Beitr Algebra Geom
DOI 10.1007/s13366-017-0353-7

ORIGINAL PAPER

Straight line motion with rigid sets

Robert Connelly[1] · **Luis Montejano**[2]

Received: 5 July 2017 / Accepted: 13 August 2017

Abstract If one is given a rigid triangle in the plane or space, we show that the only motion possible, where each vertex of the triangle moves along a straight line, is given by a hypocycloid line drawer in the plane. Each point lies on a circle which rolls around, without slipping, inside a larger circle of twice its diameter. We also prove a natural extension in Euclidean n-space.

Keywords Rigid triangle · Straight line · Hypocycloid

Mathematics Subject Classification 52C25 · 51F99

1 Introduction

Consider three straight lines in the plane or in three-space. When can you continuously move a point on each line such that all the pairwise distances between them stay constant? One case is when the three lines are parallel. Figure 1 shows another way. Are these the only possibilities? We answer the question affirmatively with Theorem 1 in the plane, and with Theorem 2 an analogue for higher dimensions is derived.

The example of Fig. 1 is known as a hypocycloid straight-line mechanism, and Fig. 2 shows a model from the Cornell Reuleaux collection; see Saylor (2017).

It is easy to see from Fig. 1 that as the inner circle rolls around the central point with a fixed radius, the triangle, formed from the other points of intersection with the

✉ Luis Montejano
luis@matem.unam.mx

1 Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

2 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, 04510 México DF, Mexico

Published online: 23 August 2017

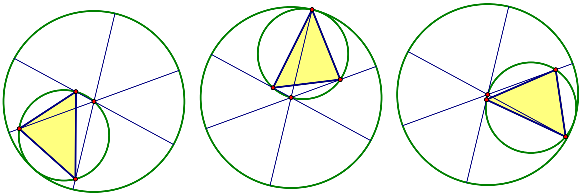

Fig. 1 An inner circle rolling inside a larger circle, without slipping, so that each vertex of the attached triangle traces out a straight line

Fig. 2 A mechanism at Cornell, where the point on the inner geared circular wheel traces out a straight line

three fixed lines, has fixed internal angles, and therefore this triangle moves rigidly. This was known at least to the Persian astronomer and mathematician Nasir al-Din al-Tusi in 1247. See Kennedy (1966).

In dimension three, one can form a cylinder rotating inside a larger cylinder of twice the diameter. But in the plane and three-space, the three lines all have a fourth line that is perpendicular and incident to all three lines.

2 The planar case

We describe the motion of a segment of fixed length d_{12} sliding between two fixed non-parallel lines, L_1 and L_2, first in the plane. We assume, without loss of generality, that L_1 and L_2 intersect at the origin and are determined by two unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$, where

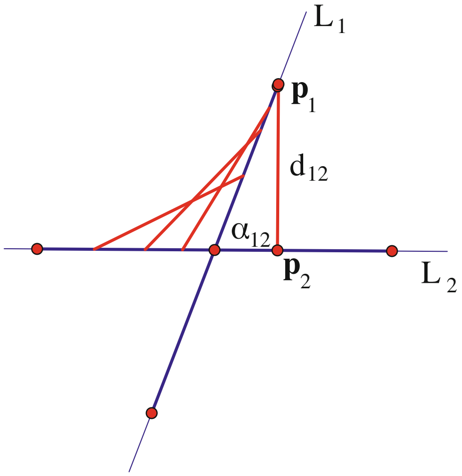

Fig. 3 As the line segment, with fixed length, moves with one point on L_1 and one point on L_2 the trace of each point is symmetric about the intersection of L_1 and L_2

$\mathbf{v}_1 \cdot \mathbf{v}_2 = \cos\alpha_{12} = c_{12}, \alpha_{12}$ being the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, and $-1 < c_{12} < 1$. Let t_1, t_2 be the oriented distance of the points $\mathbf{p}_1, \mathbf{p}_2$ from the origin, so that $\mathbf{p}_1 = t_1\mathbf{v}_1$, and $\mathbf{p}_2 = t_2\mathbf{v}_2$. Then treating the square of vector as the dot product with itself,

$$d_{12}^2 = (\mathbf{p}_1 - \mathbf{p}_2)^2 = (t_1\mathbf{v}_1 - t_2\mathbf{v}_2)^2 = t_1^2 + t_2^2 - 2c_{12}t_1t_2. \tag{1}$$

Thus in t_1, t_2 space the configurations of the line segment form an ellipse centered at the origin whose major and minor axes are at 45° from the t_1, t_2 axes. Thus there are constants $a_{12} = d_{12}/\sqrt{2(1 - c_{12})}, b_{12} = d_{12}/\sqrt{2(1 + c_{12})}$ such that

$$t_1 = a_{12}\cos\theta - b_{12}\sin\theta, \quad t_2 = a_{12}\cos\theta + b_{12}\sin\theta \tag{2}$$

describes the full range of motion of the line segment for $0 \le \theta \le 2\pi$. It is also clear from Fig. 3 that the length of the image of each $\mathbf{p}_i$ on L_i, $i = 1, 2$, is an interval of length $2d_{12}/\sin\alpha_{12}$ centered about the intersection of L_1 and L_2. We can now state the situation for the plane.

