used in this paper.

We start by choosing some matrix $\mathbf{M}^0 \in S$ with rank r. We then select r rows of $\mathbf{M}^0$ that are linearly independent. We then find an $(n_2 - r)$-by-n_2 matrix $\mathbf{H}$ (with entries in $\mathbb{Q}$) such that the selected r rows of $\mathbf{M}^0$ together with the added rows from $\mathbf{H}$, form a non-singular matrix.

For any $\mathbf{M} \in S$, let $\mathbf{M}'$ be the square matrix obtained by using the same chosen r rows from $\mathbf{M}$, vertically appended with the matrix $\mathbf{H}$ chosen above. The matrix $\mathbf{M}'$ can only be singular over some strict subvariety of S. (When S is irreducible, singularity can only happen for non-generic $\mathbf{M}$.)

Given any vector $\mathbf{x} \in \mathbb{R}^{n_2 - r}$, we define the vector $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^{n_2}$ as a vector with r leading zeros appended to $\mathbf{x}$. We now define a rational map from $S \times \mathbb{R}^{n_2 - r} \to \mathbb{R}^m$ which maps $(\mathbf{M}, \mathbf{x}) \mapsto (\mathbf{M}')^{-1} \mathbf{v}(\mathbf{x})$. Clearly, this map can be expressed using rational functions of the coordinates of $\mathbf{M}$ and $\mathbf{x}$. The map is not defined wherever $\mathbf{M}'$ is singular. Wherever the map is defined, it maps to some vector in the kernel of $\mathbf{M}$. For a fixed $\mathbf{M}$, the map is linear and injective over $\mathbf{x}$.

This procedure can be used to construct a bundle of matrices together with kernel vectors.

Lemma B.1. *Let S be a d_1-dimensional irreducible semi-algebraic set of n_1-by-n_2 matrices all of rank r. And let $d_2 := n_2 - r$ be the kernel dimension. Let B be the bundle $(\mathbf{M}, \mathbf{y})$ with $\mathbf{M}$ a matrix in S of rank r and $\mathbf{y}$ in the kernel of $\mathbf{M}$. Then B is an irreducible semi-algebraic set of dimension $d_1 + d_2$.*

Proof. We use the general construction described above (starting with a chosen $\mathbf{M}^0$). We can build an injective rational map from $(S \times \mathbb{R}^{d_2})$ to B, of the form $(\mathbf{M}, \mathbf{x}) \mapsto (\mathbf{M}, \mathbf{y})$. The image of each such rational map is an irreducible semi-algebraic set, and as an injective rational map has dimension $d_1 + d_2$.

Such a rational map may be undefined over some subvariety V of S, where the constructed linear system becomes singular, and thus its image may miss some subvariety of B. But we can always pick a different rational map, (that uses, perhaps a different set of rows, and perhaps a different $\mathbf{H}$ matrix) by starting with another $\mathbf{M}^0$, this time in V, so that the map is undefined over a different subvariety W. Since S is irreducible, V, W,

and their union, must be of lower dimension than S, and thus the region of S where both maps are defined is full dimensional. For any matrix where the map is defined, the image, as we vary $\mathbf{M}$, is the entire fiber above that matrix in B. Thus, if two maps are defined over a full dimensional region of S, then their images in B must have a full dimensional intersection. Thus, the union of these images must itself be irreducible.

The subset of S that is not defined under either map, $V \cap W$, is a strict algebraic subset of V. Due to the Noetherian property, a finite number of such rational maps is then guaranteed to have regions of definition that cover S, and thus images that cover B, which thus must be irreducible. ∎

In the next sections, we will use variations on this construction. We alter the construction when we need specific properties of the kernel vectors.

B.1. Stresses of frameworks in IR or IF$_i$

In Lemma 5.11, we wish to understand the structure of the equilibrium stress bundle over IR, a subset of $\mathbb{R}^{nd} \times \mathbb{R}^m$ consisting of pairs $(\mathbf{p}, \omega)$ where $(G, \mathbf{p}) \in \mathrm{IR}$ and ω is an equilibrium stress vector of $(G, \mathbf{p})$. This is a vector bundle over IR. We will do this by looking at a rational map that maps from a framework to each of its equilibrium stresses.

From Theorem 2.4 the dimension of IR is nd.

The rank of the rigidity matrix of any framework $(G, \mathbf{p})$ in IR is $nd - \binom{d+1}{2}$, and so the dimension of equilibrium stresses for this single framework is $m - nd + \binom{d+1}{2}$.

Then Lemma B.1 applied to the co-kernel of the rigidity matrices gives us:

Lemma B.2. *Assume G is generically infinitesimally rigid. The equilibrium stress bundle of IR is irreducible and has dimension $m + \binom{d+1}{2}$.*

Similarly, in Lemma 5.13, we wish to understand the structure of the equilibrium stress bundle over IF$_i$.

