

Research



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Periodic planar disc packings

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Several conditions are given when a packing of equal discs in a torus is locally maximally dense, where the torus is defined as the quotient of the plane by a two-dimensional lattice. Conjectures are presented that claim that the density of any collectively jammed packing, whose graph does not consist of all triangles, and the torus lattice is the standard triangular lattice, is at most $(n/(n+1))\pi/\sqrt{12}$, where n is the number of packing discs in the torus. Several classes of collectively jammed packings are presented where the conjecture holds.

1. Introduction and definitions

Packings of discs are of great importance in number theory, granular materials, algebraic number theory and who knows what else. Their rigidity properties are also important and can provide useful tools for their analysis, in particular for finding maximally dense packings, or locally maximally dense packings, in certain situations. Here, we primarily investigate the case of periodic packings of equal discs, where the period lattice is the ubiquitous triangle lattice given by the edge-to-edge tiling of equilateral triangles. One reason for choosing this lattice is that it is the basis for the most dense packing of equal discs in the plane. See, for example, the classic books [1,2], for a good discussion of these problems, as well as [3] for many examples of square and triangle tilings. Another reason for this choice is that the techniques of rigidity theory can be applied here, whereas, in many other cases, some of the tools described here are not available.

(a) Lattices

A lattice Λ in Euclidean space \mathbb{E}^d is the set of all integral linear combinations of the vectors g_1, \dots, g_D in \mathbb{E}^d ,

$$\Lambda(g_1, \dots, g_D) = \{n_1g_1 + \dots + n_Dg_n \mid n_1, \dots, n_D \in \mathbb{Z}\}. \quad (1.1)$$

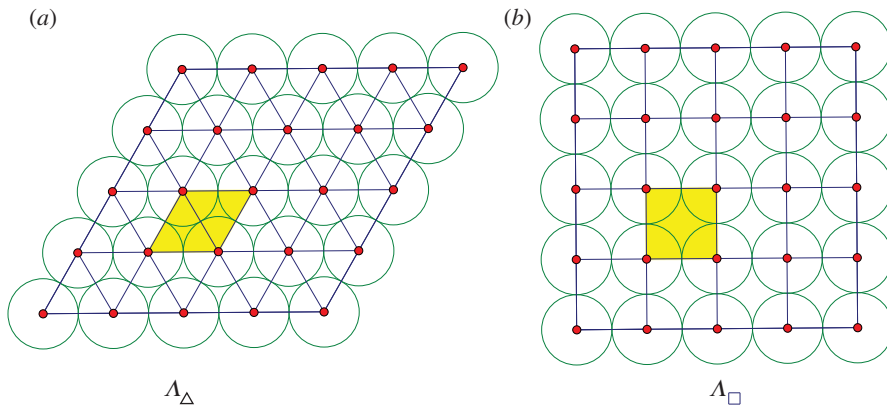


Figure 1. A portion of (a) a triangular lattice Λ_Δ and (b) the square lattice Λ_\square , along with the corresponding circle lattice packings. A fundamental region is shaded. The circle lattice packing for $\mathbb{T}^2(\Lambda_\Delta)$ has the densest packing density of any packing in the plane. (Online version in colour.)

Note that D can be greater or less than d , but the usual definition of a lattice is when $D = d$ and g_1, \dots, g_D is a basis of \mathbb{E}^d . This sort of lattice can be regarded as a projection, or injection, of the usual lattice into \mathbb{E}^d .

We also find it useful, for $d = 2$, to use complex notation, so $\mathbb{C} = \mathbb{E}^2$, and define $g(\theta) = e^{i\theta} = \cos \theta + i \sin \theta$. For notational fluidity, we define

$$g_\Delta = g\left(\frac{2\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad \Lambda_\Delta = \Lambda(1, g_\Delta),$$

$$g_\square = g\left(\frac{2\pi}{4}\right) = i \quad \text{and} \quad \Lambda_\square = \Lambda(1, g_\square),$$

which is the usual triangular lattice and the square lattice as in figure 1. We say that any lattice of the form $\Lambda(z, g_\Delta z)$ is a *triangular lattice*. In other words, there are generators of equal length such that the angle between them is $2\pi/6$. Similarly, a *square lattice* is of the form $\Lambda(z, g_\square z)$.

For any two-dimensional lattice Λ in the plane, not contained in a line, we define the corresponding torus as

$$\mathbb{T}^2(\Lambda) = \mathbb{C}/\Lambda,$$

where two points are identified if they differ by a vector in the lattice Λ . We regard $\mathbb{T}^2(\Lambda)$ as a compact metric space, where locally it is isometric to the flat Euclidean plane. Alternatively, we can regard $\mathbb{T}^2(\Lambda)$ as obtained by identifying opposite edges of a fundamental parallelogram as in figure 1.

Because $\mathbb{T}^2(\Lambda)$ is compact, it has a finite area, namely the area of any fundamental parallelogram. For example,

$$\text{Area}(\mathbb{T}^2(\Lambda(g_1, g_2))) = |g_1||g_2| \sin \theta = \text{Im}(g_1 \bar{g}_2), \tag{1.2}$$

where θ is the angle between g_1 and g_2 , \bar{z} is the complex conjugate of z and $\text{Im}(z)$ is the imaginary part of z .

We say that a *triangular torus* is any torus of the form $\mathbb{T}^2(\Lambda(z, g_\Delta z))$. In other words, a triangular torus is where the lattice used to define it is a triangular lattice. Similarly, a *square torus* is defined by a square lattice.

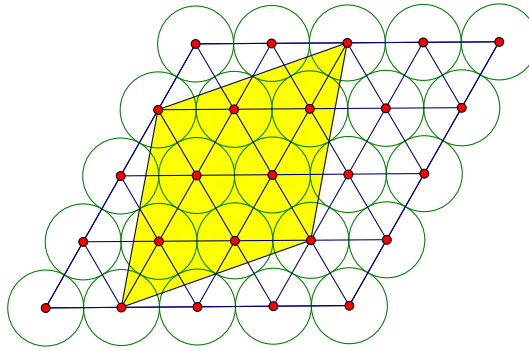


Figure 2. A triangular packing of seven equal discs in a triangular torus of maximal density $\pi/\sqrt{12}$. The fundamental region of the sublattice is indicated. (Online version in colour.)

(b) Packings

A collection of n equal (radii) discs with disjoint interiors in $\mathbb{T}^2(\Lambda)$ is called a *packing*. When $n = 1$, the packing is called a *lattice packing*. The density of a packing is δ , the ratio of the sum of the areas of the discs divided by the area of the torus. So, the density of n discs of radius r in $\mathbb{T}^2(\Lambda(z, g(\theta)z))$ is

$$\delta = \frac{n\pi r^2}{|z|^2 \sin \theta}. \quad (1.3)$$

For the triangular lattice packing when $r = 1/2$ and $n = 1$,

$$\delta_{\Delta} = \frac{\pi(1/2)^2}{\sin(2\pi/6)} = \frac{\pi}{\sqrt{12}}, \quad (1.4)$$

which is the maximum density for any packing of equal discs in the plane.

