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Rigidity of packings

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Abstract

In Ludwig Danzer's Habilitatiionsschrift [L. Danzer, Endliche Punktmengen auf der 2-Sphäre mit möglichst grossem minimalabstand, Habilitationsschrift, Göttingen, 1963] he initiated a study of the local nature of the packings from the point of view of whether their density can be increased by a small perturbation of the packing configuration. This is an abbreviated biased introduction to the theory of such locally maximally dense packings of disks in various spaces from the point of view of the theory of tensegrity structures. This has applications to jamming of granular materials as well as leading to a better understanding of jammed packings.

1. The basic idea

In [5] Ludwig Danzer addressed the problem of given a packing of disks, all with the same radius, when can you improve the density with a small perturbation, while maintaining the packing property? If a packing has such a perturbation, one could could use the perturbation to improve the density. This was applied, in many cases, by [18] to improve the density of packings of equal circles on the sphere for configurations that were conjectured to be the best possible. The idea is to linearize the problem of finding the motion by looking for an infinitesimal motion of the centers of the packing. This is a vector assigned to each center such that if the centers are moved with velocities according to those vectors, then the first derivative of the distance between those centers would not decrease. Fig. 1 shows a simple example. After a short time, one would expect that the packing disks would all lose contact, and then one could increase all the radii of the disks simultaneously, thus increasing the packing density. But some things must be kept in mind.

(1) There must be some sort of container that prevents the disks from expanding indefinitely, for example, when all the disks are on the unit sphere, in a box, or on a torus. The *density* is then

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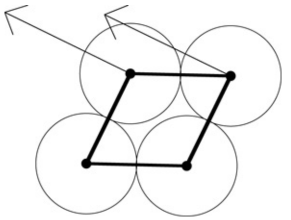

Fig. 1. The arrows here show an infinitesimal flex of this packing of four disks.

the ratio of the area (or volume) of the disks to the area (or volume) of the container. (For the case of infinite packings in the plane, the density is defined by taking a limit of the densities in increasingly large disks whose union contains the packing disks.)

(2) If the idea is to improve the density of the packing by a small perturbation, it is important that all of the packing disks be allowed to move together. There are many examples of packings where a fixed number of packing disks cannot move, if the rest are held fixed. If each packing disk is held fixed by its neighbors, we say the packing is *locally jammed*. One particular example of that is locally jammed is by [2], which has density 0. The term *stable* was used for locally jammed at that point.

(3) If the packing is infinitesimally rigid (in some appropriate container), then it is rigid, but not always conversely. In other words, when there are only the trivial infinitesimal motions of the centers coming from the derivative of congruences of the whole space restricted to the packing, then the packing is rigid. For example, the most dense packing of a single circular disk in an ellipse is rigid, but it is not infinitesimally rigid.

(4) If the packing is infinitesimally rigid, then it cannot be perturbed so that any of the contacts are broken. This means that the packing density cannot be increased with a packing that is close to the given packing. In other words, the packing is *locally maximally dense*.

(5) It may happen that the a packing has some subset of "rattlers" that can move in a non-trivial way, but that the rest of the packing is rigid, and that prevents the overall density of the packing from increasing locally. An example is the packing of Fig. 2.

(6) Even if the packing is not rigid, it could happen that the motion does not allow the density to increase. For example, the most dense packing of five congruent disks on the surface of the unit sphere has a motion, as in Fig. 3, but there are no rattlers in the sense of Comment (5.).

2. The canonical push

There are situations where some of the problems of Section 1 can be dealt with. Suppose that there is an infinitesimal motion $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$ of the configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ in Euclidean space. Each $\mathbf{p}'_i$ is a vector assigned to $\mathbf{p}_i$, which is a center of a circular disk. (The radii of the disks do not have to be the same, though.) The packing condition becomes the linear constraint

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0, \tag{1}$$

for each pair of touching disks i and j.

