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ON THE WEIGHTED KNESER–POULSEN CONJECTURE

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Abstract

Suppose that $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ are two configurations in $\mathbb{E}^d$, which are centers of balls $\mathbf{B}^d(\mathbf{p}_i, r_i)$ and $\mathbf{B}^d(\mathbf{q}_i, r_i)$ of radius r_i, for $i = 1, \dots, N$. In [9] it was conjectured that if the pairwise distances between ball centers $\mathbf{p}$ are contracted in going to the centers $\mathbf{q}$, then the volume of the union of the balls does not increase. For $d = 2$ this was proved in [1], and for the case when the centers are contracted continuously for all d in [2]. One extension of the Kneser–Poulsen conjecture, suggested in [6], was to consider various Boolean expressions in the unions and intersections of the balls, called flowers, where appropriate pairs of centers are only permitted to increase, and others are only permitted to decrease. Again under these distance constraints, the volume of the flower was conjectured to change in a monotone way.

Here we show that these generalized Kneser–Poulsen flower conjectures are equivalent to an inequality between certain integrals of functions (called flower weight functions) over $\mathbb{E}^d$, where the functions in question are constructed from maximum and minimum operations applied to functions each being radially symmetric monotone decreasing and integrable.

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1. The Kneser–Poulsen conjecture

Let $\|\cdot\|$ denote the standard Euclidean norm of the d-dimensional Euclidean space $\mathbb{E}^d$. So, if $\mathbf{p}_i, \mathbf{p}_j$ are two points in $\mathbb{E}^d$, then $\|\mathbf{p}_i - \mathbf{p}_j\|$ denotes the Euclidean distance between them. It will be convenient to denote the (finite) point configuration consisting of the points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N$ in $\mathbb{E}^d$ by $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$. Now, if $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ are two configurations of N points in $\mathbb{E}^d$ such that for all $1 \le i < j \le N$ the inequality $\|\mathbf{q}_i - \mathbf{q}_j\| \le \|\mathbf{p}_i - \mathbf{p}_j\|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. If $\mathbf{q}$ is a contraction of $\mathbf{p}$, then there may or may not be a continuous motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \dots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbb{E}^d$ for all $0 \le t \le 1$ and $1 \le i \le N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is monotone decreasing for all $1 \le i < j \le N$. When there is such a motion, we say that $\mathbf{q}$ is a *continuous contraction* of $\mathbf{p}$. Finally, let $\mathbf{B}^d(\mathbf{p}_i, r_i)$ denote the closed d-dimensional ball centered at $\mathbf{p}_i$ with radius r_i in $\mathbb{E}^d$ and let $\mathrm{Vol}_d(\cdot)$ represent the d-dimensional volume (Lebesgue measure) in $\mathbb{E}^d$. In 1954 Poulsen [10] and in 1955 Kneser [9] independently conjectured the following for the case when $r_1 = \dots = r_N$:

Conjecture 1.1. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$, *then*

$$\mathrm{Vol}_d\Big[\bigcup_{i=1}^{N} \mathbf{B}^d(\mathbf{p}_i, r_i)\Big] \ge \mathrm{Vol}_d\Big[\bigcup_{i=1}^{N} \mathbf{B}^d(\mathbf{q}_i, r_i)\Big].$$

Conjecture 1.2. *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$, *then*

$$\mathrm{Vol}_d\Big[\bigcap_{i=1}^{N} \mathbf{B}^d(\mathbf{p}_i, r_i)\Big] \le \mathrm{Vol}_d\Big[\bigcap_{i=1}^{N} \mathbf{B}^d(\mathbf{q}_i, r_i)\Big].$$

Actually, M. Kneser seems to be the one who has generated a great deal of interest in the above conjectures also via private letters written to a number of mathematicians. (For more details on this see for example [8]). The state of the art of the Kneser–Poulsen conjecture can be summarized as follows. (We refer the interested reader to [1] for a detailed description of the many partial results known.) First, Csikós [2] has proved Conjecture 1.1 (resp., Conjecture 1.2) for continuous contractions in any dimension. Second, in a recent paper [1] K. Bezdek and Connelly have managed to prove Conjecture 1.1 as well as Conjecture 1.2 in the Euclidean plane for arbitrary contractions.