When the packing is lifted to a packing of an infinite number of packing discs in the plane, this definition of packing density agrees with the usual one, where larger and larger regions are intersected with the packing to calculate the packing density.

One of our main problems is to understand the most dense packings of n equal discs in a triangular torus. For certain values of n , this determination is easy, namely when there is a triangular lattice packing inside the triangular torus. This happens when Λ_{Δ} is a lattice containing $\Lambda(z, g_{\Delta}z)$. So $z = n_1 + n_2 g_{\Delta}$, and

$$\begin{aligned} |z|^2 = z\bar{z} &= (n_1 + n_2 g_{\Delta})(n_1 + n_2 \bar{g}_{\Delta}) = n_1^2 + n_1 n_2 (g_{\Delta} + \bar{g}_{\Delta}) + n_2^2 \\ &= n_1^2 + n_1 n_2 + n_2^2. \end{aligned} \quad (1.5)$$

The ratio of the areas of the tori is given by (1.5), because the angle between the generators $2\pi/6$ is the same in both cases. Furthermore, (1.5) is the number of packing discs by (1.4), because the density is the same for both packings. Putting this together, we obtain the following theorem.

Theorem 1.1. *The maximum density of a packing of n equal discs δ_{Δ} in a triangular torus is achieved when and only when $n = n_1^2 + n_1 n_2 + n_2^2$ for n_1, n_2 integers.*

Figure 2 shows this for $n_1 = 2$, $n_2 = 1$ and $n = 7$.

We call any integer of the form (1.5) a *triangle lattice number*, which are also called Loeschian numbers in [4, p. 111] (Sloane sequence A003136). The first few numbers of this form are 0, 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39, 43, 48, 49. It is easy to see that the product of triangle lattice numbers is a triangle lattice number, and it is shown in [5] that $n > 0$ is a triangle lattice number if and only if the non-square prime factors are congruent to 1 modulo 6, or it is the prime 3.

(c) The conjectures

One of our motivations is the following conjecture of Fejes Tóth [6,7]. A packing of circles in the plane is said to be *solid* if no finite subset of the circles can be rearranged so as to form, along with the rest of the circles, a packing not congruent to the original one.

Conjecture 1.2 (L. Fejes Tóth). *The triangle packing in the plane, minus one packing disc, is solid.*

The definition of a solid packing refers to unlabelled packings, so permuting the packing elements is not considered a distinct packing. The following is a related conjecture, but, in terms of finite packings in the triangular torus, is what we call the density gap conjecture.

Conjecture 1.3. *Let n be a positive integer so that $n + 1$ is a triangle lattice number but n is not a triangle lattice number. Then, the most dense packing of n equal discs in the triangular torus is $\delta = (n/(n + 1))\delta_{\Delta} = (n/(n + 1))\pi/\sqrt{12}$, and is achieved only by the triangular packing with one disc removed.*

The point is that conjecture 1.3 implies conjecture 1.2, because a finite subpacking of a counterexample to the conjecture 1.2 can be placed in a triangular torus with a sufficiently large number of discs. We do not provide a proof of conjecture 1.3, but we do provide some evidence for it in the following sections.

Following the definitions in [8], we say that a packing is *collectively jammed* if the only continuous motion of the packing discs, fixing the lattice defining the torus and fixing the packing radius, is a continuous translation. The following is another conjecture closely related to conjecture 1.3.

Conjecture 1.4. *If a packing of n equal discs in a triangular torus is collectively jammed and is not a triangular packing, then its density is at most $\delta = (n/(n + 1))\delta_{\Delta} = (n/(n + 1))\pi/\sqrt{12}$, and the maximal density is achieved only by the triangular packing with one disc removed.*

Note that conjecture 1.4 concerns any collectively jammed packing of discs, whether or not there is a triangular packing of $n + 1$ discs available.

(d) Results for small numbers of packing discs

In [9], the locally maximally dense packings of n equal discs in a triangular torus were investigated, and all the locally maximally dense packings were found for $1 \leq n \leq 6$. Figure 3 shows those packings. For each of those values of n , there is only one locally maximally dense packing, and thus it is the maximally dense packing. For all of these packings, except $n = 5$, these are the packings expected from conjecture 1.3, namely either a triangular lattice packing or such a packing minus one packing disc. For five discs, a fundamental region is shown that has threefold dihedral symmetry, and thus the whole packing has that symmetry. Similar results were found in [10] for a square torus.

(e) Rigidity theory

For any packing, the *graph of the packing* is obtained by connecting each pair of centres of the packing discs that are tangent to each other with an edge. Indeed, the (labelled) disc centres form a configuration of points $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots)$ in the plane or quotient torus, and we consider each edge $\{j, k\}$ as a strut in the sense of a tensegrity framework as in [11–13]. An infinitesimal flex of framework $G(\mathbf{p})$ is a sequence of vectors $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \dots)$, where each \mathbf{p}'_j is associated with \mathbf{p}_j such that for each strut $\{j, k\}$

$$(\mathbf{p}_j - \mathbf{p}_k) \cdot (\mathbf{p}'_j - \mathbf{p}'_k) \geq 0. \quad (1.6)$$

Although a periodic configuration \mathbf{p} is infinite, for a fixed lattice Λ , we consider each equivalence class $\mathbf{p}_j + \Lambda$ as a single point in the quotient torus, and most of our calculations are done in the Euclidean plane. The only continuous isometries of the quotient torus are translations, so the only trivial infinitesimal flexes of our torus tensegrities are when each \mathbf{p}'_j is the same constant. With this

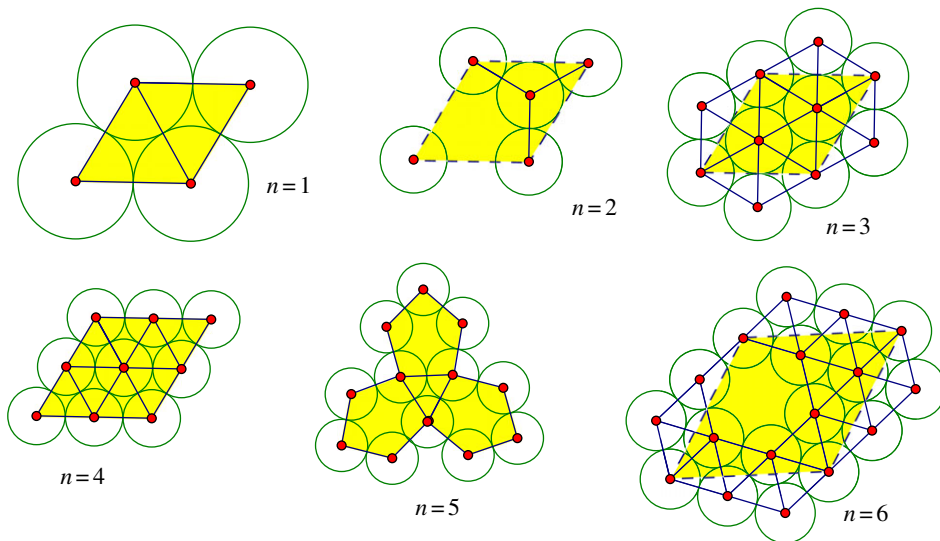


Figure 3. The locally maximally dense and maximally dense packings of equal discs in a triangular torus for $n = 1$ to 6. A fundamental region is shaded in each case. (Online version in colour.)

in mind, we say that a strut tensegrity (or more generally any tensegrity) is *infinitesimally rigid* if the only infinitesimal flexes are trivial. A strut tensegrity, in a torus, is *rigid* if the only continuous motions of the vertices not decreasing the lengths of edges are translations. The following is a useful result that can be found in [11,12].

