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Generic global rigidity of body–bar frameworks

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A B S T R A C T

A basic geometric question is to determine when a given framework $G(\mathbf{p})$ is globally rigid in Euclidean space $\mathbb{R}^d$, where G is a finite graph and $\mathbf{p}$ is a configuration of points corresponding to the vertices of G. $G(\mathbf{p})$ is *globally rigid in* $\mathbb{R}^d$ if for any other configuration $\mathbf{q}$ for G such that the edge lengths of $G(\mathbf{q})$ are the same as the corresponding edge lengths of $G(\mathbf{p})$, then $\mathbf{p}$ is congruent to $\mathbf{q}$. A framework $G(\mathbf{p})$ is *redundantly rigid*, if it is rigid and it remains rigid after the removal of any edge of G.

When the configuration $\mathbf{p}$ is generic, redundant rigidity and $(d+1)$-connectivity are both necessary conditions for global rigidity. Recent results have shown that for $d=2$ and for generic configurations redundant rigidity and 3-connectivity are also sufficient. This gives a good combinatorial characterization in the two-dimensional case that only depends on G and can be checked in polynomial time. It appears that a similar result for $d \geqslant 3$ is beyond the scope of present techniques and there are examples showing that the above necessary conditions are not always sufficient.

However, there is a special class of generic frameworks that have polynomial time algorithms for their generic rigidity (and redundant rigidity) in $\mathbb{R}^d$ for any $d \geqslant 1$, namely generic body-and-bar frameworks. Such frameworks are constructed from a finite number of rigid bodies that are connected by bars generically placed with respect to each body. We show that a body-and-bar framework is generically globally rigid in $\mathbb{R}^d$, for any $d \geqslant 1$, if and only if it is redundantly rigid. As a consequence there is a deterministic

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polynomial time combinatorial algorithm to determine the generic global rigidity of body-and-bar frameworks in any dimension.

1. Introduction

A d-dimensional (bar-and-joint) *framework* $G(\mathbf{p})$ is a pair, where G is a finite graph and $\mathbf{p}$ is a configuration of points in Euclidean space $\mathbb{R}^d$ corresponding to the vertices of G. Two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ are *equivalent in* $\mathbb{R}^d$ if corresponding edge lengths are the same. We say that $G(\mathbf{p})$ is *globally rigid in* $\mathbb{R}^d$ if when $G(\mathbf{q})$ in $\mathbb{R}^d$ is equivalent to $G(\mathbf{p})$, $\mathbf{q}$ is congruent to $\mathbf{p}$. The configurations $\mathbf{p}$ and $\mathbf{q}$ are *congruent* if there is a rigid congruence of $\mathbb{R}^d$ that takes $\mathbf{p}$ to $\mathbf{q}$.

A framework $G(\mathbf{p})$ is *rigid in* $\mathbb{R}^d$ if there is a neighborhood $U_{\mathbf{p}}$ in the space of configurations in $\mathbb{R}^d$ such that if $G(\mathbf{q})$ is equivalent to $G(\mathbf{p})$ and $\mathbf{q} \in U_{\mathbf{p}}$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

Determining global rigidity of $G(\mathbf{p})$ for a given configuration $\mathbf{p} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is NP-hard for any $d \geqslant 1$ [21], and the difficulty of determining rigidity for $d \geqslant 2$ appears to be equally hard. A natural way to address this difficulty is to consider the case when the configuration $\mathbf{p}$ is *generic*, which means that all the coordinates of all the points of the configuration $\mathbf{p}$ are algebraically independent over the rational numbers. In other words, the only polynomial with integer coefficients that is satisfied by these coordinates is the zero polynomial. This is something of an overkill, especially in the case of rigidity, since a reasonable finite set of polynomial equations, given by certain determinants, can be used in many instances. In the case of global rigidity, the equations that would determine the "bad" cases for global rigidity are much harder to determine.

With the concept of generic in mind, we define a graph G to be *generically rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is rigid at all generic configurations $\mathbf{p}$, and *generically globally rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is globally rigid at all generic configurations $\mathbf{p}$ [3,4]. It is well-known that rigidity is a generic property [30] and recent results in [4,8] prove that global rigidity is also a generic property for graphs in each dimension.

Two natural necessary conditions, observed by Hendrickson [10], for generic global rigidity in $\mathbb{R}^d$ are that the graph G be $(d+1)$-vertex-connected, and that, for a generic configuration $\mathbf{p}$, $G(\mathbf{p})$ be *redundantly rigid*, which means that $G(\mathbf{p})$ is rigid and remains rigid after the removal of any edge (Theorem 2 below).

For $d = 2$, Berg and Jordán [2] and Jackson and Jordán [11] confirm, using [4], that Hendrickson's necessary conditions are sufficient for generic global rigidity. For $d = 3$, Connelly [3] showed that the complete bipartite graph $K_{5,5}$ is generically redundantly rigid and 5-vertex-connected, but not generically globally rigid, showing that Hendrickson's necessary conditions are not sufficient. Similar examples exist for all $d \geqslant 3$, see [3,6]. Furthermore, Frank and Jiang [6] show that, for $d \geqslant 5$, there are graphs containing a complete graph K_{d+1} as a subgraph that are not generically globally rigid in $\mathbb{R}^d$, even though they satisfy Hendrickson's necessary conditions.

So it is natural to search for classes of graphs where generic global rigidity can be determined combinatorially in line with Hendrickson's necessary conditions, without recourse to matrix calculations for each graph, as in [4]. At a workshop at BIRS in 2008, two of the authors and Meera Sitharam conjectured that generic body-and-bar frameworks would give rise to one such class. Informally, a d-dimensional body-and-bar framework consists of disjoint full-dimensional rigid bodies connected by bars in such a way that the attachment points of the bars on the bodies are pairwise distinct, see e.g. [24–26,30] for more details. In this paper we treat these frameworks as special bar-and-joint frameworks and work with the following equivalent definition.

A d-dimensional *body-and-bar framework* is a bar-and-joint framework $G(\mathbf{p})$ in which the vertex set of G is partitioned into pairwise disjoint complete graphs (the *bodies*) and the remaining edges (bars), connecting these bodies, are pairwise disjoint. A graph with this structure is called a *body–bar graph*. The body-and-bar framework is *generic* if the configuration $\mathbf{p}$ is generic (that is, all of the vertices of all of the bodies are generic). We may record the connections between the bodies in a single multigraph H (without loops, but with multiple edges allowed), where each body is represented as a vertex of H and each bar connecting different bodies is represented as an edge. (Conversely, each multigraph H induces a body–bar graph in a natural way, see Section 3.)

The following is our main result (see also Theorem 16).

Theorem 1. *A body-and-bar framework is generically globally rigid in $\mathbb{R}^d$ if and only if it is generically redundantly rigid in $\mathbb{R}^d$.*

This characterization leads to a polynomial time combinatorial algorithm to determine the generic global rigidity of body–bar frameworks in any dimension. Previous results of Tay [24,25] showed that generic rigidity (and hence generic redundant rigidity) of body-and-bar frameworks in $\mathbb{R}^d$, for all $d \geqslant 1$, can be determined efficiently. We shall not rely on these earlier results in this paper, though they inspired the original conjecture.

For the proofs of previous results [2,11], and for our main theorem here, we rely on several key techniques. In [4], a sufficient condition is given in terms of the rank of a stress matrix (to be defined later), that combines with (infinitesimal) rigidity at a generic point to imply generic global rigidity in any specific dimension (see also [5]). To apply this result, certain key inductive constructions have been shown to preserve both the maximal rank of the corresponding stress matrix, and the infinitesimal rigidity. It is also necessary that these inductive constructions generate all members of the class from a generically globally rigid seed (a small complete graph).

These results have significant theoretical interest as steps towards a full theory of generic global rigidity of arbitrary frameworks. There are also a wide range of applications for the algorithms that detect global rigidity, such as localization in wireless sensor networks [1,13], molecular conformation [31], and stability of molecules. We return to possible applications of our main theorem in Section 6.