2. Flowers of balls

Flowers of balls, which are sets built from balls using the lattice operations $\cup$ and $\cap$ have been introduced in [6]. Here we recall their definition following [3]. Let f be a *lattice polynomial* that is an expression built up from some variables using the binary operations $\cup$ and $\cap$ with properly placed brackets indicating the order of the evaluation of the operations. The *sign* of f is defined as follows. If f is the union (resp., intersection) of two shorter lattice polynomials, then sgn $f = 1$ (resp., sgn $f = -1$). If f is a single variable, then sgn $f = 0$. The structure of f can be described with the help of a rooted tree T_f defined recursively on the length of f as follows. If f is a single variable, then T_f is a single vertex labelled with that variable. If sgn $f = 1$ (resp., sgn $f = -1$), then write f in the form $f_1 \cup \dots \cup f_j$ (resp., $f_1 \cap \dots \cap f_j$), where sgn $f_i \leq 0$ (resp., sgn $f_i \geq 0$) for all $1 \leq i \leq j$. Finally, to obtain T_f take the disjoint union of the trees $T_{f_i}, 1 \leq i \leq j$ and a new vertex, the root of T_f labelled with f, and draw an edge from the new vertex f to the roots of the trees $T_{f_i}, 1 \leq i \leq j$. It is clear that if we know the rooted tree T_f and the signs of its vertices, then we can reconstruct f. Also, it will be convenient to write f as $f(x_1, \dots, x_N)$ indicating the variables of f by $x_1, \dots, x_N$. We will always assume that each variable occurs exactly once in f meaning that for each $1 \leq i \leq N$ there is exactly one vertex of T_f labelled by x_i. For $1 \leq i < j \leq N$ consider the paths from the vertices x_i and x_j to the root f. These paths meet each other first at a vertex g. Let $\epsilon_{ij} = \epsilon_{ji}$ denote the sign of the lattice polynomial g. Finally, a *flower of balls* in $\mathbb{E}^d$ is a set of the form $f(\mathbf{B}^d(\mathbf{p}_1, r_1), \dots, \mathbf{B}^d(\mathbf{p}_N, r_N))$, where $f(x_1, \dots, x_N)$ is a lattice polynomial. Now, the main result of Csikós [3] can be phrased as follows.

THEOREM 2.1. *Let* $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \dots, \mathbf{p}_N(t))$ *be a continuous motion with* $\mathbf{p}_i(t) \in \mathbb{E}^d$ *for all* $0 \leq t \leq 1$ *and* $1 \leq i \leq N$ *such that* $\mathbf{p}(0) = \mathbf{p}$ *and* $\mathbf{p}(1) = \mathbf{q}$, *and* $\epsilon_{ij}\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ *is monotone decreasing for all* $1 \leq i < j \leq N$. *Then*

$$\mathrm{Vol}_d[f(\mathbf{B}^d(\mathbf{p}_1, r_1), \dots, \mathbf{B}^d(\mathbf{p}_N, r_N))] \geq \mathrm{Vol}_d[f(\mathbf{B}^d(\mathbf{q}_1, r_1), \dots, \mathbf{B}^d(\mathbf{q}_N, r_N))].$$

Furthermore we also conjecture the following discrete version of Theorem 2.1, which we call the *generalized Kneser–Poulsen Conjecture*:

CONJECTURE 2.2. *Let* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *be point configurations in* $\mathbb{E}^d$ *such that* $\epsilon_{ij}\|\mathbf{p}_i - \mathbf{p}_j\| \geq \epsilon_{ij}\|\mathbf{q}_i - \mathbf{q}_j\|$ *for all* $1 \leq i < j \leq N$, *where* $\epsilon_{ij}, 1 \leq i < j \leq N$ *are the sign coefficients assigned to the lattice polynomial* f. *Then*

$$\mathrm{Vol}_d[f(\mathbf{B}^d(\mathbf{p}_1, r_1), \dots, \mathbf{B}^d(\mathbf{p}_N, r_N))] \geq \mathrm{Vol}_d[f(\mathbf{B}^d(\mathbf{q}_1, r_1), \dots, \mathbf{B}^d(\mathbf{q}_N, r_N))].$$

The original Kneser–Poulsen Conjecture is Conjecture 2.2 when f consists of all unions. Note that Conjecture 2.2 implies Conjecture 1.1 and Conjecture 1.2 as well as Theorem 2.1. Also the main theorem of [1] is Conjecture 2.2 for the plane $d = 2$. We will see next that Conjecture 2.2 also implies some quite seemingly stronger results.