Theorem 1.5. *Any strut tensegrity in a fixed torus is rigid if and only if it is infinitesimally rigid.*

That infinitesimal rigidity implies rigidity is a general fact. The converse implication is because all the edges are struts.

If the infinitesimal flexes \mathbf{p}' that satisfy (1.6) are equalities, we say that \mathbf{p}' is an infinitesimal flex of the *bar framework*. Again, if the only infinitesimal flexes of the bar framework are trivial, then we say the bar framework is infinitesimally rigid.

Any assignment of a scalar $\omega_{jk} = \omega_{kj}$ to each edge $\{j, k\}$ of the packing graph is called a *stress* (where $\omega_{jk} = 0$ when $\{j, k\}$ is not an edge in the graph) and it is called an *equilibrium stress* or equivalently a *self-stress* if for each vertex \mathbf{p}_j

$$\sum_k \omega_{jk}(\mathbf{p}_k - \mathbf{p}_j) = 0.$$

The following result of Roth & Whiteley [14] is very useful in deciding when a strut tensegrity is infinitesimally rigid, because often solving the equality constraints of the bar conditions is somewhat easier than solving the direct inequality constraints of (1.6).

Theorem 1.6. *A strut tensegrity is infinitesimally rigid if and only if the underlying bar framework is infinitesimally rigid, and there is an equilibrium stress that is negative on all the struts.*

So the lack of an equilibrium stress, all of the same sign, or an infinitesimal motion of the bar framework, is enough to show that the discs can be moved keeping the packing property.

With these results in mind, we can model the local behaviour of a packing of discs with a strut tensegrity on the edges connecting the centres of tangent packing discs. If the underlying graph of a packing is rigid (and therefore infinitesimally rigid) as a strut graph in a fixed torus, then the packing is collectively jammed as defined in §1c.

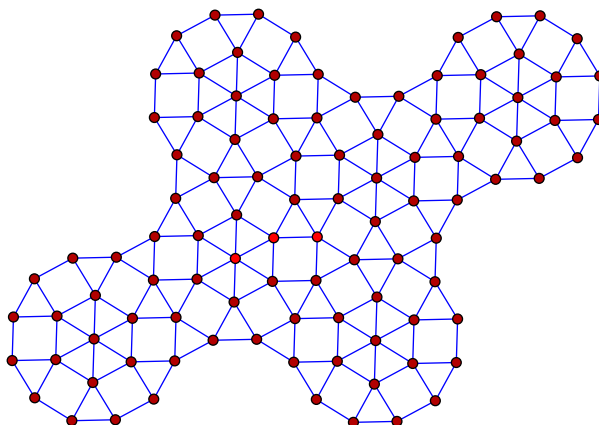


Figure 4. A portion of a square and triangle tiling of the plane. (Online version in colour.)

A consequence of theorems 1.5 and 1.6 is that, if a packing of discs is collectively jammed, then $m \geq 2n - 2 + 1 = 2n - 1$, where n is the number of packing discs, and m is the number of contacts, i.e. the number of edges in the contact graph.

We say that a packing is *locally maximally dense* if there is an epsilon $\epsilon > 0$ such that any ϵ perturbation of the packing to another packing (with possibly different common radius) has the same or less density. Another consequence of the proof of theorem 1.5 is that any locally maximally dense packing must have a subpacking that is rigid and thus infinitesimally rigid. We call such a rigid subset of the packing discs a (*rigid*) *spine* of the original packing. In other words, the spine is collectively jammed. In many cases, the spine consists of the whole packing, or there are isolated packing discs not in the spine. A packing disc not in the spine is called a *rattler*. For example, in [8], there are examples of locally maximally dense circle packings in a square torus, or a nearly square torus, that has rattlers.

The following is our main result, which is a special case of conjecture 1.3. The next sections will provide a proof.

Theorem 1.7. *Suppose that the graph of a collectively jammed packing of n discs in a triangular torus consists of not all triangles. Assume that either all the faces are rhombi, or the rhombi form a single strip, where the rest of the faces are triangles. Then, its density is $\delta < (n/(n+1))\delta_{\Delta} = (n/(n+1))\pi/\sqrt{12}$.*

2. Triangle and square tilings

In this section, we provide information about the rigidity and flexibility properties of triangle and square tilings of the plane. An example of a square and equilateral triangle tiling is shown in figure 4.

Given an equilateral triangle and square tiling of the plane, a circle packing is obtained by centring circles of radius one-half of the edge length at each of the vertices. In fact, any graph corresponding to an edge-to-edge tiling of triangles and rhombi, whose edge lengths are the same, and such that the internal angles are all greater than or equal to $2\pi/6$, will correspond to a circle packing.

Let $g = e^{2\pi i/12}$, a 12th root of unity in the plane \mathbb{C} . Then g is one of the four roots of the cyclotomic polynomial $z^4 - z^2 + 1$, which can be regarded as a defining relation for g , and all 12 powers of g are an integral linear combination of $1, g, g^2, g^3$. So, $\Lambda = \Lambda(1, g, g^2, g^3)$ is a four-dimensional lattice, and any point z in Λ is a unique integral linear combination $n_1 + n_2 g + n_3 g^2 + n_4 g^3 = z$, because the cyclotomic polynomial is irreducible over the rational field. (Alternatively, this follows since 1 and $\sqrt{3}/2$ are independent over the rationals, and they form the real and imaginary components of the four powers of g .)

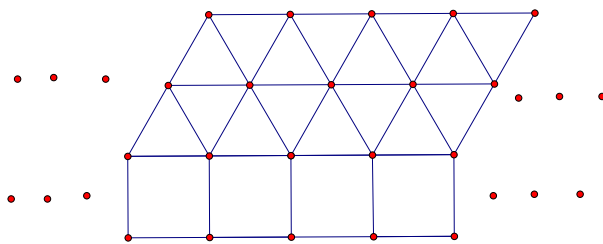


Figure 5. A strip in a square-and-triangle tiling. (Online version in colour.)