We also note that by the results in [5], graphs which are generically globally rigid in $\mathbb{R}^d$ are also generically globally rigid in spherical and hyperbolic d-space. $\mathbb{R}^d$ is the classical sample of a general class of metrics over which rigidity and generic global rigidity results are invariant.

2. Prior results on global rigidity and infinitesimal rigidity

Hendrickson [10] proved two key necessary conditions for the global rigidity of a bar-and-joint framework at a generic configuration. These properties were previously conjectured by Whiteley [29]. We say that $G(\mathbf{p})$ is *redundantly infinitesimally rigid in* $\mathbb{R}^d$ if removing any edge of $G(\mathbf{p})$ results in an infinitesimally rigid framework. (We refer the reader to [30] for the definition of infinitesimal rigidity and the rigidity matrix of a framework, and for more details on the rigidity of different kinds of frameworks.)

Theorem 2. *(See Hendrickson [10].) Let $G(\mathbf{p})$ be a globally rigid generic bar-and-joint framework in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or*

(i) *the graph G is $(d+1)$-vertex-connected, and*
(ii) *the framework $G(\mathbf{p})$ is redundantly infinitesimally rigid in $\mathbb{R}^d$.*

Note that redundant rigidity is a generic property. Thus the conditions of Theorem 2 are necessary for generic global rigidity.

One critical technique used for proving global rigidity of frameworks uses stress matrices. This technique is at the core of the proof that global rigidity is a generic property, as well as some specific inductive techniques (below).

This stress matrix approach builds on the fact that any globally rigid generic framework is dependent (redundant), with an equilibrium stress ω which is non-zero on all edges. Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$ with $G=(V,E)$. Recall that an *equilibrium stress* on $G(\mathbf{p})$ is an assignment of scalars ω_{ij} to the edges such that for each $i \in V$

$$\sum_{j|ij \in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}$$

This can also be visualized as a linear dependence of the rows of the rigidity matrix [30].

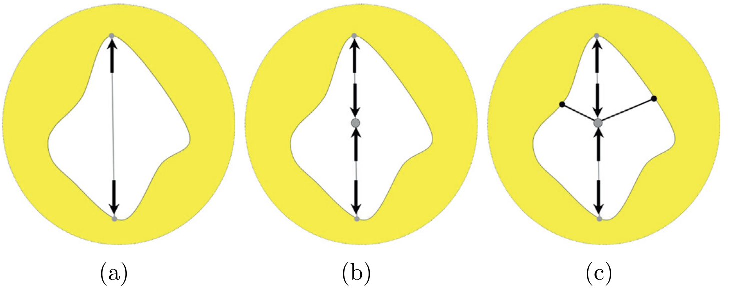

Fig. 1. An edge with an equilibrium stress (a) can be subdivided with a modified stress on the parts (b) and $d-1$ edges added (c). The result is an *edge split* which preserves infinitesimal rigidity and a stress matrix of full rank.

Given a stress, there is an associated $|V| \times |V|$ symmetric matrix Ω, the *stress matrix* such that for $i \neq j$, the i, j entry of Ω is $-\omega_{ij}$, and the diagonal entries for i, i are $\sum_{j \neq i} \omega_{ij}$. Here we follow the convention that an equilibrium stress can be extended to non-adjacent pairs i, j by putting $\omega_{ij} = 0$. Note that all row and column sums are now zero. It is easy to see that the rank of Ω is at most $|V| - d - 1$. We say that Ω is of *full rank* if its rank is equal to $|V| - d - 1$. Connelly has developed a number of properties of these stress matrices.

Theorem 3. *(See Connelly [3,4].) Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$, where $\mathbf{p}$ is a generic configuration. If $G(\mathbf{p})$ has an equilibrium stress where the rank of the associated stress matrix Ω is $|V| - d - 1$ and $G(\mathbf{p})$ is infinitesimally rigid, then $G(\mathbf{p})$ is globally rigid in $\mathbb{R}^d$.*

Let $\{i, j\}$ be a bar (edge) in framework $G(\mathbf{p})$. The *subdivision* operation removes the bar $\{i, j\}$ and adds a new vertex k and two new bars $\{i, k\}$ and $\{k, j\}$ in such a way that the position $\mathbf{p}_k$ of the new vertex k is on the line through $\mathbf{p}_i$ and $\mathbf{p}_j$, but not at $\mathbf{p}_i$ and $\mathbf{p}_j$ (Fig. 1(a), (b)). Suppose that $(\mathbf{p}_j - \mathbf{p}_i) = \alpha(\mathbf{p}_k - \mathbf{p}_i)$ for some parameter $\alpha \neq 0, 1$. It is easy to check that if there is an equilibrium stress ω on $G(\mathbf{p})$ then we obtain an equilibrium stress ω^* on the subdivided framework by setting $\omega^*_{ik} = \frac{1}{1-\alpha}\omega_{ij}$, $\omega^*_{kj} = \frac{1}{\alpha}\omega_{ij}$, and $\omega^*_{ij} = 0$. The stresses on the other bars are unchanged. The next lemma shows that by subdividing a stressed bar and adjusting the equilibrium stress as above, we increase the rank of the stress matrix by one. We shall use this lemma later with $\alpha = \frac{1}{2}$.

Lemma 4. *Let ω be an equilibrium stress on the d-dimensional framework $G(\mathbf{p})$ and let Ω be the associated stress matrix. Suppose that for bar $\{i, j\}$ we have $\omega_{ij} \neq 0$ and let $G^*(\mathbf{p}^*)$ be obtained from $G(\mathbf{p})$ by subdividing $\{i, j\}$ with parameter $\alpha \neq 0, 1$. Then* $\operatorname{rank} \Omega^* = \operatorname{rank} \Omega + 1$, *where Ω^* is the stress matrix associated with the equilibrium stress ω^* on $G^*(\mathbf{p}^*)$.*

Proof. By scaling ω, if necessary, we may suppose that $\omega_{ij} = 1$. Then $\omega^*_{ik} = \frac{1}{1-\alpha}$, $\omega^*_{kj} = \frac{1}{\alpha}$, and $\omega^*_{ij} = 0$. Thus the new stress matrix Ω^* is obtained from Ω by adding a new row and new column of zeros for vertex k, and then adding the following numbers to the entries of the row of i (resp. j, k) in the columns of i, j, and k: $(-1 + \frac{1}{1-\alpha}, 1, -\frac{1}{1-\alpha})$ (resp. $(1, -1 + \frac{1}{\alpha}, -\frac{1}{\alpha})$ and $(-\frac{1}{1-\alpha}, -\frac{1}{\alpha}, \frac{1}{1-\alpha} + \frac{1}{\alpha})$).

Observe that Ω^* can also be obtained from Ω by adding a new row and new column of zeros for vertex k, then adding the numbers $(-\frac{1}{1-\alpha}, -\frac{1}{\alpha}, \frac{1}{1-\alpha} + \frac{1}{\alpha})$ to the entries of the row of k in the columns of i, j, and k (which increases the rank by 1, since $\frac{1}{1-\alpha} + \frac{1}{\alpha} \neq 0$), and then adding $-\alpha R_k$ to the row of i and $-(1-\alpha)R_k$ to the row of j, where R_k denotes the row of k. This implies the lemma. $\square$

Note that for $d \geqslant 2$ a subdivided framework in $\mathbb{R}^d$ is never infinitesimally rigid. After subdivision it is necessary to add some additional bars to rigidify it infinitesimally (Fig. 1(c)).

Recall that, given a graph G with edge $e = \{i, j\}$, and $d-1$ additional vertices $1, \dots, d-1$, the *edge split on e in $\mathbb{R}^d$* is the addition of a new vertex k, the removal of e, and the insertion of $d+1$ new edges $\{i,k\},\{k,j\},\{k,1\},\ldots,\{k,d-1\}$. The corresponding geometric operation on $G(\mathbf{p})$ subdivides the edge e and inserts $d-1$ new bars from the new vertex positioned at $\mathbf{p}_k$. Thus we can extend a stress ω on $G(\mathbf{p})$ to the edge split framework $G^*(\mathbf{p}^*)$ by using ω^* on the subdivided edge e, and making the stresses 0 on the edges $\{k,1\},\ldots,\{k,d-1\}$. We say that the edge split on $e=\{i,j\}$ is a *general position edge split on e* if the vectors $\mathbf{p}_k-\mathbf{p}_i,\mathbf{p}_k-\mathbf{p}_j,\mathbf{p}_k-\mathbf{p}_1,\ldots,\mathbf{p}_k-\mathbf{p}_d$ span $\mathbb{R}^d$ (Fig. 1(c)). If $\mathbf{p}$ is generic, then the spanning condition will hold automatically. Even though some of the new edges have 0 stress, the following is true.