Theorem 2.1. *If there is a non-trivial infinitesimal motion* $\mathbf{p}'$*, the disks lie in Euclidean space (or a hyperbolic space), and the boundaries of the container either do not exist (as with a torus) or are flat (or concave in the sense that the container is the complement of a finite number of*

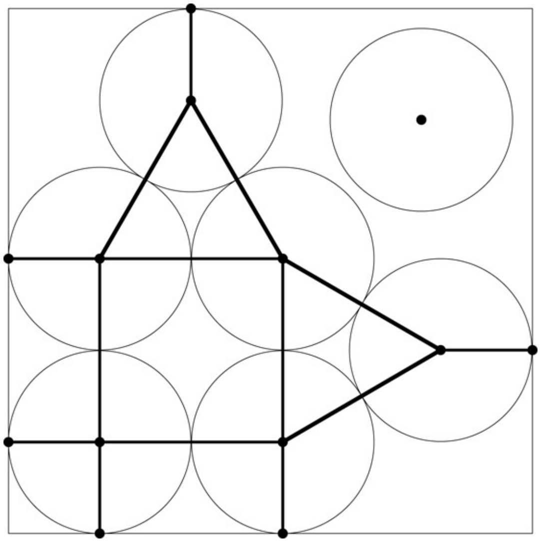

Fig. 2. This shows a packing of seven congruent disks, where six of them are held fixed by the container, which is a square. This is the most dense packing with seven congruent disks in a square, by [15].

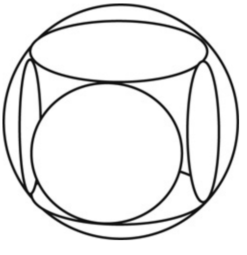

Fig. 3. This shows a packing of five congruent spherical caps on the surface of the two-dimensional sphere. Two antipodal disks allow the three others to move so that their centers slide along the bisecting great circle, keeping contact with the two antipodal circles. But this sort of behavior is *not* what is expected.

convex sets), then there is a continuous motion that increases the distance strictly between every pair of touching disks (and the walls of the container) unless in Euclidean space $\mathbf{p}'_i = \mathbf{p}'_j$*, in which case they move as a rigid body.*

Proof. The motion, which I call the *canonical push*, is given by $\mathbf{p}_i(t) = \mathbf{p}_i + t\mathbf{p}'_i$, for $t \geq 0$, so the squared distance between adjacent disks is

$$\begin{aligned} |\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2 &= |\mathbf{p}_i + t\mathbf{p}'_i - (\mathbf{p}_j + t\mathbf{p}'_j)|^2 \\ &= (\mathbf{p}_i - \mathbf{p}_j)^2 + 2t(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + t^2(\mathbf{p}'_i - \mathbf{p}'_j)^2. \end{aligned} \tag{2}$$

The conclusion follows from this calculation, since the second-order term is always positive unless $\mathbf{p}'_i = \mathbf{p}'_j$. □

There are several observations with regard to (2).

(1) Notice that the infinitesimal motion $\mathbf{p}'$ that determines the canonical push is a solution to the system of inequalities (1) and so whether to choose $\mathbf{p}'$ or $-\mathbf{p}'$ may be forced. Also the

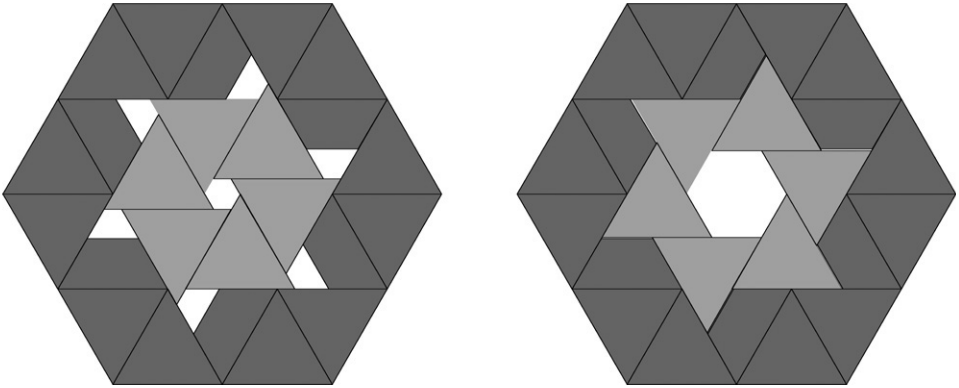

Fig. 4. This shows a configuration of six triangles enclosed by several others, where the sliding motion is given by the canonical push as in Theorem 2.1.

magnitude of how far to do the push until another contact occurs must be determined by other means.