Theorem 1 *If a triangle, with fixed edge lengths, continuously moves with each vertex* $\mathbf{p}_i$ *on a line* L_i, $i = 1, 2, 3$ *in the plane, then either all the lines are parallel or they all intersect at a point forming a hypocycloid straight line drawer as above, with the range of each point an interval of the same length on each line, while the midpoint of each range is the intersection of the three lines.*

Proof Choose any two of the the three lines, say L_1 and L_2. The parametrization discussed above shows that any position of the d_{12} segment can be described by the equations in (2). These define the positions of $\mathbf{p}_1$ and $\mathbf{p}_2$ as a function of θ, for all $0 \le \theta \le 2\pi$. Then the position of $\mathbf{p}_3$ is determined also as a function of θ, $\mathbf{p}_3(\theta)$, since it is carried along rigidly. The image of $\mathbf{p}_3$ is a continuous curve in the plane and it is symmetric about the intersection of L_1 and L_2. Actually, it is not difficult to see that it is an ellipse (see for example Pedoe 1975), but that is not needed for this part of the argument. If that image is to be in a straight line L_3, then L_3 should intersect the intersection of L_1 and L_2. So if $\mathbf{p}_3(\theta)$ satisfies the equations corresponding to (1) for 13 and 23 replacing 12, for an interval of values of θ, then it must satisfy those equations for all θ, and the image of each $\mathbf{p}_i$ in L_i is the same length for $i = 1, 2$. But applying this argument to another pair such as 13, shows that all the images are the same length. □

3 The higher dimensional case

Suppose one is given a rigid triangle in Euclidean n-space, where each vertex of the triangle moves along a straight line. In dimension 3, each pair of non-parallel lines $L_i, L_j,\ i, j = 1, 2, 3$, have two points $\mathbf{q}_{ij} \in L_i$, for $i \neq j$, such that $\mathbf{q}_{ij} - \mathbf{q}_{ji}$ is perpendicular to both L_i and L_j. The shortest distance between L_i and L_j is $|\mathbf{q}_{ij} - \mathbf{q}_{ji}| = D_{ij} = D_{ji}$. We construct six variables $t_{ij},\ i, j = 1, 2, 3,\ i \neq j$, where $\mathbf{p}_i = \mathbf{q}_{ij} + t_{ij}\mathbf{v}_i$. Then we can project orthogonally into a plane spanned by the vectors $\mathbf{v}_i, \mathbf{v}_j$ to get the analog of Eq. (1), which is the following representing the three equations for each of the three edge lengths $d_{ij} = d_{ji}$ of the triangle:

$$d_{ij}^2 - D_{ij}^2 = (\mathbf{p}_i - \mathbf{p}_j)^2 = (t_{ij}\mathbf{v}_i - t_{ji}\mathbf{v}_j)^2 = t_{ij}^2 + t_{ji}^2 - 2c_{ij}t_{ij}t_{ji}. \tag{3}$$

Similar to Eq. (2), we define constants $a_{ij} = \sqrt{d_{ij}^2 - D_{ij}^2}/\sqrt{2(1 - c_{ij})}, b_{ij} = \sqrt{d_{ij}^2 - D_{ij}^2}/\sqrt{2(1 + c_{ij})}$ such that $a_{ij} = a_{ji},\ b_{ij} = b_{ji}$ and

$$t_{ij} = a_{ij}\cos\theta_{ij} - b_{ij}\sin\theta_{ij},\quad t_{ji} = a_{ij}\cos\theta_{ij} + b_{ij}\sin\theta_{ij}, \tag{4}$$

where $\theta_{ij} = \theta_{ji}$ is the parameter as before. This gives three separate parameterizations for each line segment between each pair of lines. Furthermore the equations of (3) give a complete description of position of each $\mathbf{p}_i$, two ways for each line, in terms of t_{ij} and t_{ik}, where $|t_{ij} - t_{ik}| = |\mathbf{q}_{ij} - \mathbf{q}_{ik}|$, a constant. So $t_{ij} = t_{ik} + e_{ijk}$, where $e_{ijk} = \pm|\mathbf{q}_{ij} - \mathbf{q}_{ik}|$ is a constant. See Fig. 4 for a view of these coordinates.

Theorem 2 *If a triangle, with fixed edge lengths, continuously moves with each vertex* $\mathbf{p}_i$ *on a line* L_i, $i = 1, 2, 3$, *in a Euclidean* n*-space, then either all the lines are parallel or they intersect an affine* $(n-2)$*-plane* L *perpendicular to all of them, with the range of each point* $\mathbf{p}_i$ *an interval of the same length on each line, while the midpoint of each range is* $L_i \cap L$*, the intersection of the two lines.*

Proof If $n = 3$, start with Eq. (3) for the L_1, L_2 lines defining the variable $\theta_{12} = \theta_{21}$ which, in turn defines the positions for $\mathbf{p}_1(\theta_{12})$ on L_1 and $\mathbf{p}_2(\theta_{12})$ on L_2. Note that $t_{12} = t_{13} + e_{123}$ and $t_{21} = t_{23} + e_{213}$. So we can regard t_{13} and t_{23} as linear functions of t_{12} and t_{21}. Similarly, $t_{32} = t_{31} + e_{321}$, so we can regard t_{32} as a function of t_{31}. So if we subtract the 13 and 23 equations for (3), we are only left with linear terms in t_{31}.