Lemma 5. *(See Tay and Whiteley [27].) Let $G(\mathbf{p})$ be an infinitesimally rigid framework in $\mathbb{R}^d$ and let e be an edge of G. Then a general position edge split on e generates a new graph G^* and an extended configuration $G^*(\mathbf{p}^*)$ which is infinitesimally rigid in $\mathbb{R}^d$.*

Note also that since the new vertex $\mathbf{p}_k$ lies on the line between $\mathbf{p}_i$ and $\mathbf{p}_j$, the configuration $\mathbf{p}^*$ is not generic, even if $\mathbf{p}$ is. There is an elementary way to connect the rank calculations for particular configurations to the property of global rigidity of a framework $G(\mathbf{p})$ at a generic configuration $\mathbf{p}$. This is implicit in [4] and explicit in [5].

Lemma 6. *Suppose that $G(\mathbf{p})$ is an infinitesimally rigid framework in $\mathbb{R}^d$ and ω is an equilibrium stress on $G(\mathbf{p})$ with a stress matrix Ω of full rank $|V|-d-1$. Then there is a neighborhood $U_{\mathbf{p}}$ in the space of configurations in $\mathbb{R}^d$ such that if $\mathbf{q}\in U_{\mathbf{p}}$, then $G(\mathbf{q})$ is infinitesimally rigid in $\mathbb{R}^d$ and has an equilibrium stress ω' with a stress matrix Ω' of full rank $|V|-d-1$. Furthermore, if G is generically redundantly rigid and $\mathbf{q}\in U_{\mathbf{p}}$ is a generic configuration then ω' can be chosen so that $\omega'_{ij}\neq 0$ on all edges ij of G.*

Putting all these propositions together we get the following, which was the original method to imply generic global rigidity for bar frameworks [4]:

(i) Suppose that $G(\mathbf{p})$ is an infinitesimally rigid framework in $\mathbb{R}^d$ with a stress ω whose associated stress matrix is of full rank, and let e be an edge with non-zero stress.
(ii) Then a general position edge split on e in $\mathbb{R}^d$ generates a new framework $G^*(\mathbf{p}^*)$ which is infinitesimally rigid and has a stress ω^* with a stress matrix of full rank.
(iii) After moving to a nearby generic point we may conclude that G^* is generically globally rigid in $\mathbb{R}^d$.

We shall use this argument several times to verify generic global rigidity, for edge splitting as well as for other operations which preserve infinitesimal rigidity and a stress matrix of full rank.

We note that for the plane Jackson, Jordán and Szabadka [16] have an alternative proof that edge-splitting preserves generic global rigidity. This proof has recently been generalized to all dimensions [23].

3. An inductive construction of redundantly rigid body–bar graphs

Let $H=(V,E)$ be a multigraph with minimum degree at least one. The *body–bar graph induced by H*, denoted by G_H, is the graph obtained from H by replacing each vertex $v\in V$ by a complete graph B_v (a 'body') on $d_H(v)$ vertices and replacing each non-loop edge uv by an edge (a 'bar') between B_u and B_v in such a way that the bars are pairwise disjoint. (We use $d_H(v)$ to denote the degree of vertex v in H. A loop on v contributes to $d_H(v)$ by two.)

We shall prove our main result by an inductive argument which relies on a combinatorial result of Frank and Szegő [7]. Their result, stated as Theorem 7 below, provides an inductive construction for the multigraphs H that induce redundantly rigid body–bar graphs G_H in $\mathbb{R}^d$. By using the operations of the previous section we shall show how to construct an infinitesimally rigid framework $G_H(\hat{\mathbf{p}})$ with a full rank stress matrix, following the inductive construction of the underlying multigraph H. This will imply that G_H is generically globally rigid by Lemma 6 and Theorem 3.

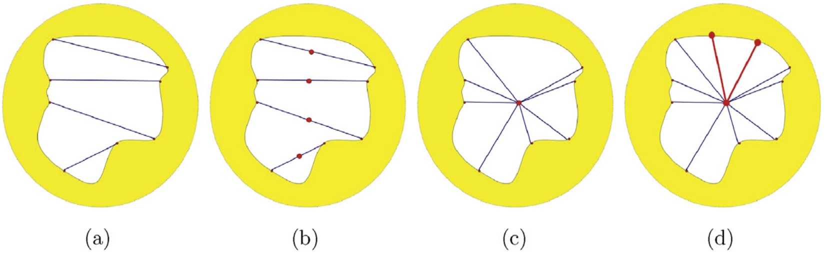

Fig. 2. A 6-split on 4 edges. The chosen edges (a), the pinch (b), (c), and the addition of 2 edges (d).

Let $H = (V, E)$ be a multigraph. For a partition $\mathcal{P}$ of V let $E_H(\mathcal{P})$ denote the set, and $e_H(\mathcal{P})$ the number of edges of H connecting distinct members of $\mathcal{P}$. We say that H is *highly m-tree-connected* if

$$e_H(\mathcal{P}) \geqslant m(t-1)+1, \tag{1}$$

for all partitions $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of V with $t \geqslant 2$. Note that a theorem of Nash-Williams [20] and Tutte [28] implies that H satisfies (1) if and only if $H - e$ contains m edge-disjoint spanning trees for all $e \in E$.

The operation *pinching k edges* (*with vertex z*) subdivides k designated edges and then contracts the k subdividing vertices into a new vertex z.

Theorem 7. *(See Frank and Szegő [7].) A multigraph H is highly m-tree-connected if and only if H can be obtained from a vertex by repeated applications of the following operations*:

(i) *adding an edge* (*possibly a loop*),
(ii) *pinching k edges* $(1 \leqslant k \leqslant m-1)$ *with a new vertex z and adding $m-k$ new edges connecting z with existing vertices.*

We call the combined operation of (ii), consisting of pinching and edge addition, an *m-split on k edges* (Fig. 2). We shall prove (Lemma 14) that if a body–bar graph G_H induced by H is generically redundantly rigid in $\mathbb{R}^d$ then H is highly $\binom{d+1}{2}$-tree-connected. Thus we shall need Theorem 7 when $m = \binom{d+1}{2}$ for each $d \geqslant 1$. With this in hand, the induction on H from Theorem 7 can be applied to generate globally rigid generic realizations of G_H.

4. Body insertion

Here we assemble several key lemmas needed for the proof of the main theorem.

We want to show that the combinatorial operation of a $\binom{d+1}{2}$-split on k edges in H can be replicated as a geometric operation on the induced body–bar framework, which adds a new body and preserves global rigidity at generic configurations. To do this, we need steps that preserve the two properties of infinitesimal rigidity and a stress matrix of full rank. We also need to preserve the structure of a body–bar framework. In particular, we need to make sure the new body inserted is connected to distinct vertices in designated bodies, as prescribed by the split operation in H.