(2) Under one of the hypotheses of Theorem 2.1, a packing of disks is locally maximally dense if and only if there is a subset of the packing disks (I call them here a *rigid spine*, the ones that are not rattlers) which is infinitesimally rigid.

(3) If the ambient space is hyperbolic space (or has constant negative Gaussian curvature), the canonical push is still defined, where each center is pushed along a geodesic path with a velocity given by the vector $\mathbf{p}_i'$ in the tangent space. In this case the separation is always strict.

(4) If the ambient space is a sphere (or has constant positive curvature), the canonical push argument does not seem to work. Ironically, for the unit 2-dimensional sphere, this was the situation of primary interest in [5]. Nevertheless, (3) of Section 1 still applies, and was used several times in [18] to improve packings of circular disks on the surface of a 2-dimensional sphere, the Tammes Problem.

(5) The canonical push is good for improving log jams, packings that are not rigid but where the direction of motion is not clear. If an infinitesimal motion exists, it is the solution to a linear programming feasibility problem, and there is a well studied technology solving such problems. On the other hand, one of the difficulties is that the actual contact graph is assumed to be known, and this may not be clear in a typical numerical simulation of a packing.

(6) The canonical push can be generalized to situations where the packing elements are arbitrary convex bodies, not just circles, BUT the allowable motions of each body are restricted to translations only. For each contact, in Eq. (1), $(\mathbf{p}_i - \mathbf{p}_j)$ is replaced by any pair of vectors, $\mathbf{p}_i$ in one body and $\mathbf{p}_j$ in the other, such that $(\mathbf{p}_i - \mathbf{p}_j)$ is perpendicular to a line separating body i from body j. This is a kind of sliding block puzzle, but even when the relative motions $\mathbf{p}_i' - \mathbf{p}_j'$ are non-zero, contact may be maintained, although still with no overlap. Fig. 4 shows an example.

3. Counting contacts

If a packing is rigid, by Theorem 2.1 it is infinitesimally rigid, as with the remarks above, and this implies that each contact corresponds to a linear inequality constraint. So if there are e contacts and N free variables, then rigidity implies $e \geq N + 1$. For example, in Fig. 2 there are

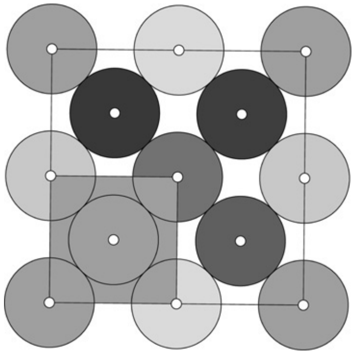

Fig. 5. Here is a square lattice with a packing indicated by the small highlighted square. It has two packing disks, one centered at the corners of the square and the other at the center of the square. When the sublattice, obtained by doubling each basis vector, is used for the covering torus, the packing of 8 disks is not rigid. For example, the 4 disks in line along an off-diagonal can move as a group.

6 disks in the rigid spine, and so $N = 2 \cdot 6 = 12$. There are 14 edges in the contact graph, so $e = 14 > 12 + 1 = N + 1$.

