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PRESTRESS STABILITY OF TRIANGULATED CONVEX POLYTOPES AND UNIVERSAL SECOND-ORDER RIGIDITY*

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Abstract. We prove that universal second-order rigidity implies universal prestress stability and that triangulated convex polytopes in 3-space (with holes appropriately positioned) are prestress stable.

Key words. rigidity, prestress stability, universal rigidity

AMS subject classifications. 52C25, 52B10

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1. Introduction.

A classic result of Cauchy [7] implies that the boundary of any convex polytope in 3-space is rigid, when each of its natural two-dimensional faces is held rigid, even though they are allowed to rotate along common edges like hinges. In [19], Dehn proved that a polytope with triangular faces is infinitesimally rigid, and therefore rigid, when the edges are regarded as fixed length bars connected to its vertices. Alexandrov [2] showed that any convex triangulated polytope, where the natural surface may consist of nontriangular faces, is still infinitesimally rigid, as long as the vertices of the triangulation are not in the relative interior of the natural faces. Connelly [8] proved that *any* convex triangulated polytope in 3-space is second-order rigid, no matter where the vertices of the triangulation are positioned, and second-order rigidity implies rigidity in general. The only trouble with this last result is that second-order rigidity is a very weak property. A stronger property, which we will now discuss, is called prestress stability.

When a framework is constructed with physical bars, if it is rigid but not infinitesimally rigid, it is often called "shaky" in the engineering literature [18]. For such a rigid, but not infinitesimally rigid, framework, if each of the bars is at its natural rest length, then the framework might deform significantly under external forces [17]. But in some situations, this shakiness can be rectified by placing some of the bars in either tension or compression. When successful, the resulting structure is at a local minimum of an internal energy functional that can be verified using the "second derivative test." Such structures will not deform greatly under external forces, even though they are infinitesimally flexible. Thus, the stiffness of a physical framework ultimately depends not just on the geometry but also on the physical properties and tensional states of the material.

With this in mind, a (geometric) bar framework is called *prestress stable* if there exists a way to place its bars in tension or compression so that the resulting structure is at a local minimum of an internal energy functional that can be verified using the second derivative test.

In this paper (Theorem 6.3), we show that any arbitrarily triangulated convex polytope in 3-space is in fact prestress stable. Indeed, extending the results of [8], we

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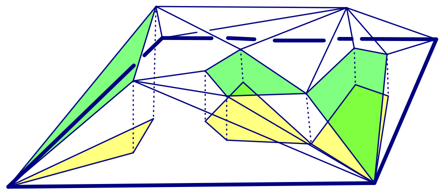


FIG. 1. *The green faces in the convex polytope project onto the yellow holes in the base face. The rest of the bottom face minus the yellow holes can be triangulated at will.*

show that there are many ways of positioning holes in the faces of the polytope so that any triangulation of the remaining surface is prestress stable. Our condition is that for each face F of the polytope, there is another convex polytope P_F projecting orthogonally onto F, with F as the bottom face, such that each hole of F is the projection of an upper face of P_F, as in Figure 1. Furthermore, for any triangulation of P, minus the holes, we assume that the boundary of each face F is infinitesimally rigid in the plane of F.

As part of obtaining this result, we first show (Corollary 4.8) that, in general, if any bar framework is universally second-order rigid in the sense that it remains second-order rigid when thought of as a (degenerate) framework in all higher dimensions, then it must be prestress stable in its original and all higher dimensions. This result is also related to the problem of characterizing when a semidefinite feasibility problem has a singularity degree of exactly one [21].

Some of these ideas were briefly sketched in the survey [10] but have not previously been given a complete and formal treatment.

2. Definitions and background. Let $(G, \mathbf{p})$ denote a (bar and joint) framework where $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_n})$ is a configuration of n points $\mathbf{p}_i \in \mathbb{R}^d$, and G is a corresponding graph, with n vertices and with e edges connecting some pairs of points of $\mathbf{p}$.

2.1. Local rigidity. Here we define a sequence of local rigidity properties.

We start with the most basic idea defining a rigid structure. We say that a framework $(G, \mathbf{p})$ is *locally rigid* in $\mathbb{R}^d$ if there are no continuous motions in $\mathbb{R}^d$ of the configuration $\mathbf{p}(t)$, for $t \geq 0$, that preserve the distance constraints

$$|\mathbf{p}_i(t) - \mathbf{p}_j(t)| = |\mathbf{p}_i - \mathbf{p}_j|$$

for all edges, $\{i, j\}$, of G, where $\mathbf{p}(0) = \mathbf{p}$, unless $\mathbf{p}(t)$ is congruent to $\mathbf{p}\ \forall\, t$. By a congruence, we mean that $\mathbf{p}(t)$ can be obtained from $\mathbf{p}$ by simply restricting a Euclidean isometry of $\mathbb{R}^d$ to the vertices.

The simplest way to confirm that a framework is locally rigid is to look at the linearization of the problem.

A *first-order flex* or *infinitesimal flex* of $(G, \mathbf{p})$ in $\mathbb{R}^d$ is a corresponding assignment of vectors $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$, $\mathbf{p}'_i \in \mathbb{R}^d$, such that for each $\{i, j\}$, an edge of G, the following holds:

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0. \tag{1}$$

The *rigidity matrix* $R(\mathbf{p})$ is the e-by-nd matrix, where

$$R(\mathbf{p})\mathbf{p}' = (\dots, (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j), \dots)^t,$$

for $\mathbf{p}' \in \mathbb{R}^{nd}$. Here each row of the matrix is indexed by an edge $\{i, j\}$ of the graph, and $()^t$ is the transpose. We write $R(\mathbf{p}, \mathbf{q}) = R(\mathbf{q}, \mathbf{p}) = R(\mathbf{p})\mathbf{q}$ for any $\mathbf{q} \in \mathbb{R}^{nd}$, which we call the *rigidity form* for the graph G in $\mathbb{R}^d$. With this, (1) can be rewritten as

$$R(\mathbf{p}, \mathbf{p}') = 0.$$

A first-order flex $\mathbf{p}'$ is *trivial* if it is the restriction to the vertices of the time-zero derivative of a smooth motion of isometries of $\mathbb{R}^d$. This is equivalent to there being a d-by-d skew-symmetric matrix A and a vector $\mathbf{b} \in \mathbb{R}^d$ such that

$$\mathbf{p}'_i = A\mathbf{p}_i + \mathbf{b} \tag{2}$$

$\forall\ i = 1, \dots, n$.

Note that the property of being a trivial infinitesimal flex is independent of the graph G.

A framework $(G, \mathbf{p})$ is called *infinitesimally rigid* in $\mathbb{R}^d$ if it has no infinitesimal flexes in $\mathbb{R}^d$ except for trivial ones.

A classical theorem states as follows.

THEOREM 2.1. *If a framework $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^d$, then it is locally rigid in $\mathbb{R}^d$.*

The converse of the theorem is false, so there is room for weaker conditions that can be used to certify local rigidity. One such notion is called prestress stability. The rough idea is to look for an energy function on configurations for which $(G, \mathbf{p})$ is a local minimum.