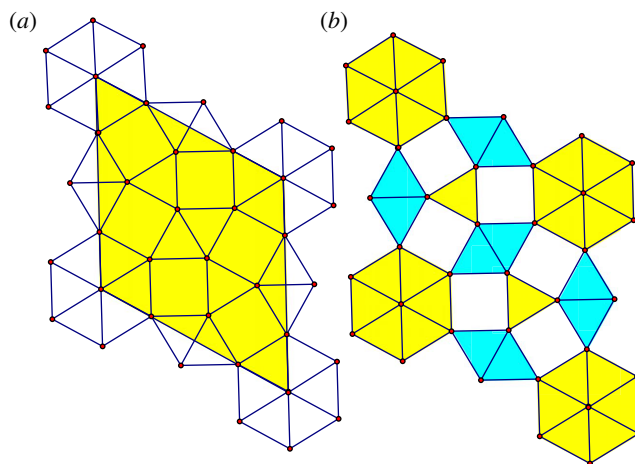


Figure 6. (a) A periodic triangle and square tiling with a period rhombus shown and the rigid groups of triangles shown. (b) The triangles with edges in the same lattice have the same shade. The threefold covering of this tiling is the one given in figure 4. (Online version in colour.)

Position the origin at one of the vertices of the tiling and align one edge of the graph with one of the 12 generators of Λ . One sees that each oriented edge in the tiling is one of the powers of g and each vertex of the tiling is in Λ . Also, g^2 is a sixth root of unity with cyclotomic polynomial $z^2 - z + 1$, so $\Lambda_1 = \Lambda(1, g^2)$ and $\Lambda_2 = \Lambda(g, g^3)$ are both sublattices of Λ . Each directed edge (directed in either direction) of each triangle in the triangle and square tiling belongs to exactly one of Λ_1 or Λ_2 . We say that two polygons in the plane, with disjoint interiors, are *edge-adjacent* or *edge-connected* if they have an edge in common. We say that they are *vertex-adjacent* if they have a vertex in common, but are not edge-adjacent. When two triangles are edge-adjacent, their edges belong to the same sublattice; and when they are vertex-adjacent to each other and edge-adjacent to the same square, their edges are in opposite lattices.

Consider an infinite edge-to-edge tiling of the plane by triangles and squares such that there are no infinite strips of squares or triangles, as in figure 5.

Theorem 2.1. *The bar graph consisting of the edges of a square and equilateral triangle tiling of the plane, with no infinite strips, flexes with one degree of freedom by rotating the edges of the lattice Λ_1 relative to the lattice Λ_2 . If the tiling is periodic with respect to a triangular lattice, it remains periodic with respect to a contracted triangular lattice. If the tiling is periodic with respect to a square lattice, it remains periodic, and the period lattice generators remain perpendicular, but they do not remain the same length.*

Proof. If an edge of the square and triangle tiling is in Λ_1 , then it must be an even power of g ; and if it is in Λ_2 , then it is an odd power of g . This is indicated by the shading in figures 6 and 7. Note that both Λ_1 and Λ_2 are two-dimensional discrete lattices, so they can be rotated by $g_\theta = e^{i\theta}$

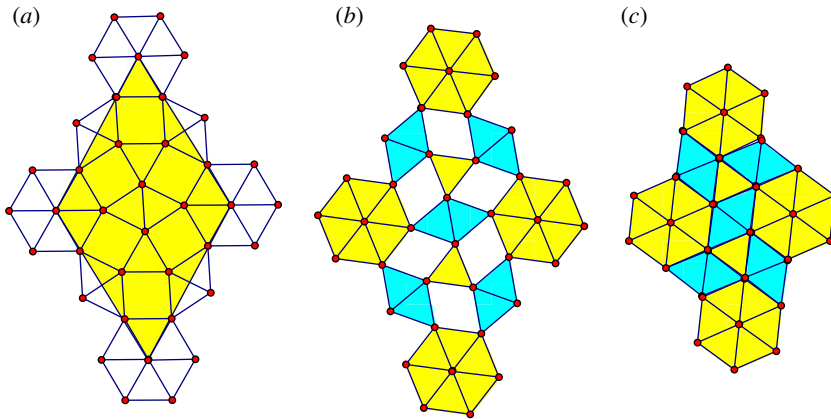


Figure 7. (a) The flexed framework of figure 6 where each square is flexed until it is the union of two triangles. (b) When the extra edge is filled in, the resulting tiling is a triangle lattice tiling. There are 13 vertices in the original tiling. (c) When it is flexed even further to collapse the squares entirely, another tiling of triangles occurs, in this case with seven vertices. (Online version in colour.)

and the same integral linear combination of the generators of Λ can be used for Λ_1 and Λ_2 . If there are no infinite strips, then there must be edges of the tiling that are part of the generators from both Λ_1 and Λ_2 . So, explicitly, if $z = n_1 + n_2 g + n_3 g^2 + n_4 g^3$ is in Λ , then the flexed z is

$$z(\theta) = n_1 + n_2 g g_\theta + n_3 g^2 + n_4 g^3 g_\theta = \lambda_1 + \lambda_2 g_\theta, \quad (2.1)$$

where $\lambda_1 = n_1 + n_3 g^2$ is in Λ_1 , and $\lambda_2 = n_2 g + n_4 g^3$ is in Λ_2 .

The next question is to determine what the lattice of the flexed configuration is. Let $\lambda_1 + \lambda_2$ represent one of the generators of a period parallelogram for a triangular lattice tiling of triangles and squares, where $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Then, $g^2(\lambda_1 + \lambda_2)$, the rotated vector by $2\pi/6$, is another generator. Then, in the flexed configuration, these generators are flexed to $\lambda_1 + \lambda_2 g_\theta$ and $g^2(\lambda_1 + \lambda_2 g_\theta)$, respectively, because $g^2 \lambda_1 \in \Lambda_1$ and $g^2 \lambda_2 \in \Lambda_2$, and clearly the length of these generators is $|\lambda_1 + \lambda_2 g_\theta| = |g^2(\lambda_1 + \lambda_2 g_\theta)|$. But for the same pair of generators, $g^4(\lambda_1 + \lambda_2)$ is the difference vector between them, and by a similar calculation it also has the same length as the other generators. Thus, the lattice is a contracted triangular lattice, as desired.

For the square lattice, there is an important difference in how the generators are flexed. If $\lambda_1 + \lambda_2$ is a generator as before, then $g^3(\lambda_1 + \lambda_2)$ is the other generator of the same length at right angles. But now $g^3 \lambda_1 \in \Lambda_2$ and $g^3 \lambda_2 \in \Lambda_1$, so the flexed generators are $g^3(\lambda_1 g_\theta + \lambda_2)$ and $g^3(\lambda_1 + \lambda_2 g_\theta)$, respectively. But nevertheless, as long as there are no strips, the infinite framework can still be flexed as before, and when $g_\theta = g_{2\pi/6}$ all the squares have been flexed to $\pi/6$ rhombi, and the vertices are part of a triangular lattice. If the period parallelogram remains a square, then it means that the square lattice is a sublattice of the triangular lattice. But that is not possible, because if $\lambda \in \Lambda_1$, then its rotation by $2\pi/4$, which is $g^3 \lambda$, would be in Λ_2 . But, these two lattices are disjoint except at 0. ■

Note that a consequence of theorem 2.1 is that any packing whose graph consists of triangles and squares, with no strips in a triangular torus, is not rigid, i.e. not collectively jammed.