We will achieve this geometric goal of splitting k edges in four steps in all dimensions d:

(i) constructing a generically infinitesimally rigid graph (body) $B_{d,k}$ on k vertices, isostatically attached to a complete graph of the appropriate size (Fig. 3);
(ii) geometrically grafting this small structure onto the previous framework across shared vertices, with the new vertices laid onto the midpoints of the edges to be split, making a larger infinitesimally rigid framework (Figs. 4, 5(a));
(iii) using simple local exchange operations based on collinear triangles to replace the edges to be split by a pair of split edges so that the resulting framework remains infinitesimally rigid and has a full rank stress matrix (Fig. 5(b), (c));

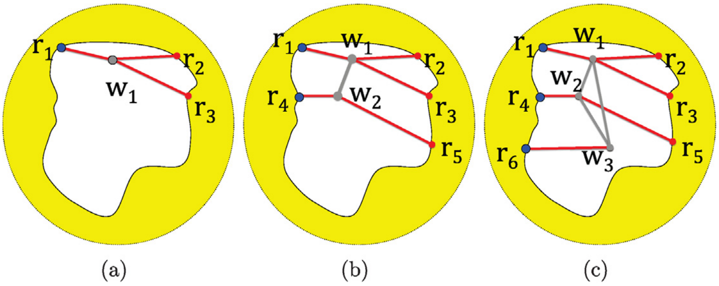

Fig. 3. Creating an inner isostatic body $B_{d,k}$ on $1 \leqslant k \leqslant d$ vertices, attached to an outer complete graph, illustrated for $d = 3$. The initial case has a d-valent addition (a) on d attachment vertices; the next case uses a second d-valent addition (b) now using $d-1$ more attachments (b), and a third example applies an additional d-valent addition with $d-2$ new attachments forming a triangle body (c).

(iv) applying additional edge splits to separate some of the attachment edges at the new body so that we recover the body–bar structure (Fig. 6).

The next subsections will describe these steps in detail. There are some minor differences depending on whether k is small ($1 \leqslant k \leqslant d$) or large ($d+1 \leqslant k \leqslant \binom{d+1}{2} - 1$), which will be indicated in each case.

4.1. Constructing the new body

For each pair of integers d, k with $1 \leqslant k \leqslant \binom{d+1}{2} - 1$ we construct a generically rigid graph $B_{d,k}$ on k vertices (the initial new body) with vertex set W, which will be attached to a complete graph on vertex set R in such a way that the combined framework $C_{d,k}$ is also generically rigid. The size of vertex set R (along which the new body will be grafted to the previous one) is $dk - \binom{k}{2}$, when $1 \leqslant k \leqslant d$, and $\binom{d+1}{2}$, when $d+1 \leqslant k \leqslant \binom{d+1}{2} - 1$.

We begin with the construction of the initial body when $1 \leqslant k \leqslant d$. As a key tool, we recall a simple operation which will be used repeatedly, see e.g. [30, Lemma 11.1.1]. Given a graph $G = (V, E)$, the *vertex d-addition* operation adds a new vertex v_0 and d new edges $v_0v_1, \dots, v_0v_d$ for some $v_i \in V$, $1 \leqslant i \leqslant d$. The corresponding geometric operation on $G(\mathbf{p})$ adds a new vertex positioned at $\mathbf{p}_0$ and inserts d new bars from $\mathbf{p}_0$ to $\mathbf{p}_i$, $1 \leqslant i \leqslant d$.

Lemma 8 *(Vertex addition lemma). Let $G(\mathbf{p})$ be a d-dimensional framework and let $G'(\mathbf{p})$ be obtained from $G(\mathbf{p})$ by a vertex d-addition. If $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_d$ are in general position in d-space then* $\operatorname{rank} R(G', \mathbf{p}) = \operatorname{rank} R(G, \mathbf{p}) + d$, *where $R(G, \mathbf{p})$ is the rigidity matrix of the framework.*

Fig. 3 illustrates the construction in the case when $1 \leqslant k \leqslant d$. For $k = 1$ we start with one inner d-valent vertex w_1 added to a complete graph on d vertices. At each additional step k, we add a d-valent vertex w_k attached to the $k-1$ previous inner vertices w_i, $i < k$, and to $d-k+1$ new attachment vertices r_j (added to the outer body making a larger complete graph). At each vertex w_i we designate the first edge, connecting w_i to R, as its *primary* attachment edge, and the remaining $d-k$ as *secondary*. If we reach $k = d$, we add a vertex w_d connected to the $d-1$ inner vertices and with a single primary attachment edge to one new vertex on the outer complete graph forming $C_{d,d}$.

Next we describe the construction of $B_{d,k}$ when $d+1 \leqslant k \leqslant \binom{d+1}{2} - 1$. We begin with $C_{d,d}$, as defined above, which has d vertices in the body (on vertex set $W_d = \{w_1, w_2, \dots, w_d\}$) attached to a complete graph on vertex set R with $|R| = \binom{d+1}{2}$. Then we add $k-d$ additional vertices $w_{d+1}, \dots, w_k$ by edge split operations, splitting secondary edges connecting the set W_d and R, so that each additional new inner vertex becomes connected to exactly one vertex in R (the outer part of the split edge) as its primary attachment edge, and also connected to $d-1$ other inner vertices in W_d. In the

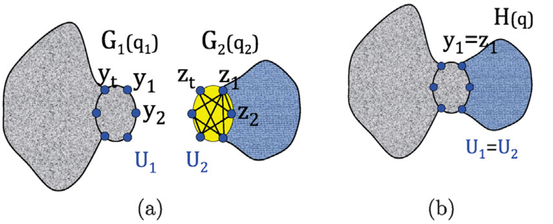

Fig. 4. Given two frameworks $G_1(\mathbf{q}_1)$ and $G_2(\mathbf{q}_2)$ with corresponding subsets of vertices U_1 and U_2 (a), we can graft $G_2(\mathbf{q}_2)$ onto $G_1(\mathbf{q}_1)$ by identifying vertices in U_1 and U_2 and deleting all edges among U_2 in G_2 (b).

final graph $C_{d,k}$ the body has vertex set $W = \{w_1, w_2, \ldots, w_k\}$, while the number of edges from W to R remains as $\binom{d+1}{2}$. Lemmas 5 and 8 imply:

Lemma 9 *(Body construction). The graphs $B_{d,k}$ and $C_{d,k}$ are both generically infinitesimally rigid in $\mathbb{R}^d$ for all pairs d, k with $1 \leqslant k \leqslant \binom{d+1}{2} - 1$. The number of vertices of $B_{d,k}$ is equal to k. The number of vertices in the outer body (which is equal to the number of edges from W to R) is equal to $dk - \binom{k}{2}$, when $1 \leqslant k \leqslant d$, and $\binom{d+1}{2}$, when $d+1 \leqslant k \leqslant \binom{d+1}{2} - 1$.*

Note that there is exactly one primary attachment edge to the outer complete graph at each inner vertex w_i.

4.2. Grafting

We now need to 'graft' the body $B_{d,k}$ and its attachments onto an infinitesimally rigid bar-and-joint framework with k designated edges to be pinched. As the grafting process is general, and geometric, we present it in a corresponding general form.

Let $G_1(\mathbf{q}_1)$ and $G_2(\mathbf{q}_2)$ be two d-dimensional frameworks and let $V_1 = \{w_1, w_2, \ldots, w_t\} \subseteq V(G_1)$ and $V_2 = \{z_1, z_2, \ldots, z_t\} \subseteq V(G_2)$ be designated vertex sets of the same cardinality satisfying $\mathbf{q}_1(w_i) = \mathbf{q}_2(z_i)$ for $1 \leqslant i \leqslant t$. By *grafting $G_2(\mathbf{q}_2)$ onto $G_1(\mathbf{q}_1)$ along V_1, V_2* we mean the operation which creates a new framework (on a new graph) $H(\mathbf{q})$ by deleting the edges of G_2 connecting vertices in V_2 and identifying w_i and z_i for $1 \leqslant i \leqslant t$. See Fig. 4.

Let $G(\mathbf{p})$ be a d-dimensional framework. For two non-adjacent vertices $u, v \in V(G)$ we say that uv is an *implied edge* of $G(\mathbf{p})$ if $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(G + uv, \mathbf{p})$. The *closure* of $G(\mathbf{p})$ is the framework obtained from $G(\mathbf{p})$ by adding all the implied edges.