To this end we define an *equilibrium stress* for a framework $(G, \mathbf{p})$ to be an assignment of a scalar $\omega_{ij} = \omega_{ji}$ to each edge $\{i, j\}$ of the graph G, such that for each vertex i of G,

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0,$$

where the nonedges of the stress $\omega = (\dots, \omega_{ij}, \dots)$ have zero stress.

In this paper, we will use the following proposition (see, e.g., [11, Lemma 2.5]).

PROPOSITION 2.2. *Any equilibrium stress $\omega \in \mathbb{R}^e$ for $(G, \mathbf{p})$ must be in the cokernel of $R(\mathbf{p})$.*

We say that a framework $(G, \mathbf{p})$ is *prestress stable* in $\mathbb{R}^d$ if there is an equilibrium stress ω for $(G, \mathbf{p})$ such that for every nontrivial first-order flex $\mathbf{p}'$ in $\mathbb{R}^d$ of $(G, \mathbf{p})$, we have $\sum_{i<j} \omega_{ij}(\mathbf{p}'_i - \mathbf{p}'_j)^2 > 0$. (When this inequality holds, we say that the stress ω *blocks* the first-order flex $\mathbf{p}'$.) From this definition it is clear that if a framework $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^d$, then, using the all-zero stress, it is automatically prestress stable in $\mathbb{R}^d$.

The following is shown in [17].

THEOREM 2.3. *If a framework $(G, \mathbf{p})$ is prestress stable in $\mathbb{R}^d$, then it is locally rigid in $\mathbb{R}^d$.*

It is also shown in [17] that this definition of prestress stability coincides with the motivating property, described in the introduction. This means that there is an energy for which $(G, \mathbf{p})$ is a minimum that can be verified with the second derivative test. In this correspondence, the coefficients, ω_{ij}, correspond to the first derivative of the energy of the associated bar with respect to changes in its squared length.

The converse of Theorem 2.3 is again false, so there is room for even weaker conditions that can be used to certify local rigidity. One such notion, used in [8] to study the rigidity of triangulated convex polytopes, is second-order rigidity. The idea is motivated by looking at the first two derivatives of some proposed continuous flex of the framework.

A *second-order flex* of $(G, \mathbf{p})$ in $\mathbb{R}^d$ is a corresponding assignment of vectors $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$, $\mathbf{p}'_i \in \mathbb{R}^d$, and $(\mathbf{p}''_1, \dots, \mathbf{p}''_n)$, $\mathbf{p}''_i \in \mathbb{R}^d$, such that for each $\{i, j\}$ an edge of G the following hold:

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0, \tag{3}$$

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) + (\mathbf{p}'_i - \mathbf{p}'_j)^2 = 0. \tag{4}$$

Using the rigidity matrix defined above, (3) and (4) can be rewritten as

$$R(\mathbf{p}, \mathbf{p}') = 0, \tag{5}$$

$$R(\mathbf{p}, \mathbf{p}'') + R(\mathbf{p}', \mathbf{p}') = 0. \tag{6}$$

We say that $(G, \mathbf{p})$ is *second-order rigid* in $\mathbb{R}^d$ if there is no second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $(G, \mathbf{p})$ in $\mathbb{R}^d$ with $\mathbf{p}'$ nontrivial as a first-order flex.

The following is proven in [8].

THEOREM 2.4. *If a framework $(G, \mathbf{p})$ is second-order rigid in $\mathbb{R}^d$, then it is locally rigid in $\mathbb{R}^d$.*

Second-order rigidity is a natural property, but it has some practical difficulties, which can be seen with a dual formulation [17].

THEOREM 2.5. *A framework $(G, \mathbf{p})$ is second-order rigid in $\mathbb{R}^d$ iff for every nontrivial first-order flex $\mathbf{p}'$ in $\mathbb{R}^d$ of $(G, \mathbf{p})$, there is an equilibrium stress ω such that $\sum_{i<j} \omega_{ij} (\mathbf{p}'_i - \mathbf{p}'_j)^2 > 0$.*

From this it is clear that if a framework $(G, \mathbf{p})$ is prestress stable in $\mathbb{R}^d$, then it is second-order rigid in $\mathbb{R}^d$. But in second-order rigidity, it can happen that no one stress *blocks* (has positive energy on) all nontrivial first-order flexes. Rather one can think of there being a "demon" living in the framework that senses any particular nontrivial first-order flex $\mathbf{p}'$ and blocks it.

Putting these together, we can summarize the state of affairs as in [17].

THEOREM 2.6. *Infinitesimally rigid in $\mathbb{R}^d$ implies prestress stability in $\mathbb{R}^d$ which implies second-order rigidity in $\mathbb{R}^d$ which implies locally rigidity in $\mathbb{R}^d$. None of these implications is reversible. (See Figure 2.)*

Remark 2.7. There are significant difficulties in attempting to define a meaningful notion of third-order rigidity [16].

One of the two main results of this paper (Theorem 6.3) is that a triangulated convex polytope in $\mathbb{R}^3$ not only is second-order rigid in $\mathbb{R}^3$ (a result of [8]) but is in fact prestress stable in $\mathbb{R}^3$.

Remark 2.8. A framework $(G, \mathbf{p})$ is called *globally rigid in* $\mathbb{R}^d$ if there are no other (even distant) frameworks $(G, \mathbf{q})$ in $\mathbb{R}^d$ having the same edge lengths as $(G, \mathbf{p})$, other than congruent frameworks. This is a much stronger property than local rigidity, but we will note in the next section that the global/local distinction vanishes in unconstrained dimensions.

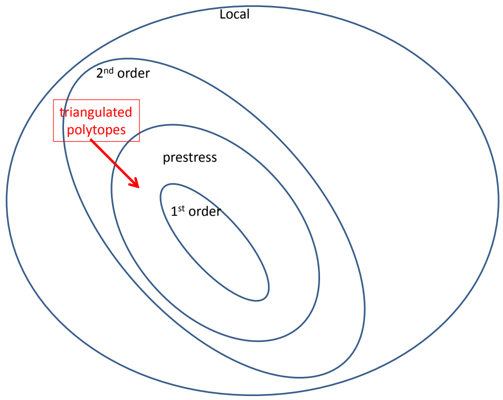


FIG. 2. *Nested space of d-dimensional local rigidity properties of frameworks. This paper shows that triangulated polytopes not only are second-order rigid in* $\mathbb{R}^3$ *but are also prestress stable in* $\mathbb{R}^3$.

2.2. Universal rigidity. In order to study the prestress stability in $\mathbb{R}^3$ of triangulated convex polytopes, and indeed for its own sake as well, we look at what happens when we regard our d-dimensional framework as realized (degenerately) in $\mathbb{R}^D \supset \mathbb{R}^d$, where $D > d$ and D is arbitrarily large. It is easy to see that we do not need to take D to be any larger than $n-1$, where n is the number of the vertices of G, but we use the symbol D instead of n to emphasize that this denotes a number of spatial dimensions. This then brings up the notion of universal rigidity.