3. The weighted Kneser–Poulsen conjecture

First, as in [4], let us introduce for real numbers x and y the following notation: $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$. Second, motivated by the above definition of flowers one can introduce a special class of real valued functions with N variables as follows. Let $h(x_1, \dots, x_N)$ be an expression built up from the variables $x_1, \dots, x_N$ using the binary operations $\vee$ and $\wedge$ with properly placed brackets indicating the order of the evaluation of the operations. Also, it is assumed that each variable x_i, $1 \le i \le N$, occurs exactly once in h. Just like in the case of a flower one can assign to h a rooted tree T_h and define the sign of h as well as introduce the sign coefficients $\epsilon_{ij} = \epsilon_{ji}$ for all $1 \le i < j \le N$. In short, we call h a *Boolean flower formula* with N variables. Third, let $\mathbf{w}(x) = (w_1(x), \dots, w_N(x))$ be a vector valued function with $w_i(x)$ being a decreasing non-negative function defined on $[0, +\infty)$ and satisfying $\int_0^{+\infty} w_i(x) x^{d-1} dx < +\infty$ for all $1 \le i \le N$ and for a fixed $d > 1$. Call such a function a *proper weight vector function* for $\mathbb{E}^d$. Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ be an arbitrary point configuration in $\mathbb{E}^d$. Then the *flower weight function* $(h \circ \mathbf{w})_{\mathbf{p}}$ of $\mathbf{p}$ generated by $h(x_1, \dots, x_N)$ and $\mathbf{w}(x)$ is defined by

$$(h \circ \mathbf{w})_{\mathbf{p}}(\mathbf{x}) := h(w_1(\|\mathbf{x} - \mathbf{p}_1\|), \dots, w_N(\|\mathbf{x} - \mathbf{p}_N\|)),$$

for all $\mathbf{x} \in \mathbb{E}^d$. Having introduced all this we are ready to phrase the following extension of the Kneser–Poulsen conjecture, which we call the *weighted Kneser–Poulsen conjecture.*

CONJECTURE 3.1. *Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ be point configurations in $\mathbb{E}^d$ such that $\epsilon_{ij}\|\mathbf{p}_i - \mathbf{p}_j\| \ge \epsilon_{ij}\|\mathbf{q}_i - \mathbf{q}_j\|$ for all $1 \le i < j \le N$, where $\epsilon_{ij}, 1 \le i < j \le N$ are the sign coefficients assigned to the Boolean flower formula h with N variables, and $\mathbf{w}$ is any proper weight vector function generating the flower weight functions $(h \circ \mathbf{w})_{\mathbf{p}}$ and $(h \circ \mathbf{w})_{\mathbf{q}}$. Then*

$$\int_{\mathbb{E}^d} (h \circ \mathbf{w})_{\mathbf{p}}(\mathbf{x}) d\mathbf{x} \ge \int_{\mathbb{E}^d} (h \circ \mathbf{w})_{\mathbf{q}}(\mathbf{x}) d\mathbf{x}.$$