By assumption the dimension of IF$_i$ is $nd - C_i$ and the dimension of equilibrium stresses for any single framework in IF$_i$ is $m - nd + \binom{d+1}{2} + F_i$. Again we conclude

Lemma B.3. *The equilibrium stress bundle of IF$_i$ is irreducible and has dimension $[nd - C_i] + [m - nd + \binom{d+1}{2} + F_i]$.*

Lemma B.4. *Let $(G, \mathbf{p}^0)$, a framework in IF$_i$ (resp. IR), have an equilibrium stress ω^0. Then any nearby framework in IF$_i$ (resp. IR) must have an equilibrium stress close to ω^0.*

Proof. Using the general construction above, starting with $\mathbf{p}^0$, we can build a rational map from $(\mathrm{IF}_i \times \mathbb{R}^{m-nd+\binom{d+1}{2}+F_i})$ to $\mathbb{R}^m$, of the form $(\mathbf{p}, \mathbf{x}) \mapsto \omega$, where ω is an equilibrium stress of $\mathbf{p}$, and such that the map is well defined in a neighborhood of $\mathbf{p}^0$. For an appropriate $\mathbf{x}^0$, we have $(\mathbf{p}^0, \mathbf{x}^0) \mapsto \omega^0$. Since this map is continuous, for a nearby $\mathbf{q}$ we must have $(\mathbf{q}, \mathbf{x}^0) \mapsto \omega$, where ω is close to ω^0. ∎

B.2. Infinitesimal flexes of frameworks in IF_i

In Lemma 5.19, we need a rational map that maps from a framework in IF_i to a non-trivial infinitesimal flex of that framework. At a fixed framework, by varying the $\mathbf{x}$ parameters, we wish the image to be an F_i-dimensional space of such flexes.

Lemma B.5. *We can define a rational map, $f(\mathbf{p}, \mathbf{x}) \colon \mathrm{IF}_i \times \mathbb{R}^{F_i} \to \mathbb{R}^{nd}$ (f stands for flex) that (over its domain/where there is no division by zero) maps to a non-trivial infinitesimal flex $\mathbf{p}' \in \mathbb{R}^{nd}$ of $\mathbf{p}$, and for a fixed $\mathbf{p}$, the map is a linear injective map over $\mathbf{x}$.*

Proof. Again, we will use our general construction above to define such a map. But we need to make special care to make sure that the image of the map does not contain any trivial flexes. This requires a bit of care in defining the extra rows to complete our square matrix, as well as how we construct the $\mathbf{v}(\mathbf{x})$ vector.

We proceed as follows: Pick $\mathbf{p}^0$, a generic configuration of IF_i. Pick a subset of $nd - \binom{d+1}{2} - F_i$ edges that are independent in $(G, \mathbf{p}^0)$. Instead of completing this matrix with a constant $\mathbf{H}$ matrix, we do the following: Add a set of F_i "fake edges" to this subset to create a graph G' so that $(G', \mathbf{p}^0)$ is infinitesimally rigid. (This can be done, one by one, as $\mathbf{p}^0$ has a full d-dimensional affine span). Finally create $\mathbf{J}$, an $\binom{d+1}{2}$-by-nd matrix of constants so that the rows of $\mathbf{J}$ form a linear complement to the rows of the rigidity matrix of $(G', \mathbf{p}^0)$.

For any $\mathbf{p}$ we define its nd-by-nd modified, non-singular, rigidity matrix $\mathbf{M}'(\mathbf{p})$, as the rigidity matrix of $(G', \mathbf{p})$ with the added rows of the $\mathbf{J}$ matrix above appended to it. We invert this to obtain $(\mathbf{M}'(\mathbf{p}))^{-1}$.

Let $\mathbf{v}(\mathbf{x})$ be the nd-vector with $nd - \binom{d+1}{2} - F_i$ leading zeros, followed by the F_i coordinates of $\mathbf{x}$, followed by $\binom{d+1}{2}$ more zeros. Then $f(\mathbf{p}, \mathbf{x}) := (\mathbf{M}'(\mathbf{p}))^{-1}\mathbf{v}(\mathbf{x})$ gives us our desired map. Any non-zero coordinates in $\mathbf{x}$ will ensure that some fake edge changes its length at first order, thus making our obtained flex, non-trivial. ∎

B.3. A framework in the kernel of a stress matrix

In section 6 we want a rational map that maps from a stress matrix Ω to a framework in its kernel with a d-dimensional affine span. We can represent such a framework as a vector in $\mathbb{R}^{nd}$. We can represent the kernel condition using the nd-by-nd matrix $\mathbf{I}_d \bigotimes \Omega$.

Again, we can then apply our general construction above. In order to obtain a framework with a full d-dimensional affine span, we set the extra $\mathbf{H}$ rows to represent the pinning of $d+1$ specific vertices. We fix the non-zero elements of the right hand side, $\mathbf{v}$, to place these pinned vertex in general affine position. In this setting, we only want one framework per stress, so there are no free variables $\mathbf{x}$.

The map will be undefined over some subvariety V of Str, (which includes, for example all of the Ω of rank strictly less than $n-d-1$, and all of the equilibrium stress matrices of frameworks where the chosen $d+1$ vertices lie in a single hyperplane).

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