Given a framework $(G, \mathbf{p})$ with a d-dimensional affine span, but realized in $\mathbb{R}^D$, we define the notions of *universal local rigidity*, *universal second-order rigidity*, and *universal prestress stability* as, respectively, local rigidity, second-order rigidity, and prestress stability in $\mathbb{R}^D$. (If $d < n-1$, then a framework with a d-dimensional affine span can never be infinitesimally rigid in $\mathbb{R}^D$, and so there is no need to define universal infinitesimal rigidity).

These three properties naturally inherit the inclusion relations of the previous section: universal prestress stability implies universal second-order rigidity, which implies universal local rigidity. However it is not clear, a priori, if any of these implications reverse. In this paper we will conclude that in fact universal prestress stability is no different than universal second-order rigidity (but that these are stronger properties than universal local rigidity).

Remark 2.9. It turns out that there is no need to define a separate notion of universal global rigidity, as it is immediately implied by universal local rigidity [9, 13]. Thus we henceforth drop the "local" qualifier from the term universal rigidity. This makes universal rigidity a very strong property indeed that, for example, implies global rigidity in $\mathbb{R}^d$.

The property of universal prestress stability is actually the same as another property called *super stability*, which we now describe.

Given an equilibrium stress $\omega = (\ldots, \omega_{ij}, \ldots)$ for a framework $(G, \mathbf{p})$ in $\mathbb{R}^D$, we define the *stress energy* as

$$E_\omega(\mathbf{q}) = \sum_{i<j} \omega_{ij} (\mathbf{q}_i - \mathbf{q}_j)^2,$$

which is a quadratic form on the configuration space $\mathbb{R}^{nD}$. The matrix of E_ω with respect to the standard basis of $\mathbb{R}^{nD}$ is $I^D \otimes \Omega$. The matrix Ω, called the *equilibrium stress matrix* corresponding to the equilibrium stress ω, is defined as the symmetric $n \times n$ matrix whose i,j entry is $-\omega_{ij}$, when $i \neq j$, and is such that all row and column sums are zero. The energy E_ω is positive semidefinite (PSD) over $\mathbb{R}^{nD}$ iff Ω is a PSD matrix.

In this paper, we will use the following proposition.

PROPOSITION 2.10 (see [11, Prop. 1.2] and [9, Cor 1]). *Suppose that the affine span of $(G, \mathbf{p})$ is d-dimensional. The kernel of any equilibrium stress matrix Ω for $(G, \mathbf{p})$ must be of dimension at least $d + 1$, and thus the rank of Ω can be at most $n - d - 1$.*

When the rank of Ω is $n-d-1$, if Ω is also an equilibrium stress matrix for another framework $(G, \mathbf{q})$, then the configuration $\mathbf{q}$ is an affine image of the configuration $\mathbf{p}$.

When the rank of Ω is $n - d - 1$ and Ω is PSD, if $\mathbf{q}$ is another configuration with zero energy under E_ω, then the configuration $\mathbf{q}$ is an affine image of the configuration $\mathbf{p}$.

Let L be an affine subspace of some Euclidean space. We say that a set of lines $\{L_1, \ldots, L_m\} \subset L$ lie on a *conic at infinity* for L if, regarding the line directions in L as points at infinity in a corresponding real projective space, they all lie on a (nontrivial) conic in that space.

Concretely, suppose we have a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ with a d-dimensional span. Then $(G, \mathbf{p})$ has its edge directions on a conic at infinity for $\langle \mathbf{p} \rangle$, the affine span of $\mathbf{p}$, iff there exists a nonzero symmetric d-by-d matrix, Q, such that for all edges $\{i, j\}$, we have $(\mathbf{p}_i - \mathbf{p}_j)^t Q (\mathbf{p}_i - \mathbf{p}_j) = 0$. From Lemma A.1 the nonzero property for Q is equivalent to the existence of some nonedge pair $\{k, l\}$, where $(\mathbf{p}_k - \mathbf{p}_l)^t Q (\mathbf{p}_k - \mathbf{p}_l) \neq 0$.

So, for a framework $(G, \mathbf{p})$ in $\mathbb{R}^D$ with a d-dimensional span, $(G, \mathbf{p})$ has its edge directions on a conic at infinity for $\langle \mathbf{p} \rangle$ iff there exists a symmetric D-by-D matrix, Q, such that for all edges $\{i, j\}$, we have $(\mathbf{p}_i - \mathbf{p}_j)^t Q (\mathbf{p}_i - \mathbf{p}_j) = 0$, but for some nonedge pair $\{k, l\}$, we have $(\mathbf{p}_k - \mathbf{p}_l)^t Q (\mathbf{p}_k - \mathbf{p}_l) \neq 0$.

Conics at infinity are important due to the following proposition.

PROPOSITION 2.11 (see [11, Prop. 4.2]). *Let $(G, \mathbf{p})$ be a framework in $\mathbb{R}^D$. There exists a noncongruent framework $(G, \mathbf{q})$ with the same edge lengths as $(G, \mathbf{p})$ and where $\mathbf{q}$ is an affine image of $\mathbf{p}$ iff the edge directions of $(G, \mathbf{p})$ lie on a conic at infinity for $\langle \mathbf{p} \rangle$.*

Following [9] we say a framework $(G, \mathbf{p})$ is *super stable* if there is an equilibrium stress ω for $(G, \mathbf{p})$ such that its associated stress matrix Ω is PSD, the rank of Ω is $n - d - 1$, where d is the dimension of the affine span $\langle \mathbf{p} \rangle$ of $\mathbf{p}$, and the edge directions do not lie on a conic at infinity of $\langle \mathbf{p} \rangle$.

It turns out that super stability is equivalent to prestress stability in any fixed dimension d' that is greater than dimension of the affine span of $\mathbf{p}$.

THEOREM 2.12. *Let $(G, \mathbf{p})$ be a framework with a d-dimensional affine span in $\mathbb{R}^{d'}$ with $d' \geq d + 1$. Then $(G, \mathbf{p})$ is super stable iff it is prestress stable in $\mathbb{R}^{d'}$.*

In particular, this means that universal prestress stability is the same as super stability. The proof of this theorem mainly involves unwinding the various definitions with some linear algebra, and we delay it to Appendix A. Note that, unlike the case of prestress stability, second-order rigidity in $\mathbb{R}^{d+1}$ does not imply super stability [5].

In this paper, our first main result (Corollary 4.8) will be that if $(G, \mathbf{p})$ is universally second-order rigid, then it is super stable.