When the container is a fixed set with a topological boundary as in Fig. 2 or Fig. 4, the count above applies. But if the container is a sphere or a torus, for example, there are always rigid congruences of the whole space to consider. The corresponding derivative of these motions are *trivial infinitesimal motions*. In this situation, the packing is rigid if the only motions are the rigid congruences of the ambient space, and only the trivial infinitesimal motions. For the 2-dimensional torus, the dimension of the space of trivial motions is 2. We regard the torus $T = \mathbb{E}^2/L$ as the quotient space obtained from the plane by a lattice $L = \{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \mid m_1, m_2 \in \mathbb{Z}\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent vectors in $\mathbb{E}^2$. So if there are n packing disks in T, in order for the packing to be rigid we must have at least $2n - 2 + 1 = 2n - 1$ contacts.

The idea of using a torus is equivalent to having an infinite packing, where the packing is forced to be periodic, and the periods are given by the lattice L. This problem is interesting for various lattices. For example, see [11,10,13].

But there is an unpleasant property that occurs with packing in a torus determined by a fixed lattice. When the torus is "unwrapped" to a finite covering, a rigid packing can become not rigid. This corresponds to replacing $T = \mathbb{E}^2/L$ with $T' = \mathbb{E}^2/L'$, where $L' \subset L$ is a sublattice. Suppose that the number of contacts is the minimum $e = 2n - 1$, where there are n disks in T, and the index of L' in L is k. In other words, there are k disks in T' corresponding to each disk in T. So the number of disks in T' is $n' = kn$, and similarly the number of contacts in T' is $e' = ke$. But then $2n' - 1 = 2kn - 1 = k(e + 1) - 1 = e' + k - 1 < e'$ for $k > 1$. By the canonical push, Theorem 2.1, this packing will now not be rigid, and the density can be increased by a local perturbation. Fig. 5 shows an example of this behavior.

4. Shearing a torus

In order to address the problem of packing in a torus in Section 3, we can allow more degrees of freedom for the packing configurations. This was an idea suggested by A. Rogers. Allow the lattice that defines the torus to vary, but constrain the area (volume in higher dimensions) not to increase under any perturbation of the configuration. Effectively, this allows the packing to be sheared as a whole as well as allowing the individual motions as before. If there is no allowable infinitesimal motion in this situation, then there is no motion and the packing is rigid, as before. But the real usefulness is an analogue of the canonical push. This extends a result of [16].

Let $\Lambda = [\mathbf{v}_1, \dots, \mathbf{v}_d]$ be a basis for the lattice L in $\mathbb{E}^d$, and let $\Lambda' = [\mathbf{v}'_1, \dots, \mathbf{v}'_d]$ be the corresponding infinitesimal deformations of the corresponding lattice generators. We consider infinitesimal motions $\mathbf{p}'$ of the configuration $\mathbf{p}$, that are consistent with the infinitesimal motion Λ' of the lattice, and obey the constraints (1) as well as having the time 0 derivative of the volume of the torus being non-positive. Note that if Λ' is the time 0 derivative of an orthogonal motion (where the motion starts at the identity), it corresponds to a trivial infinitesimal motion and $\Lambda' = S\Lambda$, where $S = -S^{\mathrm{T}}$ is a skew symmetric matrix, and $()^{\mathrm{T}}$ denotes the transpose. Furthermore, for any infinitesimal motion $\mathbf{p}'$ with Λ' we can replace Λ' with $\Lambda' + S\Lambda$, where S is skew symmetric and the configuration $\mathbf{p}'$ with $\mathbf{p}' + S\mathbf{p}$ as well.

Theorem 4.1. *If Λ' and $\mathbf{p}'$ represent an infinitesimal motion of a lattice and its configuration that satisfies (1) with non-positive volume change, then there is a smooth motion of the lattice with its configuration that strictly increases adjacent distances and decreases the volume unless $\mathbf{p}'_i = \mathbf{p}'_j$ for adjacent disks i and j, and Λ' is trivial.*