3. Triangle and rhombus tilings

We next consider the case when the quadrilaterals are not necessarily squares, but rhombi. For the case when the tiling is the graph of a packing of equal circles, we assume that each internal angle of each rhombus is strictly between $2\pi/6 = \pi/3$ and $(2/6)2\pi = 2\pi/3$. If the angle is either of those

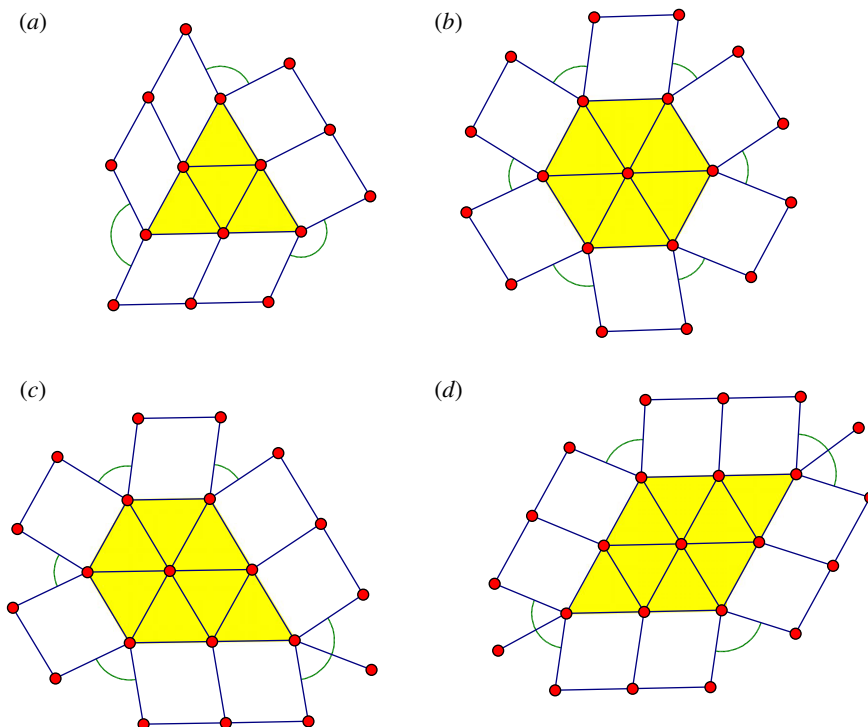


Figure 8. The shaded polygons are examples of components of triangles. The edge-adjacent rhombi are shown as well. The extra edges for cases (c) and (d), when the internal angles are $\pi/6$, are there because of the degree five hypothesis. (Online version in colour.)

extreme angles, then we assume that the rhombus is subdivided into two equilateral triangles. If the graph has that property, then we say that it is *completed*.

Theorem 3.1. *Suppose an (edge-to-edge) tiling of the plane into (equilateral) triangles, rhombi, is such that each vertex that is adjacent to a triangle has degree five or six, and there are no infinite strips. Then, all the rhombi are congruent, and the tiling can be flexed to a square and triangle tiling as in theorem 2.1.*

Proof. The edge-connected triangle components form convex polygons with between three and six vertices, with internal angles other than π , and such that the internal angle at each other vertex is either $2\pi/3$ or $2\pi/6$. There are no non-convex vertices because of the completed property. Figure 8 shows examples of these polygons for all the cases from three to six.

On each edge of each of these polygons, there is a rhombus or adjacent rhombi such that they form a parallelogram edge-adjacent to one edge of the triangle component. The external angles of the triangle component are such that when you count the angles, not coming from the edge-adjacent rhombi, they add to 2π . In cases (b), (c) and (d), there are six such angles, so the average angle is $2\pi/6 = \pi/3$. But, because each angle must not be less than $\pi/3$, they are all $\pi/3$. In the case (a), there are three such angles, the average is $2\pi/3$ and each angle cannot be greater than $2\pi/3$. So, all the angles are $2\pi/3$. Thus, each triangle component is vertex-adjacent to another triangle component at each vertex (not including the vertices whose interior and exterior angles are π). Furthermore, as one proceeds around the boundary of each triangle component, the rhombi are all congruent, with the same internal angles (when they are not squares), appearing in the same order with respect to a clockwise order, say, along the boundary. So, the rhombus components are all parallelograms with those same angles, because there are no infinite components. So, each triangle component has its edge directions in one of two possible sets of directions, where each set consists of three directions $2\pi/3$ apart. Thus, one set can rotate relative to the other. ■

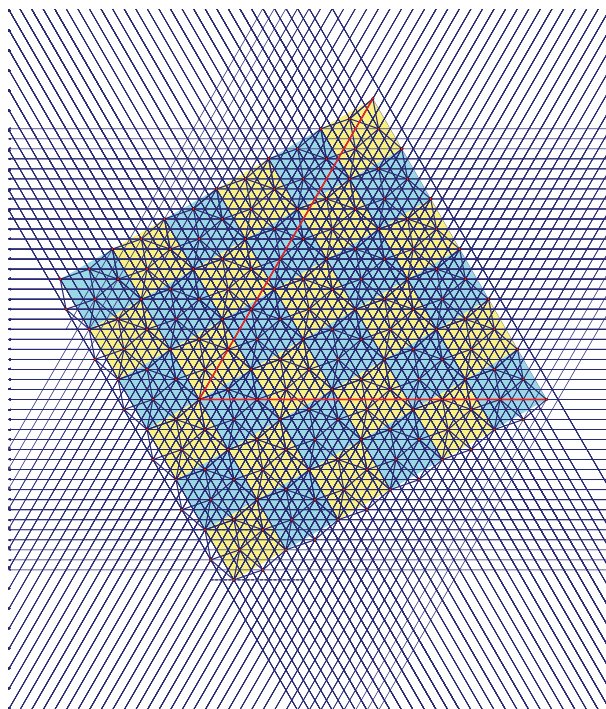


Figure 9. Two solid line segments here represent the generators of the Λ_Δ lattice. (Online version in colour.)

Remark. In a previous draft of this paper, it was claimed that if the original tiling was periodic with respect to a triangular lattice, then the flex of theorem 3.1 would preserve this triangular periodicity. This is not necessarily true. Figure 9 shows an example. The vertices of the packing graph lie on an underlying triangular lattice Λ_ϵ (which must be true for this sort of example) as shown in figure 9. The lattice Λ_ϵ provides coordinates for the packing graph, and it allows one to verify that the configuration exists as shown. The whole graph is a covering graph of a four-vertex graph on another lattice Λ that is not a triangular lattice. This graph is a triangle and rhombus tiling, but the rhombi are not squares, although the rhombus shape is close to a square. The triangular sublattice Λ_Δ of Λ is indicated in figure 9, and again the Λ_ϵ lattice provides coordinates to verify that Λ_Δ is a triangular lattice. But when the packing graph is flexed as in theorem 3.1, it is easy to verify that the Λ_Δ lattice ceases to be triangular. We thank Alex Smith and Jeff Shen for pointing out the flaw in the proof of our previously incorrect statement of theorem 3.1.