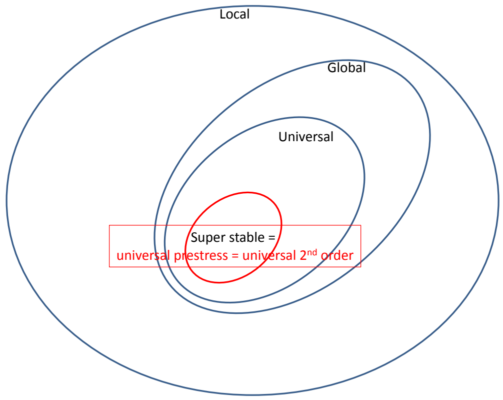


FIG. 3. *Nested space of local, global, and various universal rigidity properties of frameworks. This paper shows that universal second-order rigidity is the same as universal prestress stability. The relationship between the properties shown here to those of Figure 2 are not obvious, other than the fact that universal prestress stability implies prestress stability in* $\mathbb{R}^d$.

Since it is known that there are frameworks that are universally rigid but not super stable [13], this completely describes the relationship between these properties in the universal setting. See Figure 3.

3. Farkas.

Our central argument will rely on a basic Farkas-like duality principle for closed convex cones.

DEFINITION 3.1. *Let Y be a closed convex cone in $\mathbb{R}^m$ for some m. Its dual cone Y^* is defined as $\{\omega \in \mathbb{R}^m \mid \langle \omega, \mathbf{y}\rangle \geq 0 \ \ \forall\, \mathbf{y} \in Y\}$, where $<,>$ is the usual inner product. Note that* $\mathrm{int}(Y^*)$ *consists of the ω such that $\langle \omega, \mathbf{y}\rangle > 0$ for all nonzero $\mathbf{y}$ in Y.*

LEMMA 3.2. *Let Y be a closed convex cone in a finite dimensional real space. Let L be a linear space with $Y \cap L = 0$. Then there is an element $\omega \in L^\perp$ such that $\omega \in \mathrm{int}(Y^*)$.*

Proof. We prove the contrapositive: Suppose there is no such interior element ω. Then $L^\perp$ is disjoint from the interior of Y^*. Thus there is a "supporting" hyperplane Z^* for Y^* such that $Z^* \supset L^\perp$.

To see this we note that $\mathrm{int}(Y^*)$ is convex and is disjoint from $L^\perp$. Thus we can use the weak separation theorem to find a hyperplane Z^* (through the origin), that weakly separates $L^\perp$ from $\mathrm{int}(Y^*)$ and thus from Y^*. This Z^* is a supporting hyperplane for Y^*. Meanwhile, since $L^\perp$ is linear, it must be contained in Z^*.

We can associate with the supporting hyperplane $Z^* =: \mathbf{y}^\perp$ a nonzero dual linear functional (i.e., a primal vector), $\mathbf{y}$, such that $\langle Y^*, \mathbf{y}\rangle \geq 0$. Thus by definition of a dual cone, $\mathbf{y} \in (Y^*)^* = cl(Y) = Y$. Also we must have $\mathbf{y} \in L$. Thus $Y \cap L \neq 0$. ∎

The above proof is based on [23]; we include it here for completeness.

4. Universal second-order rigidity.

We are interested in universal second-order rigidity, but it will be helpful to look at the case where some subset of the vertices are pinned to first order.

DEFINITION 4.1. *Given framework $(G, \mathbf{p})$ in $\mathbb{R}^D$. Let G_0 be a subset of the vertices of G. We regard $(G, G_0, \mathbf{p})$ as the framework, where the vertices in G_0 are pinned "to first order" as described next.*

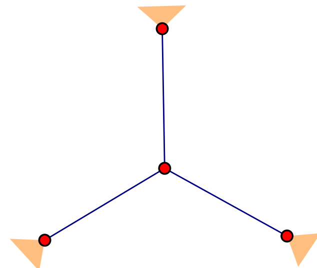


FIG. 4. *This framework is not universally second-order rigid when the outer three vertices are only pinned to first order.*

We say that $(G, G_0, \mathbf{p})$ is pinned universally second-order rigid if there is no second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $(G, \mathbf{p})$ in $\mathbb{R}^D$, with $\mathbf{p}'$ nontrivial as a first-order flex in $\mathbb{R}^D$ and with $\mathbf{p}'_j = 0$ when $\mathbf{p}_j$ corresponds to a vertex in G_0. Note that there is no pinning constraint imposed on $\mathbf{p}''$.

We should be careful to realize that we assume the "pinned" vertices are only pinned to the first order. For example, Figure 4 is not pinned universally second-order rigid, when the indicated vertices are pinned only to the first order. In this example, there is a $\mathbf{p}'$ moving the central vertex orthogonal to the plane in 3-space, and there is a corresponding $\mathbf{p}''$ of the pinned vertices pointing toward the central vertex. When bars are inserted between the first-order pinned vertices, then the framework becomes pinned universally second-order rigid. These inserted bars can even be subdivided as long as the subdividing vertices are also pinned to first order.

Although this in not, on its own, the most natural concept, pinned universal second-order rigidity will be exactly what we need, using duality, to establish the existence of an equilibrium stress matrix that is positive definite when acting on an appropriate subspace. When this construction is applied to frameworks with appropriately chosen pins, we will be able to reason about prestress stability.

DEFINITION 4.2. *Given a graph G, a dimension d, and a chosen subset of "pinned" vertices G_0 we define*

$$\mathcal{C}'(d, G_0) := \{\mathbf{p}' \in \mathbb{R}^{nd} \mid \mathbf{p}'_j = 0 \ \forall \ \mathbf{p}_j \ \textit{corresponding to vertices in } G_0\}.$$

DEFINITION 4.3. *Given a framework $(G, \mathbf{p})$, let*

$$Y := \{R(\mathbf{p}', \mathbf{p}') \mid \mathbf{p}' \in \mathcal{C}'(D, G_0)\} \subset \mathbb{R}^e,$$

where $R(\cdot, \cdot)$ is the rigidity form for the graph G, and D is sufficiently large.

Let L be the linear space defined as the linear span of the columns of $R(\mathbf{p})$, the rigidity matrix.

LEMMA 4.4. *Y is a closed convex cone.*

Proof. Let Y^+ be the set $\{R(\mathbf{p}', \mathbf{p}') \mid \mathbf{p}' \in \mathbb{R}^{nD}\}$. This set is isomorphic to the projection of the convex cone of Euclidean distance matrices on to the coordinates corresponding to the edges of the graph. From [21, Thm. 3.2], the set Y^+ is a closed convex cone. Our set Y is obtained by intersecting Y^+ with the linear subspace where the edge lengths between the vertices in G_0 are all 0. This too must be a closed convex cone. ∎

LEMMA 4.5. *Suppose that the vertices of G_0 in $\mathbf{p}$ have an affine span that agrees with the affine span of $\mathbf{p}$. Let $(G, G_0, \mathbf{p})$ be pinned universally second-order rigid. Then there is no nontrivial intersection of L and Y.*