4. Indicator functions

For any set X, the corresponding *indicator function* $I_X\colon X \to \{0,1\}$ is defined by

$$I_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}.$$

Note that $I_{\mathbf{B}^d(\mathbf{p}_i,r_i)}(\mathbf{x}) = I_{[0.r_i]}(\|\mathbf{x}-\mathbf{p}_i\|)$. Hence when $w_i = I_{[0.r_i]}$, for $i = 1,\dots,N$,

$$\begin{aligned}\int_{\mathbb{E}^d}(h\circ\mathbf{w})_{\mathbf{p}}(\mathbf{x})d\mathbf{x} &= \int_{\mathbb{E}^d}(h\circ(I_{\mathbf{B}^d(\mathbf{p}_1,r_1)}(\mathbf{x}),\dots,I_{\mathbf{B}^d(\mathbf{p}_N,r_N)}(\mathbf{x}))d\mathbf{x}\\ &= \mathrm{Vol}_d[f(\mathbf{B}^d(\mathbf{p}_1,r_1),\dots,\mathbf{B}^d(\mathbf{p}_N,r_N))],\end{aligned}$$

where h is a Boolean flower formula for $\vee$ and $\wedge$ corresponding to the lattice polynomial f for unions and intersections as indicated in Section 3.

This shows that Conjecture 3.1 implies Conjecture 2.2, which, in turn implies Conjecture 1.1 (resp., Conjecture 1.2) as a special case. Indeed, we show in the next section that Conjecture 3.1 follows from Conjecture 2.2 and, so the two are equivalent, but the weighted Kneser–Poulsen conjecture extends to a larger set of circumstances.

5. Extensions

One natural extension of Conjecture 2.2, which is also a logical consequence, is to take positive linear sums of terms of the sort, where distance constraints are consistent for each of those terms. If f and g are two lattice polynomials on the same variables such that the signs of f are the same as the signs of g, then we say f and g are *compatible*. This can be extended to the case when the set of variables for f and g are not the same by insisting that the signs be the same on the set of variables that they do share. This gives the following:

Theorem 5.1. *Suppose that $f_1,\dots,f_M$ are a set of pairwise compatible lattice polynomials, and Conjecture 2.2 holds in dimension d for those polynomials. Let $\mathbf{p} = (\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1,\mathbf{q}_2,\dots,\mathbf{q}_N)$ be point configurations in $\mathbb{E}^d$, such that $\epsilon_{ij}\|\mathbf{p}_i-\mathbf{p}_j\| \geq \epsilon_{ij}\|\mathbf{q}_i-\mathbf{q}_j\|$ for all $1 \leq i < j \leq N$, where $\epsilon_{ij}, 1 \leq i < j \leq N$ are the sign coefficients assigned to the lattice polynomials $f_1,\dots,f_M$. Then for all $a_k \geq 0$, $k = 1,\dots,M$,*

$$\begin{aligned}&\sum_{k=1}^{M} a_k \mathrm{Vol}_d\left[f_k(\mathbf{B}^d(\mathbf{p}_1,r_1),\dots,\mathbf{B}^d(\mathbf{p}_N,r_N))\right]\\ &\qquad \geq \sum_{k=1}^{M} a_k \mathrm{Vol}_d[f_k(\mathbf{B}^d(\mathbf{q}_1,r_1),\dots,\mathbf{B}^d(\mathbf{q}_N,r_N))].\end{aligned} \tag{1}$$

For the time being we assume that the weight functions of Section 3 are monotone decreasing step functions with a finite number of steps. In other words, we assume that for each $i = 1, \dots, N$ there are positive real numbers $b_{i1}, \dots, b_{iM_i}$ and $r_{i1} < r_{i2} \cdots < r_{iM_i}$ such that

$$w_i(x) = \sum_{j=1}^{M_i} b_{ij} I_{[0,r_{ij}]}(x).$$

For the configuration $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$, replace each $\mathbf{p}_i$ with M_i copies of itself to get $\tilde{\mathbf{p}}_i$, and so to get $\tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_N)$, and create the corresponding configuration of balls $\mathbf{B}^d(\mathbf{p}_i, r_{i1}), \dots, \mathbf{B}^d(\mathbf{p}_i, r_{iM_i})$, $i = 1, \dots, N$, where for each point $\mathbf{p}_i$ in the configuration $\mathbf{p}$ we have M_i concentric balls namely $\mathbf{B}^d(\mathbf{p}_i, r_{i1}), \dots, \mathbf{B}^d(\mathbf{p}_i, r_{iM_i})$. Let $c_{ik} = \sum_{j=1}^{k} b_{ij}$. Then replace each w_i with M_i others, where each $w_{ik} = c_{ik} I_{[0,r_{ik}]}, k = 1, \dots, M_i$ to get $\tilde{\mathbf{w}}$. Then

$$w_i(x) = \sum_{j=1}^{M_i} b_{ij} I_{[0,r_{ij}]}(x) = \bigvee_{k=1}^{M_i} c_{ik} I_{[0,r_{ik}]}(x).$$