Proof. By the remarks above we can replace Λ' with $\Lambda' + S\Lambda$ so that $\Lambda^{-1}\Lambda' + S$ is a symmetric matrix, where S is a skew symmetric matrix. So we assume that $\Lambda^{-1}\Lambda'$ is a symmetric matrix. Then the motion, as with the canonical push, is $\mathbf{p}(t) = \mathbf{p} + t\mathbf{p}'$, and the motion of the lattice is $\Lambda(t) = \Lambda + t\Lambda'$, for $t \geq 0$. The volume of the torus determined by the lattice is $\det(\Lambda(t))$. We calculate this volume as

$$\det(\Lambda(t)) = \det(\Lambda + t\Lambda') = \det(\Lambda)\det(I + t\Lambda^{-1}\Lambda'),$$

where I is the identity matrix and det is the determinant function. Since $\Lambda^{-1}\Lambda'$ and thus $I + t\Lambda^{-1}\Lambda'$ are symmetric, the determinant remains unchanged when they are conjugated by an orthogonal matrix A. By the spectral theorem, we can choose A so that $A(I + t\Lambda^{-1}\Lambda')A^{\mathrm{T}}$ is a diagonal matrix with, say, diagonal (eigen) values $1 + t\lambda_1, \dots, 1 + t\lambda_d$. Then we have

$$\begin{aligned}\det(\Lambda(t)) &= \det(\Lambda)(1 + t\lambda_1)\dots(1 + t\lambda_d)\\ &= \det(\Lambda)\left(1 + t(\lambda_1 + \dots + \lambda_d) + 2t^2\sum_{i<j}\lambda_i\lambda_j + O(t^3)\right).\end{aligned}$$

If the trace $\lambda_1 + \dots + \lambda_d < 0$, then the volume of the torus will decrease strictly for t sufficiently small. Otherwise $\lambda_1 + \dots + \lambda_d = 0$ and

$$0 = (\lambda_1 + \dots + \lambda_d)^2 = \lambda_1^2 + \dots + \lambda_d^2 + 2\sum_{i<j}\lambda_i\lambda_j.$$

This implies that

$$\sum_{i<j}\lambda_i\lambda_j < 0,$$

unless all $\lambda_i = 0$ in which case Λ' is trivial. In either case, the motion is strict, as was to be shown. □

This proof appeared in a condensed form in [6].

To calculate the degrees of freedom in the plane, suppose that one of the packing elements is fixed. Then there are $2(n-1) = 2n - 2$ degrees of freedom for the other disks. Consider

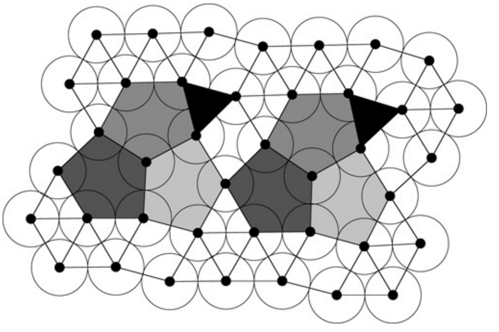

Fig. 6. This is an example of a periodic packing of disks that is strictly jammed. The periods of the lattice can be seen from the solid triangles, for example.

the lattice based at the center of the fixed disk. That adds 3 more degrees of freedom, after the rotations of the lattice about the fixed point are taken into account. Since there are inequality constraints, one more constraint is needed. Theorem 4.1 adds one more inequality constraint on the area of the torus. So if there are e contacts in a rigid packing of n disks in a 2-dimensional torus, then $2n-2+3+1 \leq e+1$, or $2n+1 \leq e$. In this case, when we take a finite covering space of the torus together with the packing disks, it has enough contacts to be rigid, even allowing the lattice to change. Call such a packing that is rigid in a torus, while allowing the lattice to change, *strictly jammed*. I conjecture that if a circle packing in a torus is strictly jammed, then so is any packing coming from a cover given by a sublattice.

At one point I also conjectured that any strictly jammed packing of congruent circular disks in a 2-dimensional torus could be obtained by removing a finite number of packing disks from a triangular lattice packing (the lattice generated by $(1, 0)$ and $(1/2, \sqrt{3}/2)$). That conjecture was incorrect. The packing in Fig. 6, found with the help of some numerical simulations of A. Donev, S. Torquato, and F. Stillinger, is strictly jammed, but it is not part of the triangular lattice.