Proof. Suppose that $\mathbf{y}$ was some nontrivial intersection point. Since $\mathbf{y} \in L$, then we have $\mathbf{y} = -R(\mathbf{p}, \mathbf{p}'')$ for some $\mathbf{p}''$. Since $\mathbf{y} \in Y$, then we have $\mathbf{y} = R(\mathbf{p}', \mathbf{p}')$ for some nonzero $\mathbf{p}' \in \mathcal{C}'(D, G_0)$. Since D is a sufficiently large dimension, we can, without loss of generality, assume that $\mathbf{p}'$ is orthogonal to the d-dimensional affine span of $\mathbf{p}$. Thus $\mathbf{p}'$ is a first-order flex for $(G, \mathbf{p})$. Since $\mathbf{p}'$ is nonzero, but is zero on a set of vertices with a full affine span, then this flex is nontrivial in $\mathbb{R}^D$. Thus $(\mathbf{p}', \mathbf{p}'')$ is a nontrivial second-order flex. This contradicts the assumed pinned universal second-order rigidity. ∎

LEMMA 4.6. *Any $\omega \in L^\perp$ is an equilibrium stress vector for $(G, \mathbf{p})$. Any $\omega \in \mathrm{int}(Y^*)$ must correspond to an Ω that is positive definite on $\mathcal{C}'(1, G_0) \subset \mathbb{R}^n$*

Proof. The vector ω must be in the cokernel of $R(\mathbf{p})$ and thus must be an equilibrium stress vector for $(G, \mathbf{p})$ (Proposition 2.2). Any $\omega \in \mathrm{int}(Y^*)$ has the property that for any nonzero $\mathbf{p}' \in \mathcal{C}'(D, G_0)$ we have

$$\sum_{i<j} \omega_{ij} (\mathbf{p}'_i - \mathbf{p}'_j)^2 > 0.$$ ∎

THEOREM 4.7. *Let $(G, G_0, \mathbf{p})$ be pinned universally second-order rigid. Then it must have an equilibrium stress Ω that is positive definite on $\mathcal{C}'(1, G_0)$.*

Proof. This follows immediately using Lemmas 4.5, 4.4, 3.2, and 4.6. ∎

We can now obtain our first main result about (unpinned) universal second-order rigidity.

COROLLARY 4.8. *If $(G, \mathbf{p})$ is universally second-order rigid and has a d-dimensional affine span, then it must have an equilibrium stress matrix Ω that is PSD (acting on $\mathbb{R}^n$) and of rank $n - d - 1$. Thus it must be super stable/universally prestress stable.*

Proof. Pick any subset of $d + 1$ vertices with a full d-dimensional affine span in the configuration $\mathbf{p}$ to be the subset G_0. If $(G, \mathbf{p})$ is universally second-order rigid, then $(G, G_0, \mathbf{p})$ must be pinned universally second-order rigid. From Theorem 4.7, it must have an equilibrium stress Ω that is positive definite on $\mathcal{C}'(1, G_0)$. Since G_0 is of size $d + 1$, then Ω must have rank at least $n - d - 1$, and as this is maximal (Proposition 2.10), it must have rank equal to $n - d - 1$ and also must be PSD.

Since it is universally second-order rigid it is universally (globally) rigid. Thus it cannot have any noncongruent frameworks with the same edge lengths, let alone one that arises through an affine transform. So from Proposition 2.11 it cannot have its edge directions on a conic at infinity. Thus it is super stable and, from Theorem 2.12, also universally prestress stable. ∎

4.1. Generalizations. There are a few directions for generalizing Corollary 4.8.

A framework $(G, \mathbf{p})$ with a d-dimensional affine span is called *dimensionally rigid* if there is no other framework that has the same edge lengths and has an affine span of dimension greater than d [3]. If a framework is dimensionally but not universally rigid, then it is not locally rigid in $\mathbb{R}^D$, but all other equivalent frameworks can be obtained from $\mathbf{p}$, through the restrictions of an affine transform acting on $\mathbb{R}^D$ [4]. (The edge directions of $(G, \mathbf{p})$ lie on a conic at infinity.) Like universal rigidity, dimensional rigidity can often be certified by a single PSD equilibrium stress matrix Ω of rank $n-d-1$ but such a single-matrix certification does not always exist [13].

It is easy to see that Corollary 4.8 also provides a characterization when such a single certification of dimensional rigidity exists. Suppose that in all dimensions, there are no second-order flexes other than those where $\mathbf{p}'$ arises from an affine transform of $\mathbf{p}$. Then following the proof of the corollary, we pick any subset of $d+1$ vertices with a full d-dimensional affine span in the configuration $\mathbf{p}$ to be a subset G_0. Then $(G, G_0, \mathbf{p})$ will be pinned universally second-order rigid, and duality will give us our desired Ω.

We can also consider a general PSD feasibility problem [21]

$$F := \{X \in \mathcal{S}^n_+ : \mathcal{M}(X) = \mathbf{b}\},$$

where $\mathcal{S}^n_+$ is the cone of PSD n-by-n matrices and $\mathcal{M}$ is a linear mapping from $\mathcal{S}^n$ to $\mathbb{R}^e$, for some e, and $\mathbf{b} \in \mathbb{R}^e$. Let r be the highest rank among the solution set F, obtained, say, by some solution X_0. If $r = n$, this problem has a positive definite feasible solution, and we say that the problem has a singularity degree of zero.

Suppose that $r < n$; then (due to a Farkas duality) it must have a (dual) PSD matrix Ω such that $\langle \Omega, X\rangle = 0 \ \forall \ \{X : \mathcal{M}(X) = \mathbf{b}\}$. If we can find such an Ω of rank $n-r$, then we say that the problem has a singularity degree of one. The pair (X_0, Ω) forms a certificate that the maximal rank for this problem is indeed r.

Otherwise, the singularity degree is higher and can be found by appropriately iterating the above process, called facial reduction [6, 13]. Higher singularity degrees can result in numerical instability.

Corollary 4.8 gives a characterization for singularity degree one, when $\mathcal{M}$ corresponds to computing the squared edge length measurement over the edges of a graph G, given the Gram matrix of a configuration $\mathbf{p}$. (Here r would correspond with $d+1$.) Can we generalize this to get a characterization for a general semidefinite programming feasibility problem to have singularity degree one?

Indeed, most of the ideas generalize very naturally. Let us begin with the appropriate generalization to (6). Suppose we can factor $X_0 = P^tP$, where P is a rank r n-by-n matrix.

The analogue of the linear space $R(\mathbf{p}, \mathbf{p}'')$, where $\mathbf{p}''$ is allowed to be any configuration, is then $\mathcal{M}(P^tP'' + (P'')^tP)$, where P'' is allowed to be any n-by-n matrix. Interestingly, the space spanned by $P^tP'' + (P'')^tP$ is, in fact, $\tan(X_0, \mathcal{S}^n_+)$, the tangent to $\mathcal{S}^n_+$ at X_0 (see [25] for definitions). And thus our analogous linear space is $\mathcal{M}(\tan(X_0, \mathcal{S}^n_+))$. The analogue to an equilibrium stress vector is a vector $\omega \in \mathbb{R}^e$ such that $\mathcal{M}^*(\omega)P^t = 0$, where $\mathcal{M}^*$ is the adjoint mapping back up into $\mathcal{S}^n$.