Lemma 10 *(Grafting). Let $G_1(\mathbf{q}_1)$ and $G_2(\mathbf{q}_2)$ be infinitesimally rigid d-dimensional frameworks and let $H(\mathbf{q})$ be the framework obtained by grafting $G_2(\mathbf{q}_2)$ onto $G_1(\mathbf{q}_1)$ along V_1, V_2, where $V_1 \subseteq V(G_1)$ and $V_2 \subseteq V(G_2)$ with $|V_1| = |V_2| = t \geqslant d$. If the points $\mathbf{q}_1(w_i) = \mathbf{q}_2(z_i)$, $1 \leqslant i \leqslant t$, are in general position in $\mathbb{R}^d$ then $H(\mathbf{q})$ is infinitesimally rigid.*

Proof. Let $H = (U, F)$. We shall prove that for each pair $u, v \in U$ the edge uv is in the closure of $H(\mathbf{q})$, which will imply that $\operatorname{rank} R(H, \mathbf{q}) = \operatorname{rank} R(K_{|U|}, \mathbf{q})$, from which it follows that $H(\mathbf{q})$ is infinitesimally rigid. (Here $K_{|U|}$ denotes the complete graph on $|U|$ vertices.)

Since $G_1(\mathbf{q}_1)$ is infinitesimally rigid, for each pair $u, v \in V(G_1)$ the edge uv is an implied edge (of $G(\mathbf{q}_1)$ and hence) of $H(\mathbf{q})$. Since $G_2(\mathbf{q}_2)$ is infinitesimally rigid, it follows that for each pair $u, v \in V(G_2)$ the edge uv is implied in $H(\mathbf{q})$. Now consider a pair u, v with $u \in V(G_1) - V_1$ and $v \in V(G_2) - V_2$.

We shall use the fact that if X is a set of $d+1$ points in general position in $\mathbb{R}^d$ and y is another point then there is a set $X' \subset X \cup \{y\}$ of $d+1$ points with $y \in X'$ in general position.

First suppose that the points corresponding to the vertices of $V_2 \cup \{v\}$ contain a set X of $d+1$ points in general position. This is the case when $t \geqslant d+1$, or $t = d$ and $\mathbf{q}_2(v)$ is not in the hyperplane

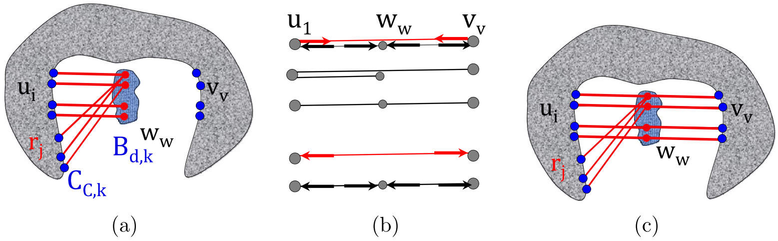

Fig. 5. Given a framework $G_H(\mathbf{p})$ with k designated edges, we can graft on $C_{d,k}(\mathbf{p}')$ so that the new vertices w_i are placed onto the designated edges (a). We can then use triangle exchanges (b) to make these new vertices effectively split the original designated edges (c).

spanned by the points $\mathbf{q}_2(z_i)$, $1 \leqslant i \leqslant d$, of V_2. By choosing a set $X' \subset X \cup \{\mathbf{q}_2(v)\}$ of $d+1$ points with $\mathbf{q}_2(v) \in X'$ in general position and applying Lemma 8 we obtain that each edge connecting a vertex in $V(G_1)$ and v is implied, in particular, uv is implied in $H(\mathbf{q})$.

It remains to consider the case when $t = d$ and $\mathbf{q}_2(v)$ is in the hyperplane L spanned by the points $\mathbf{q}_2(z_i)$, $1 \leqslant i \leqslant d$, of V_2. The infinitesimal rigidity of $G_2(\mathbf{q}_2)$ implies that v has a neighbor v' in G_2 for which $\mathbf{q}_2(v')$ is not in L. Lemma 8 implies that each edge connecting a vertex in $V(G_1)$ and v' is implied. Then we may use the general position argument above, by choosing X to be the points corresponding to the vertices of $V_2 \cup \{v'\}$, and Lemma 8 to deduce that uv is an implied edge in $H(\mathbf{q})$. $\square$

We are ready to graft the framework on $C_{d,k}$ onto a selected framework $G_H(\mathbf{p})$ with the k edges $u_1v_1, \dots, u_kv_k$ to be pinched. For each of the k inner vertices w_i of $C_{d,k}$ there is a unique primary attachment edge incident with it. Let r_i, $1 \leqslant i \leqslant k$, denote the other end-vertex of the primary attachment edge incident with w_i and let $r_{k+1}, \dots, r_s$ denote the remaining vertices of attachment of the inner body, where $s = dk - \binom{k}{2}$, when $1 \leqslant k \leqslant d$, and $s = \binom{d+1}{2}$, when $d+1 \leqslant k \leqslant \binom{d+1}{2} - 1$. Suppose that S is a set of s designated vertices of G_H with $u_i \in S$ and $v_i \notin S$, $1 \leqslant i \leqslant k$.

Let $\mathbf{p}'$ be a configuration for $C_{d,k}$ for which the vertices of the outer complete graph are on top of the s designated vertices of $G_H(\mathbf{p})$ with $\mathbf{p}'(r_i) = \mathbf{p}(u_i)$ for $1 \leqslant i \leqslant k$, and with each $\mathbf{p}'(w_i)$ at the midpoint of the edge u_iv_i, $1 \leqslant i \leqslant k$.

We shall graft $C_{d,k}(\mathbf{p}')$ onto $G_H(\mathbf{p})$ along S, R. See Fig. 5. We call this graft and placement *placing $C_{d,k}$ onto the k edges and s vertices*. The set S of vertices will be determined by the split operation on H, when we apply the next lemma.

Lemma 11 *(Placing the body). Given an infinitesimally rigid generic framework $G_H(\mathbf{p})$ with k designated edges $u_1v_1, \dots, u_kv_k$ and a set S of s distinct designated vertices with $u_i \in S$ and $v_i \notin S$, $1 \leqslant i \leqslant k$,* placing $C_{d,k}$ onto the k edges and s vertices *produces an extended infinitesimally rigid framework $\bar{G}_H(\bar{\mathbf{p}})$.*

Proof. Since $G_H(\mathbf{p})$ is generic, it follows that $C_{d,k}(\mathbf{p}')$ is also a generic framework. (This holds because there is a rational homeomorphism between the configuration of points u_i, v_i and the configuration, where the v_i are replaced by the w_i, the midpoints.) Thus $C_{d,k}(\mathbf{p}')$ is infinitesimally rigid by Lemma 9.

Since $G_H(\mathbf{p})$ is generic, the points of S are in general position. We also have $|S| \geqslant d$. Thus, by Lemma 10, the grafted framework $\bar{G}_H(\bar{\mathbf{p}})$ is infinitesimally rigid. $\square$

4.3. Triangle exchange

The next step is to modify $\bar{G}_H(\bar{\mathbf{p}})$ so that the vertices w_i split the k edges, preserving infinitesimal rigidity and raising the rank of the stress matrix by k, for the k new vertices. This is done by exchanges on collinear triples of vertices u_i, w_i, v_i, replacing the bars u_iw_i and u_iv_i with the two bars u_iw_i and w_iv_i. We call this operation a *triangle exchange* (on the collinear pair u_iw_i, u_iv_i) and the resulting framework $G^*_H(\mathbf{p}^*)$ the *inserted framework*.

Lemma 12 *(Insertion of body). Let $G_H(\mathbf{p})$ be an infinitesimally rigid framework with a full rank stress matrix with a non-zero stress on all edges and let $\hat{G}_H(\hat{\mathbf{p}})$ be obtained from $G_H(\mathbf{p})$ by placing $C_{d,k}$ on k edges u_iv_i and s vertices. Suppose that $G^*_H(\mathbf{p}^*)$ is obtained from $\bar{G}_H(\bar{\mathbf{p}})$ by k triangle exchange operations on the collinear pairs u_iw_i, u_iv_i, $1 \leqslant i \leqslant k$. Then*

(i) *$G^*_H(\mathbf{p}^*)$ is infinitesimally rigid;*
(ii) *$G^*_H(\mathbf{p}^*)$ has a full rank stress matrix.*

Proof. We have placed the grafted framework so that u_i, w_i, v_i are collinear, $1 \leqslant i \leqslant k$.