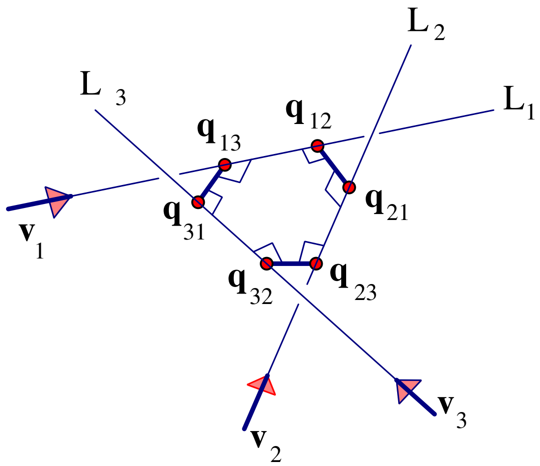

Fig. 4 Three skew lines in three-space, and for each pair of lines, shows their pair of nearest points

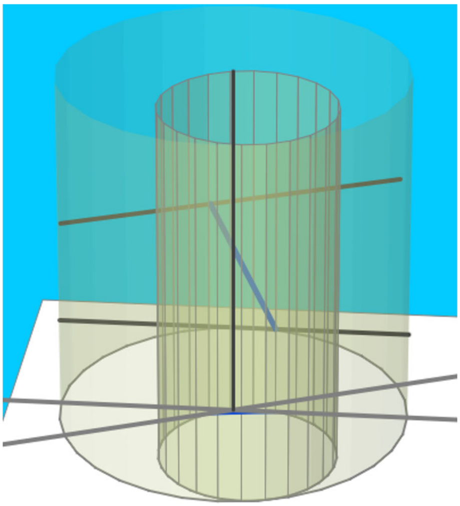

Fig. 5 A large vertical cylinder with two skew lines through its axis of symmetry, which are then projected onto two lines in the plane. A smaller parallel vertical cylinder, with half the diameter of the larger cylinder and containing the line of symmetry of the larger cylinder, is also shown

The squared terms have cancelled. Thus either t_{31} has no term in the difference of the two equations, or t_{31} is a non-zero rational function of t_{12} and t_{21} and thus θ_{12}.

On this last case, the image of $\mathbf{p}_1$ in L_1 is symmetric about $\mathbf{q}_{12}$. Also, if the linear t_{31} term disappears, then the 12 and 13 equations imply the 23 equation, which means that the 12 edge of the triangle makes a full 360° turn and again the image of $\mathbf{p}_1$ in L_1 is symmetric about $\mathbf{q}_{12}$.

Applying the above argument to each equation of (3) for ij, we see that each image of $\mathbf{p}_i$ in L_i is symmetric about $\mathbf{q}_i$. If all the lines lie in a three-dimensional Euclidean space, and no two lines L_i are parallel, then the midpoints of the images of each $\mathbf{p}_i$ must lie on a line perpendicular to all of the L_i, as claimed. So these lines are just a three-dimensional "lift" of the two-dimensional case.

If the lines span a higher dimensional space, then there is a non-zero vector perpendicular to each $\mathbf{v}_i$, for $i = 1, 2, 3$, and the lines L_i can be projected orthogonally into a three-dimensional space, and the projection of the points on the projected lines will be a mechanism, one dimension lower. Then similarly the three-dimensional mechanism comes from a two-dimensional mechanism, from the argument above. See Fig. 5 □

The following are some immediate corollaries.

Corollary 1 *If a polygon in the plane is continuously moves as a rigid body so that each vertex stays on a straight line, then those vertices all lie on a circle or the three lines are parallel.*

Corollary 2 *Given three lines L_1, L_2, L_3 on a Euclidean 3-space such that they are all perpendicular to a fourth line, then there are at most 8 configurations of a triangle $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ with fixed edge lengths, that continuously moves with $\mathbf{p}_i$ on L_i, $i = 1, 2, 3$.*

This is because each edge length is determined by a degree two equation, and Bézout's theorem (Kendig 2011) implies that there are at most $2^3 = 8$ individual solutions.

References

Kendig, K.: A Guide to Plane Algebraic Curves, Volume 46 of the Dolciani Mathematical Expositions. Mathematical Association of America, Washington, DC (2011). (MAA Guides, 7)

Kennedy, E.S.: Late medieval planetary theory. Isis **57**(3), 365–378 (1966). http://en.wikipedia.org/wiki/Tusi-couple

Pedoe, D.: The ellipse as an hypotrochoid. Math. Mag. **48**(4), 228–230 (1975)

Saylor, J.: Kinematic Models for Design, Digital Library (KMODDL). http://kmoddl.library.cornell.edu/facets/moon61899.htm