If h is a Boolean flower formula for $\mathbf{p}$, replace each term for $\mathbf{p}_i$ with the wedge of M_i copies of $\mathbf{p}_i$, as above, to get a corresponding Boolean flower formula $\tilde{h}$. Then

$$\int_{\mathbb{E}^d} (h \circ \mathbf{w})_{\mathbf{p}}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{E}^d} (\tilde{h} \circ \tilde{\mathbf{w}})_{\tilde{\mathbf{p}}}(\mathbf{x}) d\mathbf{x}.$$

The point is that now each weight function is a positive constant times the indicator function for some ball. We can now state the corresponding result for general weight functions, where $\mathbf{w}(x) = (w_1(x), \dots, w_N(x))$ is a vector valued function with $w_i(x)$ being a decreasing non-negative function defined on $[0, +\infty)$ and satisfying $\int_0^{+\infty} w_i(x) x^{d-1} dx < +\infty$ for all $1 \le i \le N$ and for a fixed $d > 1$.

THEOREM 5.2. *Suppose that f is a lattice polynomial, and Conjecture* 2.2 *holds in dimension d for $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ point configurations in $\mathbb{E}^d$ for all possible radii of the balls. Then Conjecture* 3.1 *holds for the corresponding Boolean flower formula h. In other words, if $\epsilon_{ij} \|\mathbf{p}_i - \mathbf{p}_j\| \ge \epsilon_{ij} \|\mathbf{q}_i - \mathbf{q}_j\|$ for all $1 \le i < j \le N$, where $\epsilon_{ij}, 1 \le i < j \le N$ are the sign coefficients assigned to the lattice polynomial f, then*

$$\int_{\mathbb{E}^d} (h \circ \mathbf{w})_{\mathbf{p}}(\mathbf{x}) d\mathbf{x} \ge \int_{\mathbb{E}^d} (h \circ \mathbf{w})_{\mathbf{q}}(\mathbf{x}) d\mathbf{x}.$$

PROOF. By approximating the given monotone decreasing weight functions $\mathbf{w}$ with monotone decreasing step functions we can assume that $\mathbf{w}$ is made of step functions. The argument above shows that by replacing the configurations $\mathbf{p}$ and $\mathbf{q}$ with $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ we can assume that the step functions w_i are constant on each ball.

We proceed by induction on the number of balls in the Boolean flower formula h. If there is only one ball, or if all the weight functions w_i are equal to each other, then the conclusion is clearly equivalent to Conjecture 2.2. Otherwise, let m be the minimum positive value of all of the w_i. Then $mI_{[0,r_i]}(x) \leq w_i(x)$ for all $x \geq 0$. So we can replace each w_i with $w_i - mI_{[0,r_i]}$, and we will still have valid weight functions, but with some balls having 0 weight, thus effectively removing them from the formula. Hence by induction, the conclusion of Conjecture 3.1 holds. □

6. Known results

Although the following inequalities are similar to the inequalities proved by Gordon [5] and a generalization of [7], they seem to be new. From what we know, we can state the following, which follows from the main result of [1].

THEOREM 6.1. *Conjecture* 3.1 *holds for dimension $d = 2$ for any Boolean flower formula h and any proper weight vector function* $\mathbf{w}$.

The following follows from Theorem 2.1, the main result of [3].