(i) The observation that a collinear triangle is a minimally dependent framework (circuit) in all dimensions (Fig. 5(b)) is equivalent to saying that the rows for any two of the bars generate the row for the third bar, by row reduction in the rigidity matrix. In turn, this implies that exchanging in w_iv_i to replace the edge u_iv_i, given the presence of u_iw_i, preserves the rank of the rigidity matrix and therefore preserves the infinitesimal rigidity (Fig. 5(c)).

(ii) Given the non-zero stress on u_iv_i in $G_H(\mathbf{p})$, an exchange operation in w_iv_i to replace u_iv_i produces an extended self-stress (Fig. 5(b), (c)). Moreover, by the same argument that was used in Lemma 4 for subdividing an edge, this increases the rank of the stress matrix by one for each of the k added vertices w_i. $\square$

4.4. Separating attachments

At this point G^*_H is not yet a body–bar graph: some vertices w_i in the new body on W are connected to at least two vertices that belong to other bodies. Since in a body–bar graph each vertex is connected to at most one other body, we need to separate the attachments. The following lemma shows how to get rid of the secondary attachment edges and also confirms that the final body–bar structure is globally rigid for generic configurations.

Lemma 13 *(Separating attachments). Let H be a multigraph for which the body–bar graph G_H is generically rigid in $\mathbb{R}^d$. Let $\mathbf{p}$ be a generic configuration for which $G_H(\mathbf{p})$ has an equilibrium stress ω with a stress matrix of full rank. Suppose that we have a set of designated edges u_iv_i, $1 \leqslant i \leqslant k$, and non-negative integers $s_1, s_2, \dots, s_n$ assigned to the n bodies of G_H, where $1 \leqslant k \leqslant \binom{d+1}{2} - 1$ and $\sum_{\ell=1}^{n} s_\ell = \binom{d+1}{2} - k$. Then we can construct an extended body–bar graph G_{H^*} and an extended generic configuration $\hat{\mathbf{p}}$ with one added body b^* on $\binom{d+1}{2} + k$ vertices, with s_i new vertices added to each existing body b_i, $1 \leqslant i \leqslant n$, and by replacing the k designated edges by $\binom{d+1}{2} + k$ disjoint edges connecting b^* to the added vertices and to vertices u_i, v_i, $1 \leqslant i \leqslant k$, such that*

(i) *$G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid, and*
(ii) *$G_{H^*}(\hat{\mathbf{p}})$ has a stress matrix of full rank.*

Proof. It follows from Theorems 2 and 3 that $G_H(\mathbf{p})$ is redundantly rigid. Thus, by Lemma 6, we may suppose that ω is non-zero on all edges. We start the construction of G_{H^*} by extending each body b_ℓ by s_ℓ new vertices. We do this by a sequence of s_ℓ edge splitting operations on stressed edges within b_ℓ. This way we can maintain a stress matrix of full rank and, since $\mathbf{p}$ is generic, we can also preserve infinitesimal rigidity, by Lemmas 4 and 5. These new vertices are labeled as r_j, $k+1 \leqslant j \leqslant \binom{d+1}{2}$.

Then we move to a nearby generic configuration and apply Lemma 6 to obtain an initial infinitesimally rigid framework $G'_H(\hat{\mathbf{p}}')$ with an equilibrium stress with a full rank stress matrix, such that every edge has a non-zero stress. We have two cases to consider.

(i) First suppose that $1 \leqslant k \leqslant d$. Apply Lemma 12 to the resulting framework, by using vertices $u_1, u_2, \dots, u_k$ and the first $dk - \binom{k+1}{2}$ $(\leqslant \binom{d+1}{2} - k)$ vertices r_j as designated vertices, to obtain an initial infinitesimally rigid framework $G^*_H(\hat{\mathbf{p}})$ with all new vertices placed on stressed edges. By Lemma 4 this implies that $G^*_H(\hat{\mathbf{p}})$ also has an equilibrium stress with a full rank stress matrix. Moving to a generic configuration preserves all of these properties, and ensures that every edge has a non-zero stress by Lemma 6.

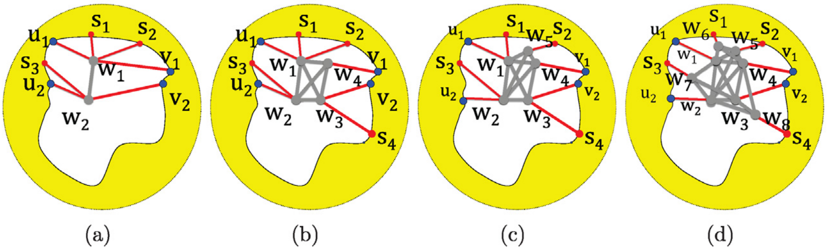

Fig. 6. Given an extended framework with a small body inserted on k edges (a) we separate the secondary attachments by edge splits (illustrated for $k = 2$, $d = 3$).

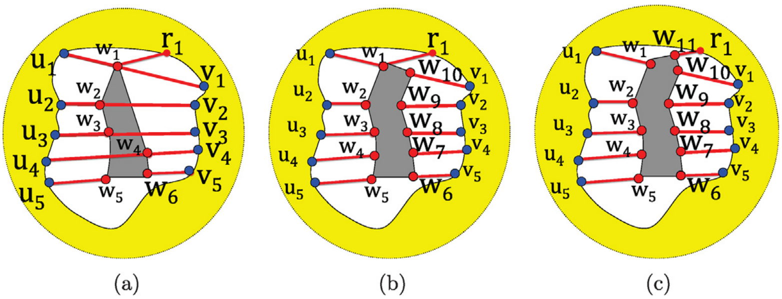

Fig. 7. Given the inserted body with $> d$ vertices, we separate the attachments by edge splits (a), (b), (c), until no two attachments share a vertex on the body.

Note that after the grafting step we have $dk - \binom{k}{2} - k$ secondary attachments incident with W, which grows to $dk - \binom{k}{2}$ after the k triangle exchange operations. Now we can apply a further sequence of edge splits on secondary attachment edges to separate these attachments at the new body and make sure that each vertex on the new body is connected to other bodies by precisely one bar. See Fig. 6(b), (c) for some stages of this procedure. In each split, we connect the new vertex to as many other w_i as possible and create additional secondary attachments as necessary. After the first $d - k$ steps, we will have d inner vertices, $\binom{d+1}{2} - d + k$ secondary attachment edges, and $\binom{d+1}{2} + k$ attachment vertices. We continue to split on secondary attachment edges, reducing the number of secondary attachment edges at each further step without creating new attachment vertices. We will end with $\binom{d+1}{2} + k$ inner vertices w_i (Fig. 6(d)).

Each of these edge splits is on a stressed edge, provided we continue to move each new vertex w_i, $i > k$, to a generic position. Therefore, this sequence of splits preserves both key properties: (i) the infinitesimal rigidity; and (ii) the full rank stress matrix.

(ii) Next suppose $d + 1 \leqslant k \leqslant \binom{d+1}{2} - 1$. As in the previous case, we apply a sequence of edge splits to eliminate the secondary attachment edges (Fig. 7). Note that after the grafting step we have $\binom{d+1}{2} - k$ secondary attachments incident with W, which grows to $\binom{d+1}{2}$ after the k triangle exchange operations. Thus we have $\binom{d+1}{2}$ secondary attachment edges and an inserted body on k vertices, when we start this phase. To eliminate all secondary attachment edges we add $\binom{d+1}{2}$ additional vertices to the new body, by using edge splits, which gives $\binom{d+1}{2} + k$ in total, as required. At each stage the framework is infinitesimally rigid, and has an equilibrium stress with a stress matrix of full rank by Lemmas 4 and 5. We conclude that the final framework $G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid with a full rank stress matrix, as required.

In both cases we can further extend the added body by edge insertions to make it a complete graph on its $\binom{d+1}{2} + k$ vertices. Hence the constructed graph G_{H^*} is a body–bar graph which is induced by a multigraph H^*, obtained from H by a $\binom{d+1}{2}$-split on k edges. □

5. Globally rigid body–bar graphs

5.1. Redundantly rigid implies highly tree-connected

First we give a direct proof, in terms of bar-and-joint frameworks, for the fact that if a body–bar graph G_H is redundantly rigid then H is highly tree-connected. It can also be deduced from Tay's characterization of generically rigid body–bar frameworks [24,25].