Let F_0 be the face of X_0 in $\mathcal{S}^n_+$. Let us define a "complement face," $\bar{F}_0$, to be any face of $\mathcal{S}^n_+$ that includes some matrix of rank $n-r$ and has no nontrivial intersection with F_0. With these defined, the general analogue to our cone $R(\mathbf{p}', \mathbf{p}')$ is $\mathcal{M}(\bar{F}_0)$.

Let us assume that $\mathcal{M}(\bar{F}_0)$ happens to be closed. Then the main ideas of this section allow us to conclude that if the constructed linear space has only the trivial intersection with the constructed cone, then our PSD feasibility problem has singularity degree one.

Unfortunately, in the case that $\mathcal{M}(\bar{F}_0)$ is not closed, then the Farkas lemma, Lemma 3.2, cannot be applied and so it is less clear if we can reason about singularity degree one.

5. Triangulated convex polygons with holes. We have two goals for this section and the next. One goal is to prove that arbitrary triangulations of certain

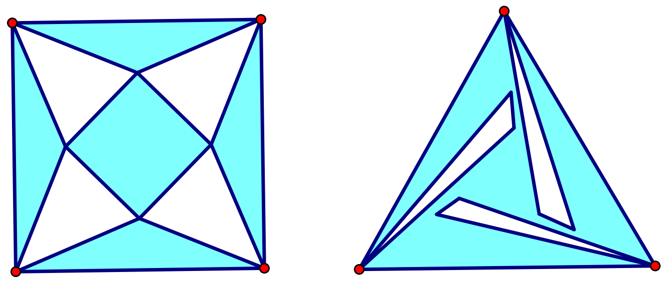


FIG. 5. *When the corner vertices of* (a) *are pinned, and any triangulation of the blue two-dimensional polyhedron is taken, the framework is super stable. For* (b) *there is a triangulation that is a finite mechanism in* 3*-space.*

polyhedral spaces are universally rigid, with some of the vertices pinned to first order. Another goal is to prove that arbitrary triangulations of certain polyhedral spaces are prestress stable in $\mathbb{R}^3$. These two goals are closely related.

A motivating example for the first goal is when the polyhedral space is the underlying space of an embedded simplicial complex where some of its vertices are pinned to the first order. But we will require that the space satisfies certain geometric conditions. For example, a two-dimensional convex planar polygon with its boundary vertices pinned to first order, and some "holes" removed from its interior, will be universally second-order rigid under any triangulation if the holes are placed correctly. Figure 5(a) is a case when the holes are properly placed, whereas in Figure 5(b) the holes are not properly placed.

DEFINITION 5.1. *Following* [22] *a* polyhedron $X = |K|$ *is the underlying space of a simplicial complex* K *in some Euclidean space and* K *is called a triangulation of* X*. If* X *is a polyhedron, with a subpolyhedron* $X_0 \subset X$*, we say that* (X, X_0) *is a* spider set *if for every point* $\mathbf{x} \in X - X_0$*, there is a framework* $(G_{sp}, \mathbf{q})$ *with* $\mathbf{x}$ *a vertex of* $\mathbf{q}$*, whose edges lie in* X*, and there is an equilibrium stress that is positive on all the edges that have at least one vertex in* $X - X_0$ *(and no zero length edges). We call* $(G_{sp}, \mathbf{q})$ *a* spider tensegrity *corresponding to* $\mathbf{x}$ *and the spider set* (X, X_0)*.*

Note that a polyhedron is allowed to be of "mixed dimension."

The definition of a spider set does not depend on a particular triangulation. For example, when a polyhedral set X is a subset of a convex planar polygon F, with X_0 the boundary of the polygon, the question as to whether it is a spider set can be determined by considering the infamous Maxwell–Cremona correspondence as discussed in [10] and shown in Figure 1. Suppose that P_F, a convex polytope, projects, orthogonally by the projection π, onto the polygon F which coincides with the "bottom" two-dimensional face of P_F. Let $P_F^{(1)}$ be the one-skeleton of P_F. Let X consist of the projection $\pi(P_F^{(1)})$ and the projection of any chosen subset of the two-dimensional top faces of P_F. Then such an (X, X_0) must be a spider set. To see this, if $\mathbf{x}$ is the projection of a vertex of P_F, then the projection of the one-skeleton serves as the spider tensegrity for X. If $\mathbf{x}$ is the projection of a point in the relative interior of one of the faces of P_F, it is easy to lift that point slightly above the convex hull of the other vertices, adjusting the polytope P_F. If $\mathbf{x}$ is the projection of a point in the relative interior of an edge of P_F, it is easy to adjust the stress to accommodate the subdivided edge.

THEOREM 5.2. *Let (K, K_0) be any triangulation of a spider set (X, X_0) and $(G, G_0, \mathbf{p})$ the corresponding framework of the one-skeleton of (K, K_0), where G_0 corresponds to those vertices of G that are in X_0. Then $(G, G_0, \mathbf{p})$ is universally second-order rigid when the vertices of G_0 are pinned to the first order.*

Proof. Let $\mathbf{p}_1$ be any vertex of $(G, G_0, \mathbf{p})$, not in G_0. By the definition of a spider set, there is a spider tensegrity $(G_{sp}, \mathbf{q})$ of (X, X_0), where $\mathbf{q}_1 = \mathbf{p}_1$ is a vertex in that tensegrity and $(G_{sp}, \mathbf{q})$ has an equilibrium stress ω, positive on all the edges with at least one vertex in $X - X_0$. Let $(\mathbf{p}', \mathbf{p}'')$ be a second-order flex of $(G, G_0, \mathbf{p})$, where $\mathbf{p}'_j = 0$ for vertices in G_0. We will show that $\mathbf{p}'_1 = 0$. Applying this to all the vertices of G not in G_0 will show our result.

For any simplex σ of K and edge τ of the tensegrity $(G_{sp}, \mathbf{q})$, the sets $\sigma \cap \tau$, which are nonempty, provide a subdivision of the edges of $(G_{sp}, \mathbf{q})$, say, $(\overline{G_{sp}}, \mathbf{r})$. The second-order flex of $(G, G_0, \mathbf{p})$ extends naturally to a corresponding second-order flex of each of the simplices of K and in particular to the segments $\sigma \cap \tau$. Thus there is a corresponding second-order flex $(\mathbf{r}', \mathbf{r}'')$ of $(\overline{G_{sp}}, \mathbf{r})$, where $\mathbf{p}'_1 = \mathbf{q}'_1 = \mathbf{r}'_1$, say, and $\mathbf{r}'_j = 0$ for $\mathbf{r}_j \in X_0$. Similarly there is a corresponding equilibrium stress $\bar{\omega}$ for $(\overline{G_{sp}}, \mathbf{r})$ that is positive on all the edges in $X - X_0$. Then for the rigidity matrix $R(\mathbf{r})$ for $(\overline{G_{sp}}, \mathbf{r})$, we have $\bar{\omega}^t R(\mathbf{r}) = 0$, by Proposition 2.2. This gives us the contradiction