4. Rigid rhombus and triangle strip tilings

When all the faces of a packing graph are rhombi, we provide an explicit description of what they look like in §4*a*. Then, in §4*c*, we calculate the maximum density of such packings for a triangular torus.

(a) A description of a collectively jammed rhombus packing

Another case of tiling corresponding to collectively jammed packing graphs in a triangular torus is when all of the faces are rhombi. We first determine the abstract graph of such a packing. For any edge-to-edge tiling of a torus, let v_j be the number of vertices of degree j for $j = 3, 4, \dots$, and

let f_j be the number of faces with j sides, for $j = 3, 4, \dots$. Define the averages

$$\bar{v} = \frac{1}{v} \sum_3^{\infty} jv_j, \quad \bar{f} = \frac{1}{f} \sum_3^{\infty} jf_j,$$

where $v = \sum_3^{\infty} v_j$ is the total number of vertices, and $f = \sum_3^{\infty} f_j$ is the total number of faces. (Note that all the sums are finite since there are only a finite number of vertices and faces. The bar notation, used here for averages, is not to be confused with complex conjugation.) Let e be the total number of edges in the graph. Then, counting the edge-vertex adjacencies and the edge-face adjacencies, we obtain

$$v\bar{v} = 2e = f\bar{f}.$$

Assuming that each face is simply connected, and because the Euler characteristic of a torus is zero, we obtain

$$0 = v - e + f = \frac{2e}{\bar{v}} - e + \frac{2e}{\bar{f}} \quad \text{and} \quad \frac{1}{\bar{v}} + \frac{1}{\bar{f}} = \frac{1}{2}.$$

So, when all the faces are quadrilaterals, $\bar{f} = 4$, and thus $\bar{v} = 4$ as well. But if there are any vertices of degree three and the graph is a packing graph, the three angles at any such vertex must each be $2\pi/3$, implying that there is a triangle in the graph. Thus, $v_3 = 0$, and so all the vertices must be of degree four. Summing up, we have shown the following result.

Proposition 4.1. *Any packing graph of equal circles in a torus, where each face is a rhombus, has each vertex of degree four.*

So far, we have not assumed that the torus is a triangular torus, or even that the packing is jammed in any sense. We next consider when the packing is collectively jammed.

If the graph of a collectively jammed packing consists of only rhombi, then in the lift to the plane, there are infinite strips of rhombi, where two opposite sides of each rhombus are in a fixed direction. The opposite sides of each rhombus are adjacent to the next rhombus in the strip. The other pair of sides is each adjacent to another rhombus in another strip of all different rhombi, where each rhombus in the second strip is edge-adjacent to a rhombus in the first strip. Figure 10 shows an example of such strips.

By theorems 1.5 and 1.6, a necessary condition on the graph of a packing to be collectively jammed is that there is an equilibrium stress on the edges that is strictly negative on each edge. One way of picturing this stress is to draw the *reciprocal graph*, which is another graph whose vertices correspond to the faces of the original graph, whose faces correspond to the vertices of the original graph, and whose edges correspond to the edges of the original graph and are perpendicular to the original edges. This reciprocal graph is indicated in figure 10 with dashed lines. In the case when all the edges of the original graph are the same length, the stress on an edge corresponds to (minus) the length of the edge of the reciprocal. See [11,12] for a discussion of this with regard to packings.

Because the packing is periodic, the stress and the reciprocal extend to the infinite packing in the lift in the plane. But each reciprocal polygon in the lift must be a convex polygon, which in our case is another quadrilateral, and the straight lines in the reciprocal, corresponding to the parallel sides of each adjacent strip, must not cross. So, this means that the lines are parallel. Doing this for all non-crossing strips gives the following result.

Proposition 4.2. *If the graph of any collectively jammed circle packing in a torus consists of only rhombi, then the rhombi are all congruent and translates of a single rhombus.*

Proposition 4.2 supplies the condition for the graph of the packing to have a proper stress that is necessary according to theorem 1.6, but we still need conditions for the graph to be infinitesimally rigid as a bar graph.

We now know that each edge of the rhombus tiling is in one of two directions. As one proceeds from one vertex to the next on the torus in one of those directions, the vertices are visited one after the other until one gets back to the starting vertex. Suppose that p_1, p_2, \dots, p_k is this *tour* of vertices

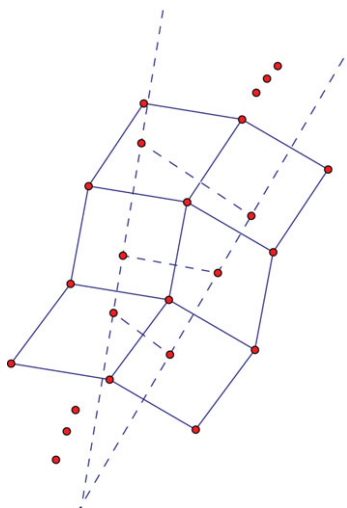


Figure 10. A portion of an infinite rhombus tiling in the plane, where an attempt is started to find a reciprocal graph, indicated in dashed lines, but cannot be continued. (Online version in colour.)

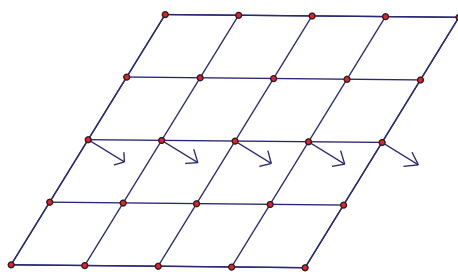


Figure 11. An example of a periodic flex when the tour of centres is not all the vertices of the graph of a rhombus packing. (Online version in colour.)

and it does not include all the vertices on the torus. Then, there is a non-zero infinitesimal flex $p'_1 = p'_2 = \dots = p'_k$ of the bar graph that is zero on all the other vertices not on this tour, where each p'_j is perpendicular to the direction of the other edge of the original rhombus. Figure 11 shows an example of this. It is easy to check that this is the only infinitesimal flex possible. Hence, we obtain the following theorem.

Theorem 4.3. *If the packing graph of a circle packing in a torus consists of only rhombi, the packing is collectively jammed if and only if all the rhombi are congruent with the sides in only two directions and the tour of vertices in each direction consists of all the vertices in the packing.*

Figure 12 shows an example of the case when the packing graph consists of rhombi and each tour in each direction visits each vertex.