Lemma 14. *Let $H=(V,E)$ be a multigraph with $|V|\geqslant 2$ and suppose that the body–bar graph G_H induced by H is generically redundantly rigid in $\mathbb{R}^d$. Then H is highly $\binom{d+1}{2}$-tree-connected.*

Proof. For a contradiction suppose that $e_H(\mathcal{P})\leqslant\binom{d+1}{2}(t-1)$ for a partition $\mathcal{P}=\{X_1,X_2,\dots,X_t\}$ of V with $t\geqslant 2$. Let $Y_i=\cup\{V(B_v)\colon\ v\in X_i\}$, for $1\leqslant i\leqslant t$, and let $\mathcal{Q}=\{Y_1,Y_2,\dots,Y_t\}$ be the corresponding partition of $V(G_H)$. The redundant rigidity of G_H implies that each vertex of G_H has degree at least $d+1$. Hence $|V(B_v)|\geqslant d+1$ and also $|Y_i|\geqslant d+1$ for all $v\in V$ and $1\leqslant i\leqslant t$. Observe that $e_{G_H}(\mathcal{Q})=e_H(\mathcal{P})\neq\emptyset$.

Let $S\subseteq E(G_H)$ be a maximal set of independent edges in G_H, i.e. a base in the d-dimensional generic rigidity matroid of G_H. Since G_H is rigid and G_H has more than $d+1$ vertices, we have $|S|=d|V(G_H)|-\binom{d+1}{2}$. Thus, by using the fact that each subset $Y\subseteq V(G_H)$ with $|Y|\geqslant d+1$ induces at most $d|Y|-\binom{d+1}{2}$ edges of S, we obtain

$$\begin{aligned} d\bigl|V(G_H)\bigr|-\binom{d+1}{2} &= |S|\leqslant\sum_{1}^{t}\left(d|Y_i|-\binom{d+1}{2}\right)+e_{G_H}(\mathcal{Q})\\ &= d\bigl|V(G_H)\bigr|-\binom{d+1}{2}t+e_H(\mathcal{P})\leqslant d\bigl|V(G_H)\bigr|-\binom{d+1}{2}. \end{aligned}$$

Thus we have equality everywhere. In particular, for all edges $e\in E_{G_H}(\mathcal{Q})$ and all bases S we must have $e\in S$. This implies that e is not redundant, and hence G_H is not redundantly rigid, a contradiction. Hence each partition of V satisfies (1) and the lemma follows. $\square$

5.2. Highly tree-connected implies a globally rigid realization

Next we show by induction, using the body insertion lemmas, that highly tree-connected graphs have globally rigid generic realizations.

Lemma 15. *Let $H=(V,E)$ be a highly $\binom{d+1}{2}$-tree-connected multigraph. Let $G_H=K_{d+2(l+1)}$, when $|V|=1$ and E is a set of $l\geqslant 0$ loops, and otherwise let G_H be the body–bar graph induced by H. Then there exists an infinitesimally rigid realization $G_H(\mathbf{p})$ of G_H in $\mathbb{R}^d$ with an equilibrium stress ω for which the associated stress matrix Ω has rank $n-d-1$, where $n=|V(G_H)|$.*

Proof. The proof is by induction on $|V|+|E|$. In the base case, when $|V|=1$ and $E=\emptyset$, G_H is a complete graph on $d+2$ vertices. In this case it is easy to construct an infinitesimally rigid realization $G_H(\mathbf{p})$ with an equilibrium stress ω for which the associated stress matrix has full rank.

Now consider a highly $\binom{d+1}{2}$-tree-connected multigraph $H=(V,E)$ and suppose that the lemma holds for all highly $\binom{d+1}{2}$-tree-connected multigraphs H' with $|V(H')|+|E(H')|<|V(H')|+|E(H')|$. By Theorem 7 H can be obtained from a smaller highly $\binom{d+1}{2}$-tree-connected multigraph H' by adding an edge or by a $\binom{d+1}{2}$-split on k edges, for some $1\leqslant k\leqslant\binom{d+1}{2}-1$. By induction, there exists an infinitesimally rigid realization $G_{H'}(\bar{\mathbf{p}})$ of $G_{H'}$ with an equilibrium stress ω for which the associated stress matrix Ω has rank $n'-d-1$, where $n'=|V(G_{H'})|$. By Lemma 6 we may suppose that $\bar{\mathbf{p}}$ is generic and ω is non-zero on all edges.

First suppose that H is obtained from H' by adding a new edge uv, possibly a loop. Then we may construct a realization $G_H(\mathbf{p})$ from $G_{H'}(\bar{\mathbf{p}})$ by performing two edge splits within B_u and B_v, respectively, which create two new vertices of degree $d+1$, followed by edge additions, which connect the new vertices and which make the two enlarged bodies complete. Note that the definition of $G_{H'}$ and the assumption on H' implies that each body in $G_{H'}$ has at least $d+1$ vertices. These operations preserve infinitesimal rigidity and the property of having a stress matrix of full rank by Lemmas 4 and 5. Thus the lemma follows by Lemma 6.

Next suppose that H is obtained from H' by a $\binom{d+1}{2}$-split on k edges. Lemma 13 confirms that there is an extended framework $G_H(\mathbf{p})$ which is infinitesimally rigid and has a stress matrix of full rank. $\square$

5.3. The main theorem

We can now assemble the pieces to give a full proof of the main theorem.

Theorem 16. *Let $H=(V,E)$ be a multigraph with $|V|\geqslant 2$ and $|E|\geqslant 2$ and let G_H be the body–bar graph induced by H. Let $d\geqslant 1$ be an integer. Then the following are equivalent*:

(a) *G_H is generically globally rigid in $\mathbb{R}^d$,*
(b) *G_H is generically redundantly rigid in $\mathbb{R}^d$,*
(c) *H is highly $\binom{d+1}{2}$-tree-connected.*

Proof. Since $|V|\geqslant 2$ and $|E|\geqslant 2$, it follows that G_H is not a complete graph. Thus (a) $\to$ (b) follows from Theorem 2.

(b) $\to$ (c) follows from Lemma 14.

(c) $\to$ (a) follows from Lemma 15 and Theorem 3. $\square$

6. Further remarks

6.1. Algorithmic implications

Theorem 16 gives rise to a polynomial time algorithm to determine whether a body–bar graph is generically globally rigid in $\mathbb{R}^d$. This follows from the fact that, as we noted earlier, a multigraph H is highly m-tree-connected if and only if $H-e$ contains m edge-disjoint spanning trees for all $e\in E(H)$. Thus efficient tree-packing algorithms can be used to test whether a given multigraph is highly m-tree-connected. We refer the reader to [22, Chapter 51] for a complexity survey for tree packing algorithms.

By using similar techniques one can also compute the maximal highly m-tree-connected subgraphs of H. The vertex sets of these subgraphs form a partition of $V(H)$. See [14] for more details (where these subgraphs are called the *m-superbricks* of H).

Another algorithmic observation is that one can easily test whether a given graph G is a body–bar graph. To see this suppose, for simplicity, that each body has at least three vertices. Then a pair u, v of vertices belongs to the same body if and only if they are adjacent and they have a common neighbor. The following method is based on this fact: consider the subgraph H of G consisting of those edges uv of G for which u and v have a common neighbor in G. It follows that G is a body–bar graph if and only if H is a collection of disjoint complete graphs covering $V(G)$ and the complement of H in G is a matching.

6.2. Globally linked pairs

Given the characterization of globally rigid graphs in the plane, the methods have recently been extended to characterize *globally linked* pairs of vertices in some classes of graphs in $\mathbb{R}^2$ [16,17]. These are pairs of vertices whose distance is the same in all frameworks which are equivalent to any given generic framework of the graph. One can ask the analogous question for body–bar graphs G_H in $\mathbb{R}^d$. We *conjecture* that a pair of vertices is globally linked in G_H if and only if there is a globally rigid subgraph of G_H which contains both (or equivalently, if they are adjacent or the vertices of H corresponding to their bodies belong to the same $\binom{d+1}{2}$-superbrick of H). This conjecture is open even for $d = 2$, in which case it is consistent with the more general conjecture for the plane, see [16, Conjecture 5.9].