5. Granular materials

One motivation for studying packings of disks is that it can serve as a model for granular materials, which can be piles of stones, grain in a barrel, even data compression for packings in higher dimensions, just to name a few examples. If one wishes to do numerical simulations of such things, there are several factors to consider or ignore. For example,

(1) What is the shape of the particles? They could be spherical, convex, irregular, different sizes, etc.
(2) What are the physical properties of the particles? For example, does friction play a role?
(3) What are the mechanical properties of the particles? Are they deformable, elastic, plastic, etc?
(4) What are the external forces that are operating? Is there an external load applied to the packing, say through the container?
(5) What is the history of the packing? If it is stable, how was stability achieved?
(6) What is the stress distribution in the packing? The internal stresses can be quite important is determining how the packing behaves with regard to external loads.

If all of the considerations above were taken into account in a serious way, it would be difficult to say too much of value. Some assumptions are in order, and this allows some geometry to be brought to bear. Suppose that the particles are infinitely hard, friction is negligible, each particle is a sphere, all the same size. Also, we can do our analysis in the plane as well as in three-space. One also tries to minimize boundary effects, and one way to do this is to restrict to periodic packings. But even then, the defining lattice can have an effect as with discussion in Section 4. This is the reason for defining the notion of strictly jammed, since locally jammed does not correspond to what would be a reasonable notion of being jammed.

If one takes a large number of marbles, all the same size, pours them into a container, and measures the resulting density, it comes out to be about 62%, 63%, or 64%, depending how you do the experiment (numerically or physically) and how slowly you allow the configuration to increase its density. This is often called a "random close packing", but there is a problem in defining it rigorously. For one thing, it depends on the way the way the material is packed. See [17] for example. For numerical simulations, the algorithm used is a factor as well, even if it seems to model some physical process. One popular algorithm for numerical packing densification is due to [12], which allows the packing disks to grow slowly, move and lose energy slowly with collisions. This seems to converge to configurations that are mechanically stable, while the rate of energy loss can be controlled.

But there is a very basic problem with convergence. How do you determine the contact information? If one is given which packing disks touch which others, by the analysis above, one can determine whether the packing is rigid, modulo rattlers, of course. But as the number of disks increases, the number of near misses increases, and the numerical tolerance, which must be part of the modeling, will be such that it is ambiguous. I call this *contact ambiguity*. In the plane it can lead to the following problem in Fig. 7.

In a physical system, contact ambiguity manifests itself not so much as to whether there is a contact, but how much force is transmitted, if any, between two adjacent disks. And, of course, when these small tolerances are relevant, assumptions about deformability of the particles, as in (3), come into question, even for very hard particles. This, in turn, leads to the isostatic conjecture. We state the following geometrically, still thinking of hard spheres or disks. A packing is said to be *polydispersed* if the radii are generic, which means that they are algebraically independent and do not satisfy any non-zero polynomial equation with integer coefficients.

Conjecture 5.1. *If a polydispersed packing of disks are in a convex container and are at rest under gravity, then there are no more than the minimal number of contacts present (modulo rattlers) and when the graph of contacts is replaced by bars, it is isostatic in the sense that any external load can be resolved by a unique internal stress.*

An example is shown in Fig. 8, where the generic hypothesis is needed for the isostatic condition. I have a proof of the Isostatic Conjecture 5.1 in the plane using ideas closely related to [14] and a classical theorem of [1], but the three-dimensional case seems to be harder.