$$\begin{aligned} 0 &= R(\mathbf{r})\mathbf{r}'' + R(\mathbf{r}')\mathbf{r}' \\ &= \bar{\omega}^t R(\mathbf{r})\mathbf{r}'' + \bar{\omega}^t R(\mathbf{r}')\mathbf{r}' \\ &= \bar{\omega}^t R(\mathbf{r}')\mathbf{r}' \\ &= \sum_{i<j} \bar{\omega}_{ij} (\mathbf{r}'_i - \mathbf{r}'_j)^2 > 0, \end{aligned}$$

unless $\mathbf{p}'_1 = \mathbf{q}'_1 = \mathbf{r}'_1 = 0$. ∎

Note that even when the spider set is a triangle with its vertices pinned, there are triangulations of the interior that are not "spiderwebs" as shown in the classic twisted example in Figure 6(a) from [14]. This means that any equilibrium stress must be negative on some of the internal edges. Figure 6(b) shows how the spider construction in the proof of Theorem 5.2 would work for the center vertex of Figure 6(a).

Together with Theorem 4.7, this gives us the following.

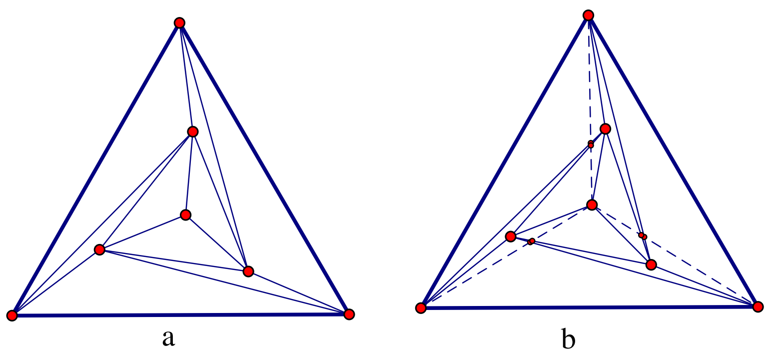


FIG. 6. (a) *The set X is a single triangle and X_0 is its boundary. The framework $(G, G_0, \mathbf{p})$ is the one-skeleton of a triangulation (K, K_0) of (X, X_0).* (b) *The outer three edges and the subdivided dashed edges form a subdivided spider tensegrity $(\overline{G_{sp}}, \mathbf{r})$ of (X, X_0) corresponding the center vertex.*

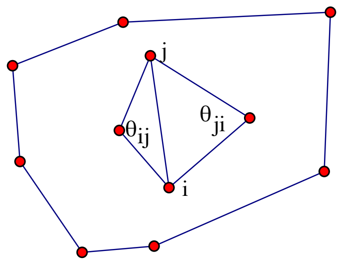


FIG. 7. *The cotangent construction assigns $\omega_{ij} := \cot(\theta_{ij}) + \cot(\theta_{ji})$ to each internal edge.*

COROLLARY 5.3. *Let (K, K_0) be any triangulation of a spider set (X, X_0) and $(G, G_0, \mathbf{p})$ the corresponding framework of its one-skeleton, where G_0 corresponds to those vertices of G that are in X_0. Then $(G, G_0, \mathbf{p})$ must have an equilibrium stress ω such that the energy, E_ω, is positive definite on $\mathcal{C}'(1, G_0)$ (see Definition 4.2).*

The existence result of Corollary 5.3 is somewhat related to the notion of a discrete Laplacian operator [28]. In that setting, given a triangulation of a polygon F in $\mathbb{R}^2$ (which need not even be convex), one looks for a vector $\omega \in \mathbb{R}^e$ with the property that E_ω is positive away from the boundary vertices and that for all *internal* vertices (not on the polygon boundary) we have $\sum_{j \in N(i)} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0$. Such an ω acts as an equilibrium stress for $(G, \mathbf{p})$ under the added assumption that all of the boundary vertices are fully pinned to all orders. Under these weaker requirements, there exist well-known constructive approaches for generating such an ω.

One such construction uses the so-called cotangent weights [26], which assign $\omega_{ij} := \cot(\theta_{ij}) + \cot(\theta_{ji})$ to each internal edge (see Figure 7). Depending on the geometry of the triangulation, some of these weights may be negative. They derive this energy in the context of a Dirichlet energy computation. There is also another derivation for the cotangent weights that uses Heron's formula for the area of a triangle, the law of sines and the law of cosines. See [20], for example. But when the polygon F has more than three vertices, this construction cannot be used to generate an ω that is in equilibrium *at the boundary vertices*.

A related question is if a "random" set of holes is put in a membrane with its boundary clamped, is there a critical threshold with a "phase change" where the membrane becomes not rigid in $\mathbb{R}^3$? It turns out that for the continuous case [15] and the triangulated case [24], the answer is no. With high probability the membrane will have some flexible parts.

6. Triangulated convex polytopes with holes.

We now wish to consider a convex polytope P in $\mathbb{R}^3$ and a bar framework $(G, \mathbf{p})$, where all the vertices and bars are contained in the two-dimensional boundary surface of P.

DEFINITION 6.1. *Let P be a convex polytope in $\mathbb{R}^3$. Let $P^{(1)}$ be the underlying point set of the one-skeleton of P. Let H be a polyhedral subset of the boundary surface of P. We say that H is holeyhedron if the following properties hold for every (two-dimensional) face F of P:*

(a) *The one-skeleton $P^{(1)} \subset H$.*
(b) *Any infinitesimal flex in the plane of F of any triangulation of $H \cap F$ is trivial on vertices that are in $P^{(1)} \cap F$.*
(c) *The polyhedron $(H \cap F, P^{(1)} \cap F)$ is a spider set.*

Note that the complete boundary surface of a convex polytope (without any holes) is always a holeyhedron.

The following is a slight generalization of [8, Theorem 5.2] using Definition 6.1(6.1). This relies deeply on Alexandrov's theorem about the infinitesimal rigidity of certain triangulated convex polytopes.

LEMMA 6.2. *Let* $(G, \mathbf{p})$ *be the bar framework of a triangulation of a holeyhedron. Let* $\mathbf{p}'$ *be a first-order flex in* $\mathbb{R}^3$. *Then* $\mathbf{p}'$, *when restricted to vertices in* $P^{(1)}$, *is a trivial first-order flex.*

We now state our second main result of this paper.

THEOREM 6.3. *Any triangulation* $(G, \mathbf{p})$ *of a holeyhedron,* H, *is prestress stable in* $\mathbb{R}^3$.

Proof. From Corollary 5.3, we have for each triangulated $F \cap H$ of our face an ω_F such that E_{ω_F} is positive definite on all vectors that vanish on $P^{(1)} \cap F$. By simply adding together all of these ω_F, we obtain an ω that is an equilibrium stress for $(G, \mathbf{p})$ with an associated stress matrix Ω that must be positive definite on any vector $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$ that vanishes on $P^{(1)}$.