(b) Strips of rhombi and triangles

Another case of collectively jammed packings in a triangular torus is when there is a single strip of rhombi, and a corresponding strip of triangles of arbitrary thickness in the graph of the packing. This includes, as a special case, the situation in §4a of all rhombi. We call a strip of rhombi in a tiling of a torus a *strip* if each rhombus is edge-adjacent with exactly two other rhombi and all lines containing the intersections between rhombi are parallel when lifted to the plane. So, these

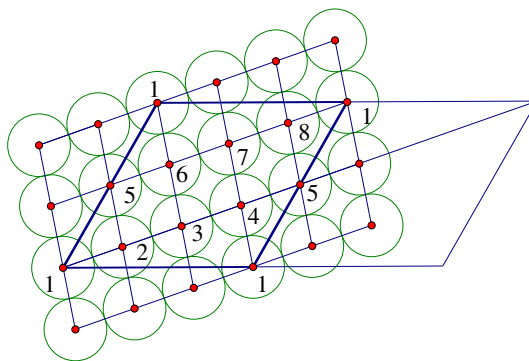


Figure 12. An example of a collectively jammed packing on a triangular torus, where one tour of vertices is indicated. The other tour is 1, 4, 7, 2, 5, 8, 3, 6. (Online version in colour.)

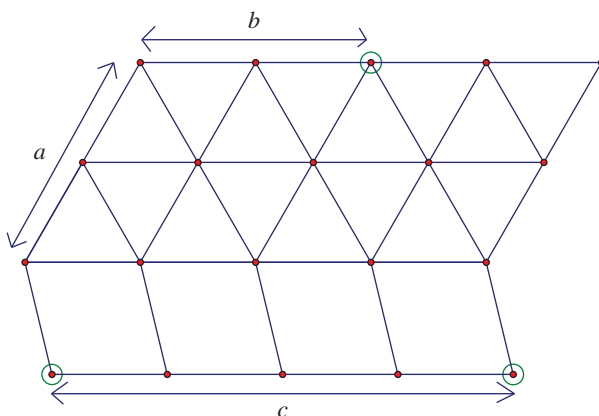


Figure 13. A strip tiling of triangles and rhombi, where c is the length of the strip of rhombi, a is the width of the strip of triangles and b represents the displacement of the identification of the bottom to the top. Three of the lattice points are indicated. (Online version in colour.)

rhombi form a cyclic chain in the torus. In the case here, except for the case when all the faces are rhombi, we assume that there is only one such strip of rhombi, and all the other faces of the graph are (equilateral) triangles. This is indicated in figure 13. In §4a, we assumed that all the faces were rhombi, and there were only two strips of rhombi, each one corresponding to one of the two parallel classes of edges in the lift. If there are two parallel strips of rhombi, it is easy to show that the graph would not be infinitesimally rigid and the packing would not be collectively jammed.

Place one of the lattice vectors, the origin, at the lower left corner of one of the rhombi. Then, as you proceed in the direction of the strip, you must come back to the same point on the torus for the first time, and thus you must hit another lattice point in the lift to the plane. Let a be the number of layers of triangles, and b the number of steps needed before you hit another lattice point on the upper layer, as indicated in figure 13. Thus, the total number of vertices on the torus is $n = c(a + 1)$. The case when $a = 0$ is the situation when all the faces are rhombi as in §4a.

In order to do calculations of density, we orient the packing so that the strips are all aligned along the real axis, the edges all have unit length, the rhombus edge direction, not along the real axis, is given by the unit-length complex number g , where $\text{Im}(g) > 0$, and the internal angle at the origin for the rhombus is between $\pi/3$ and $2\pi/3$. The edge directions of the triangles are g_Δ and g_Δ^2 as before.

(c) Calculating the density of triangle–rhombus strip packings

Theorem 4.4. *The density δ of an equal circle packing whose graph is a triangle–rhombus strip tiling in a triangular torus is always such that $(n/(n+1))\delta_\Delta > \delta$, where n is the number of packing discs.*

Proof. Because the area of the unit-length rhombus is $\text{Im}(g)$, the total area of the torus is $(\text{Im}(g) + a\sqrt{3}/2)c$ and the density of the corresponding packing is

$$\delta = \frac{\pi(1/2)^2 n}{(\text{Im}(g) + a\sqrt{3}/2)c} = \frac{\pi}{4} \frac{c(a+1)}{(\text{Im}(g) + a\sqrt{3}/2)c} = \frac{\pi}{4} \frac{a+1}{\text{Im}(g) + a\sqrt{3}/2}. \quad (4.1)$$

The next task is to use the condition that the packing is in a triangular torus. Let v be a generating vector for a lattice that defines the triangular torus, so that the lattice is $\Lambda(v, g_\Delta v)$. Because $\Lambda(c, g + ag_\Delta + b)$ is the same lattice, we have

$$\begin{aligned} c &= n_1 v + n_2 g_\Delta v = (n_1 + n_2 g_\Delta)v, \\ g + ag_\Delta + b &= n_3 v + n_4 g_\Delta v = (n_3 + n_4 g_\Delta)v, \end{aligned}$$

where n_1, n_2, n_3, n_4 are integers. Then, paying attention to the order of the lattice generators,

$$\det \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} = n_1 n_4 - n_2 n_3 = 1. \quad (4.2)$$

Then, we can solve for v and simplify, to give

$$v = \frac{c}{n_1 + n_2 g_\Delta} = \frac{c(n_1 + n_2 g_\Delta^{-1})}{n_1^2 + n_1 n_2 + n_2^2}.$$

Similarly, we solve for g :

$$\begin{aligned} g &= -ag_\Delta - b + \frac{c(n_1 + n_2 g_\Delta^{-1})(n_3 + n_4 g_\Delta)}{n_1^2 + n_1 n_2 + n_2^2} \\ &= -ag_\Delta - b + \left(\frac{c}{n_1^2 + n_1 n_2 + n_2^2} \right) (n_1 n_3 + n_2 n_4 + n_1 n_4 g_\Delta + n_2 n_3 g_\Delta^{-1}). \end{aligned} \quad (4.3)$$

We are primarily interested in the imaginary part of g , which is

$$\begin{aligned} \text{Im}(g) &= \left[-a + \frac{c}{n_1^2 + n_1 n_2 + n_2^2} (n_1 n_4 - n_2 n_3) \right] \frac{\sqrt{3}}{2} \\ &= \left[-a + \frac{c}{n_1^2 + n_1 n_2 + n_2^2} \right] \frac{\sqrt{3}}{2} \end{aligned}$$

by (4.2). Thus,

$$\text{Im}(g) + a \frac{\sqrt{3}}{2} = \frac{c}{n_1^2 + n_1 n_2 + n_2^2} \frac{\sqrt{3}}{2}.$$

Using (4.1), we calculate the density of the corresponding packing,

$$\begin{aligned} \delta &= \frac{\pi}{4} \frac{a+1}{(c/(n_1^2 + n_1 n_2 + n_2^2))\sqrt{3}/2} = \frac{\pi}{2\sqrt{3}} \frac{(a+1)(n_1^2 + n_1 n_2 + n_2^2)}{c} \\ &= \delta_\Delta \frac{(a+1)(n_1^2 + n_1 n_2 + n_2^2)}{c}. \end{aligned}$$