If one relaxes the condition that the packing disks are circular or spherical, then there are some interesting circumstances that happen. For example, if all the packing disks are ellipses or ellipsoids, the expected number of contacts in a jammed packing is significantly less than what is needed for the packing to be infinitesimally rigid. This is in direct contrast to the situation for circular disks or round spheres, where the canonical push is available. A very simple example is a single ellipse in a triangle, as in Fig. 9. A non-circular ellipse in the plane, being a rigid body, is described by 3 parameters, say the position of its center and angle of rotation. But it can touch the edges of a triangle in, at most, 3 points and if it is to be infinitesimally rigid, it needs another

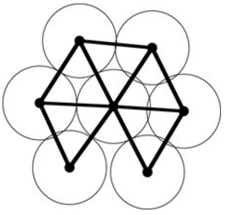

Fig. 7. This shows what are supposed to be 7 congruent circles, one surrounded by 6 others. In the outer ring, they should each touch the two adjacent neighbors, but because of numerical approximations, one of the contacts is missing.

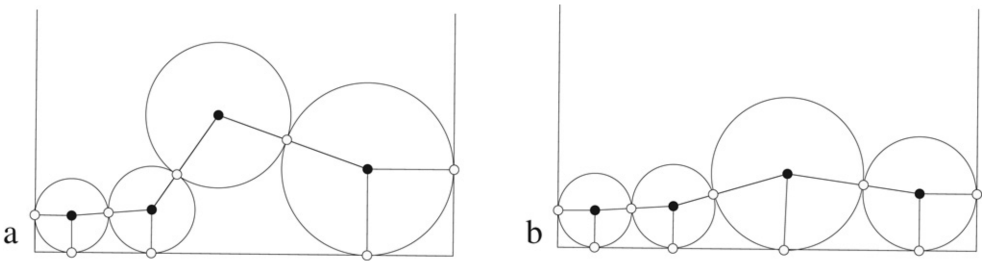

Fig. 8. Figure (a) shows 4 disks sitting in a box where there is a vertical force on each disk that stabilizes and rigidifies the whole packing. Here the radii are generic and there are the minimal number of contacts, namely 8 for the 4 disks. Figure (b) shows a similar packing of 4 disks, but the radii are not generic, and there are 9 contacts, which means that, although the packing is rigid, the redundancy prevents the calculation of resolving forces.

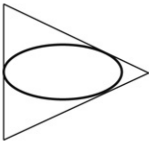

Fig. 9. This shows an ellipse of maximal size, with given axis ratio, inside a triangle. It has an infinitesimal rotation about the point where the normals to the tangent points meet.

contact. Indeed, it is easy to see that if the ellipse is of maximal size, with a fixed axis ratio, then it has an infinitesimal motion that does not extend to a continuous motion.

By a classical theorem of [9], the most dense packing of congruent ellipses in the plane is when they are axis aligned and each is adjacent to 6 others, and in a torus, this is infinitesimally rigid. On the other hand, most locally maximally dense packings of ellipses are rigid in their container, but they are not infinitesimally rigid, and they have fewer than the minimal number of contacts to be infinitesimally rigid. For example, in a 3-dimensional container with ellipsoids having 3 different axis lengths, each ellipsoid body has 6 degrees of freedom. Suppose the graph of the packing has e edges corresponding to e contacts, and v vertices corresponding to v packing ellipsoid disks. Then $v\bar{v} = 2e$, where $\bar{v}$ is the average degree of a vertex, which is the average number of contacts per packing ellipsoid. So $v\bar{v} = 12v + c$, where c is a constant depending on the number of trivial motions in the container. So for large v, $\bar{v}$ should never be less than 12 if the packing is infinitesimally rigid. But numerical simulations and physical experiments in [7,8] show that $\bar{v}$ is significantly less than 12. What this means is that the packing is at a local minimum of some energy functional, and suggests the structure is effectively prestress stable as described in [4]. See also [3] for general information about rigidity theory.

6. Open problems

The following are what I consider some of the most important open problems related to this subject, although they may be quite difficult.

(1) Is there a reasonable sense of stability and a $\delta > 0$, such that if a packing has such stability, then its density is at least δ. The result of [2] shows that the property of being held fixed by its neighbors is not a "reasonable" definition of stability. Is the property of being strictly jammed reasonable?
(2) Is the isostaic conjecture true in dimension three for spherical disks?
(3) Is there a reasonable method to assign a unique stress naturally to a strictly jammed packing of circular disks?

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