6.3. Connectivity

We did not directly refer to Hendrickson's $(d+1)$-connectivity condition of Theorem 2 in our proofs. This is because high vertex-connectivity follows for 'free' for body–bar graphs G_H induced by highly $\binom{d+1}{2}$-tree-connected multigraphs. Another related observation is that if the multigraph H is $d(d+1)$-edge connected, then the body–bar graph G_H is generically redundantly rigid in $\mathbb{R}^d$, see e.g. [30]. This now implies that G_H is globally rigid in $\mathbb{R}^d$. There are examples showing that the bound $d(d+1)$ on the edge-connectivity of H cannot be improved.

In general, it has been conjectured that $d(d+1)$-vertex-connectivity is sufficient for generic rigidity for arbitrary bar-and-joint frameworks in $\mathbb{R}^d$ [19]. We can extend this and *conjecture* that $d(d+1)$-vertex-connectivity is sufficient for global rigidity of bar-and-joint frameworks in $\mathbb{R}^d$.

6.4. Body–hinge and molecular frameworks

The infinitesimal rigidity results for body–bar frameworks have been generalized to body–hinge frameworks [15,24,27]. This suggests the following generalization of Theorem 16. For a graph G and integer k we use kG to denote the multigraph obtained from G by replacing each edge e of G by k parallel copies of e.

Conjecture 1 *(Body–hinge global rigidity conjecture). A graph G is generically globally rigid in $\mathbb{R}^d$ as a body–hinge framework if the graph $(\binom{d+1}{2} - 1)G$ is generically redundantly rigid as a body–bar framework in $\mathbb{R}^d$. Equivalently, a graph G is generically globally rigid in $\mathbb{R}^d$ as a body–hinge framework if the multigraph $(\binom{d+1}{2} - 1)G$ is highly $\binom{d+1}{2}$-tree-connected.*

For generic rigidity, there was a further conjecture, which in its various forms has been called the Molecular Conjecture [32] and has recently been verified by Katoh and Tanigawa [18]. We have an extended conjecture for the global rigidity of molecular frameworks.

Conjecture 2 *(Molecular global rigidity conjecture). A graph G is generically globally rigid in $\mathbb{R}^d$ as a molecular–hinge framework if and only if the multigraph $(\binom{d+1}{2} - 1)G$ is highly $\binom{d+1}{2}$-tree-connected.*

In many contexts, including the study of infinitesimal rigidity, there is an equivalence between the molecular–hinge structure on G and an associated bar-and-joint framework on the *square* G^2 of G in $\mathbb{R}^3$. However, for small cycles (of length at most 4) the shift between structures does not preserve equilibrium stresses (or redundance). Thus it may not preserve global rigidity, as the following example from [12] shows: consider two four-cycles with a common vertex. For this graph G we have that $5G$ is highly 6-tree-connected but G^2 is not even redundantly rigid in $\mathbb{R}^3$.

Conjecture 3. *Suppose that G has no cycles of length at most 4. Then G^2 is generically globally rigid in $\mathbb{R}^3$ as a bar-and-joint framework if the multi-graph $5G$ is highly 6-tree-connected.*

6.5. Body–bar frameworks with identifications

During the construction for the main theorem, we carefully did additional splits to separate the end-vertices of all bars incident with the same body, to obtain a body–bar structure. However, the

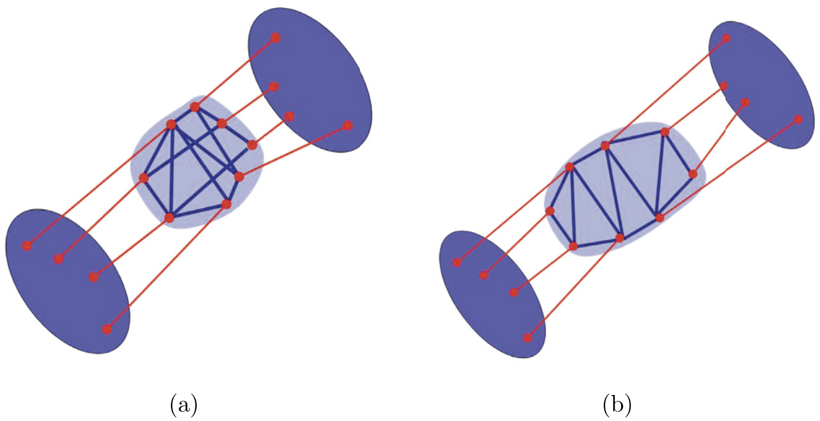

Fig. 8. There are isostatic frameworks which generate a globally rigid framework (a), but others (b), when substituted do not generate a globally rigid framework, due to failed connectivity. The first framework is globally rigid in the plane but the second one is not – it is not 3-connected.

framework was already globally rigid before doing these additional splits. So some identification of end-vertices will still preserve global rigidity. On the other hand, too much identification of the end-vertices will destroy even first-order rigidity, as the 'double banana' can be cast as two bodies joined by six bars, where two triples of bars share end-vertices.

An *identified body–bar framework* is a body–bar graph, with additional data for each body, which partitions the incident bars into classes which will share a vertex of attachment. It may be interesting to characterize which identifications preserve global rigidity (or even first-order rigidity) of a body–bar graph.

6.6. Isostatic frameworks for bodies

It is not difficult to see, by rereading the proofs of the main lemmas, that if H is a highly $\binom{d+1}{2}$-tree-connected multigraph on at least two vertices then it is possible to replace each 'body' of the globally rigid body–bar graph G_H by *some* isostatic graph preserving global rigidity in $\mathbb{R}^d$. This follows by observing that the edge addition steps within the bodies are not necessary to ensure global rigidity, and that the other operations, when restricted to the individual bodies, build up isostatic graphs by edge splits and vertex additions.

For infinitesimal rigidity it is known that in all dimensions one can replace any isostatic subframework with any other isostatic subframework on the same vertices and preserve infinitesimal rigidity. However, the same general *isostatic replacement* does not necessarily preserve global rigidity. This issue most clearly arises in the steps of the insertion lemmas when we are separating the attachment points. While a careful separation (as used in our proof) does preserve global rigidity, a general replacement can easily break down the simple necessary $(d+1)$-connectivity condition. See Fig. 8, which, in the plane, breaks the required 3-connectivity.

We do not currently have a conjecture for which isostatic replacements for bodies would preserve global rigidity. So there is a residual puzzle about how to detect whether a bar and joint framework in which we have 'identified' bodies with isostatic subframeworks, and distinct edges joining them, has the required structure to apply this theorem and claim global rigidity.

6.7. Universal rigidity

Several recent papers have explored a stronger uniqueness property: a d-dimensional framework $G(\mathbf{p})$ is called *universally rigid* if it is a unique realization of G, with the given edge lengths, in all spaces $\mathbb{R}^{d'}$ for $d' \geqslant d$. Recent results of Gortler and Thurston [9] show that for generic $\mathbf{p}$ in $\mathbb{R}^d$ this property is equivalent to the existence of a self-stress for which the associated stress matrix Ω is of full rank and positive semi-definite. Notice that this property is not generic, but the property of having a positive semi-definite stress matrix of full rank does hold for an open set among the generic configurations.

Without discussing the details here we remark that the key operations used in the inductive proof of our main theorem (edge splitting and triangle exchange) appear to preserve universal rigidity, provided we follow some additional rules when we insert new vertices. For example, if we adjust the (grafting operation and hence the subsequent) triangle exchange, so that we select whether to insert the vertex splitting an edge in its interior, or externally, we can ensure that the triangle exchange not only increases the rank of the stress matrix, but takes the positive semi-definite matrix to a larger one which is also positive semi-definite. With these additions the same proof method can be used to derive an extension of Lemma 15, in which the stress matrix is positive semi-definite, too. Together with Theorem 16 this implies, for each $d \geqslant 1$, that every generically globally rigid body–bar graph G_H in $\mathbb{R}^d$ has a generic realization in $\mathbb{R}^d$ which is universally rigid. This gives an affirmative answer to the more general [9, Question 1.16] in the case of body–bar graphs.

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