Because the packings coming from strip tilings are not a triangular lattice packing, $\delta < \delta_\Delta$, and thus

$$c > (a+1)(n_1^2 + n_1 n_2 + n_2^2) \geq n_1^2 + n_1 n_2 + n_2^2 \quad (4.4)$$

and

$$\frac{n_1^2 + n_1n_2 + n_2^2}{c} < 1.$$

Then, we can improve the inequality (4.4) to

$$\begin{aligned} c &> (a+1)(n_1^2 + n_1n_2 + n_2^2) + \frac{n_1^2 + n_1n_2 + n_2^2}{c} \\ c^2 &> c(a+1)(n_1^2 + n_1n_2 + n_2^2) + n_1^2 + n_1n_2 + n_2^2 \\ c^2 &> [c(a+1) + 1](n_1^2 + n_1n_2 + n_2^2). \end{aligned}$$

Then, going back to the total number of packing elements $n = c(a+1)$, we obtain

$$\frac{n}{n+1} = \frac{c(a+1)}{c(a+1)+1} > \frac{(a+1)(n_1^2 + n_1n_2 + n_2^2)}{c}.$$

Thus, we obtain

$$\frac{n}{n+1} \delta_\Delta > \delta_\Delta \frac{(a+1)(n_1^2 + n_1n_2 + n_2^2)}{c} = \delta. \quad \blacksquare$$

Note that it is not used that such triangle–rhombus strip packings are collectively jammed, but they are. It is easy to find a self-stress on the (strut) framework. For the edges that are in the direction of g_Δ , choose a positive stress ω_1 ; for the direction g_Δ^2 , choose ω_2 ; and for the direction g , choose ω_3 such that $\omega_1g_\Delta + \omega_2g_\Delta^2 + \omega_3(-g) = 0$. Then, any constant positive stress on the horizontal members will do.

When $a > 0$, the connected component of triangles is infinitesimally rigid all by itself, because it is connected in the torus. The case when $a = 0$ was essentially handled in §4*a* and equation (4.2).

So, theorem 4.4 verifies the density gap conjecture 1.4 in the case of strips of rhombi and triangles.

In order to believe that these calculations correspond to actual packings on a triangular torus, we present a specific example for all values of $a = 0, 1, \dots$. For any non-negative integer a , let $b = 4a + 5$, $c = (a+1)7 + 1 = 7a + 8$, $n_1 = 2$, $n_2 = 1$, $n_3 = 1$ and $n_4 = 1$. Then

$$\begin{aligned} n_1n_4 - n_2n_3 &= 1, \\ n_1^2 + n_1n_2 + n_2^2 &= 7, \\ v &= \frac{7a+8}{7}(2 + g_\Delta^{-1}), \\ g &= \frac{1}{7} + \frac{8\sqrt{3}}{7}i. \end{aligned}$$

One can verify that equation (4.3) holds for any $a \geq 0$. Thus, this corresponds to collectively jammed strip packings. Figure 12 corresponds to the case when $a = 0$.

5. Relation to previous work

One of the starting points for this work was from an example of a packing provided by Ruggero Gabbrielli, who suggested that the packing in figure 14 was at least collectively jammed. At first, it was difficult to determine if, for this packing, it was true, but it was shown by the first author at a talk at the Fields Institute [15] that it was not collectively jammed. At the next talk by Whiteley [16], it was shown that by looking at parallel classes of edges of a graph one can create a flex, which is a basic idea of theorem 2.1. It was also shown in [17, section 4.9 and figs 21 and 22] how the rotating motion described works, and it was shown that there is such a flex when there are two classes of edges as there is here.

Triangle–square tilings have been of some interest in the physics literature as in [18–21]. Indeed, the idea of associating to a triangle–square tiling a graph, where every vertex is of degree

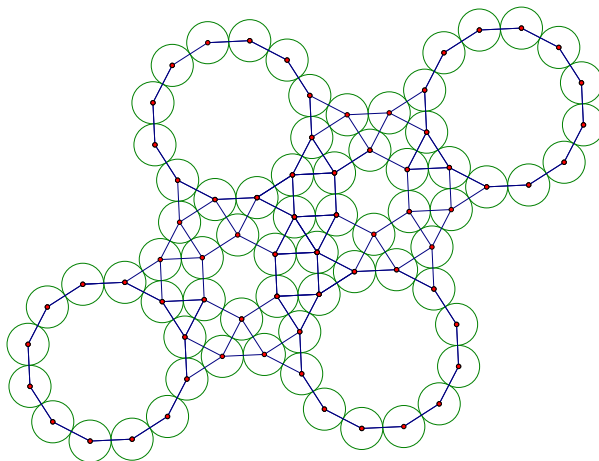


Figure 14. A packing in a triangular torus with a 12-gon face that for a generic configuration of the packing graph is rigid, but it is not rigid as a packing graph since it is a subset of the packing in figure 4. (Online version in colour.)

six, and identifying it with a triangle lattice, is given in [19]. In [19] and [18], the authors estimated the ratio of squares to triangles in a ‘random’ square-and-triangle tiling. Square lattice tilings were quite popular. Suppose that one side of a period square of a square lattice has a polygonal path involving the vectors $1, g^2, g, g^{-1}$, where $g = \sqrt{3}/2 + i/2$ as in §2. So, the sum of the edge vectors that go from one corner to the next horizontal corner of the square will be

$$s = a + b(g + g^{-1}) = a + b\sqrt{3},$$

where a and b are integers, because $1, g^2, g, g^{-1}$ is another basis for the lattice determined by the edge vectors. But the area of the square torus is

$$s^2 = a^2 + 3b^2 + 2ab\sqrt{3} = f_4 + \frac{\sqrt{3}}{4}f_3,$$

where f_3 is the number of triangles and f_4 is the number of squares. So

$$f_3 = 8ab \quad \text{and} \quad f_4 = a^2 + 3b^2.$$

Thus,

$$\frac{f_4}{f_3} = \frac{a^2 + 3b^2}{8ab} > \frac{1}{8} \left(\sqrt{3} + \frac{3}{\sqrt{3}} \right) = \frac{\sqrt{3}}{4},$$

because the minimum of $x + 3/x$ for $x > 0$ is when $x = \sqrt{3}$ as any calculus student knows.

A similar analysis for the triangular torus shows that $\sqrt{3}/4$ is a maximum for the ratio f_4/f_3 .

The ratio $\sqrt{3}/4$ for f_4/f_3 was claimed as the maximum entropy for triangle–square tilings with 12-fold symmetry in [18].

6. Future work

It would be helpful to understand the case when the graph of the packing consists of triangles and rhombi more generally than in theorem 3.1. There is the question of what kind of packing graphs can occur.

More ambitiously, it would be very useful to understand what kind of collectively jammed packing graphs are that have faces of degree larger than four. One idea is to complete the graph of the packing to a triangulation, where the degree of each vertex is six. Then, there are ways

to compare the density of such packings to the density δ_Δ and $(n/(n+1))\delta_\Delta$ using some simple techniques from the theory of the global rigidity of tensegrities.

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