THEOREM 6.2. *Conjecture* 3.1 *holds for all dimensions $d \geq 2$ for any Boolean flower formula h and any proper weight vector function* $\mathbf{w}$ *when the configuration* $\mathbf{q}$ *is a continuous monotone repositioning of* $\mathbf{p}$ *consistent with h.*

7. Remark and example

Although Theorem 5.1 is natural and follows easily, it is not covered by the Boolean flower formulas in Theorem 5.2. For example, the following graph of Figure 1 indicates a sign pattern on four vertices that does not come from any Boolean flower formula. By Theorem 5.1 if each of the terms in some sum of area functions is such that change in the distances are consistent and each of the terms has the Kneser–Poulsen property, then the areas will change in the expected way.

The following is an example of a Boolean flower formula $(w_1 \vee w_3) \wedge w_4$, where the weight functions are such that $w_1 = 4$ on D_2, $w_1 = 2$ on $D_1 - D_2$, $w_3 = 1$ on D_3, and $w_4 = 3$ on D_4, where $\mathbf{p_1}$ is the center of D_1 and D_2, $\mathbf{p_3}$ is the center of D_3, and $\mathbf{p_4}$ is the center of D_4. Figure 2 shows how the integral of the flower weight

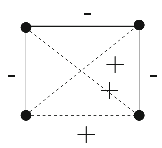

FIGURE 1. *Here the graph on four vertices is indicated. Edges with a + indicate that the sign coefficient is +1 and a − indicates that the sign is −1. The edges with a +1 label are not permitted to increase, and the edges with a −1 label are not permitted to decrease*

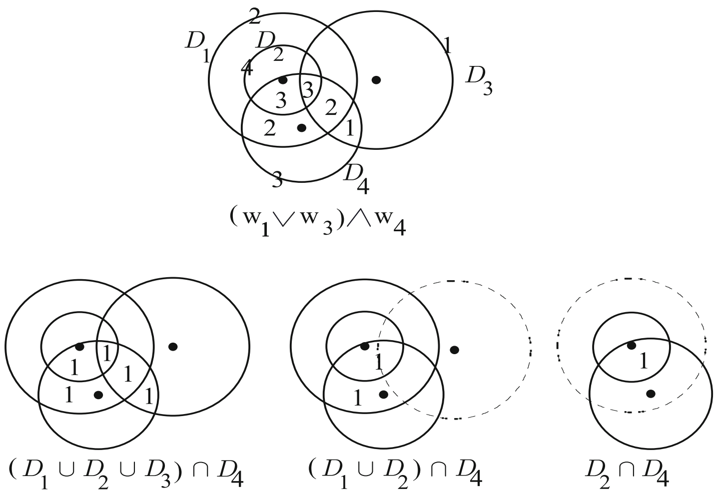

FIGURE 2. *This shows how a flower weight function can be written as a linear combination of indicator functions for the appropriate disks*

function is decomposed into integrals of the indicator functions of the appropriate disks.

References

[1] K. Bezdek and R. Connelly, Pushing disks apart — the Kneser–Poulsen conjecture in the plane, *J. Reine Angew. Math.*, **553** (2002), 221–236.

[2] B. Csikós, On the volume of the union of balls, *Discrete Comput. Geom.*, **20** (1998), 449–461.

[3] B. Csikós, On the volume of flowers in space forms, *Geom. Dedicata*, **86** (2001), 59–79.

[4] Y. Gordon, Elliptically contoured distributions, *Probab. Theory Related Fields*, **76/4** (1987), 429–438.

[5] Y. Gordon, Majorization of Gaussian processes and geometric applications, *Probab. Theory Related Fields*, **91** (1992), 251–267.

[6] Y. Gordon and M. Meyer, On the volume of unions and intersections of balls in Euclidean space, *Oper. Theory Adv. Appl.*, **77** (1995), 91–100.

[7] M. Gromov, Monotonicity of the volume of intersection of balls, *Geometrical aspects of functional analysis* (1985/86), Lecture Notes in Math. 1267, Springer, Berlin, 1987, 1–4.

[8] V. Klee and V. S. Wagon, *Old and new unsolved problems in plane geometry and number theory*, MAA Dolciani Mathematical Expositions, 1991.

[9] M. Kneser, Einige Bemerkungen über das Minkowskische Flächenmass, *Arch. Math.*, **6** (1955), 382–390.

[10] E. T. Poulsen, Problem 10, *Math. Scand.*, **2** (1954), 346.