Meanwhile, from Lemma 6.2, all first-order flexes $\mathbf{p}'$ in $\mathbb{R}^3$ for $(G, \mathbf{p})$ have $\mathbf{p}'$ trivial on $P^{(1)}$. By adding an appropriate trivial flex to $\mathbf{p}'$ we can make $\mathbf{p}'$ vanish on $P^{(1)}$. In light of (2), this addition will not change $E_\omega(\mathbf{p}')$, as ω is an equilibrium stress for $\mathbf{p}$. Thus, if $\mathbf{p}'$ is any nontrivial first-order flex of the triangulation, we have $E_\omega(\mathbf{p}') > 0$, making the triangulation $(G, \mathbf{p})$ of H prestress stable in $\mathbb{R}^3$. ∎

7. Extensions and related results.

If one considers only polyhedral subsets H of the boundary of a convex polytope P in 3-space, where one is allowed to triangulate H at will, then it is important, for each face, F, that no hole has a vertex on the interior of any of the natural edges of P (although there can be holes that touch the natural vertices). For example, Figure 24 of [8] is a tetrahedron with a small slit on one face touching the relative interior of one edge that can be subdivided and flexed as a finite mechanism to be flat in the plane. See also [1] for methods for decreasing the size of the slit.

In more detail, suppose one vertex $\mathbf{x}$ of one of the holes of H lies on a natural edge of the polytope. And suppose the interval $[\mathbf{x}, \mathbf{y}]$ is part of the boundary of that hole, interior to its face in P, as in Figure 8. Then any vertex, say, $\mathbf{z}$, on the interior interval

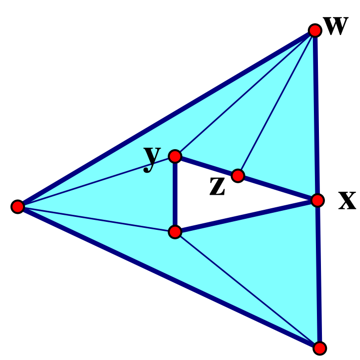


FIG. 8. *Any vertex* $\mathbf{z}$ *of a subdivision of face on the interval between the points* $\mathbf{x}$ *and* $\mathbf{y}$, *as shown, cannot be part of a spider tensegrity. All the edges adjacent to* $\mathbf{z}$ *must have zero stress.*

$[\mathbf{x}, \mathbf{y}]$ cannot be part of a spider tensegrity, because of the positivity requirement on the equilibrium stress for all interior edges of a spider tensegrity. For example, one cannot obtain equilibrium at $\mathbf{z}$ if there is positive stress on an edge, say, from $\mathbf{z}$ toward $\mathbf{w}$. And one cannot obtain equilibrium at $\mathbf{x}$ if there is positive stress on an edge from $\mathbf{x}$ toward $\mathbf{z}$. This positivity is required by Definition 6.1(c).

Alternatively, suppose the surface H has a hole with an edge that coincides completely with a natural edge of the polytope. Then when that edge is subdivided (which is allowed in a triangulation of H), then this triangulation of the set $H \cap F$ will not have the first-order rigidity property of Definition 6.1(b).

With the above comments in mind it seems that if an arbitrary triangulation of a polyhedral subset of the surface of a convex polytope P is prestress stable, each natural edge e of P must have at least a small "flange" that lies in each side of e in each of the two adjacent faces so that the relative interior of e is in the topological interior relative to the two-dimensional surface, as in Figure 5(a).

7.1. Prestressed fixed frameworks. We can extend the notion of a spider tensegrity as in Definition 5.1 to say that a framework $(G, G_0, \mathbf{p})$ is a *spider tensegrity* if there is an equilibrium stress for $(G, \mathbf{p})$ that is positive on all the edges with at least one vertex in $G - G_0$.

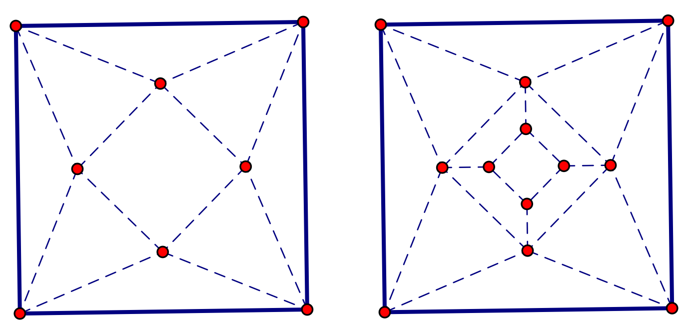
If one is given a fixed bar (or tensegrity) framework, we can use some of the techniques described previously to generate prestress stable structures as follows.

COROLLARY 7.1. *Suppose a bar framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is such that there are vertices G_0 of G such that $(G, G_0, \mathbf{p})$ is a spider tensegrity, and G_0 is contained in a subgraph G_1 of G such that $(G_1, \mathbf{q})$ is first-order rigid, with $\mathbf{q}$ corresponding to the vertices of G_1. Then $(G, \mathbf{p})$ is prestress stable in $\mathbb{R}^d$.*

Note that the framework $(G, \mathbf{p})$ above may not be first-order rigid as is the case for Figure 9(b), but the boundary is part of the framework in Figure 9(a), which is first-order rigid in the plane.

COROLLARY 7.2. *If each face F of a convex polytope in $\mathbb{R}^3$ contains a spider tensegrity such that the vertices on each boundary of F are first-order rigid in the plane of F, as in Corollary* 7.1, *then the union of these spider tensegrities is prestress stable in $\mathbb{R}^3$.*

One may apply the Roth–Whiteley criterion [27] for the first-order rigidity of tensegrity frameworks to the framework in Figure 9(a). Namely, the underlying bar



a b

FIG. 9.

framework is first-order rigid, and there is an equilibrium stress, positive on all the cables and negative on all the struts.

Appendix A. Super stability. In this section we provide a proof of Theorem 2.12. In this section d' will be any fixed dimension greater than d.

To prove that super stability implies prestress stability in $\mathbb{R}^{d'}$, we will use the fact that a PSD equilibrium stress matrix Ω of rank $n-d-1$ must block all nontrivial infinitesimal flexes except for ones arising from affine transforms of $\mathbf{p}$. Then we will argue, using the assumption of no conic at infinity, that any affine infinitesimal flex must be trivial.

As a warmup, we start with the following lemma, which we mentioned in section 2.2

LEMMA A.1. *Let $(G, \mathbf{p})$ be a framework with a d-dimensional span in $\mathbb{R}^d$. Suppose Q is a d-by-d symmetric matrix such that for all pairs, $\{k, l\}$, we have $(\mathbf{p}_k - \mathbf{p}_l)^t Q(\mathbf{p}_k - \mathbf{p}_l) = 0$. Then Q must be the zero matrix.*

Proof. Let us pick a subset S of $d+1$ vertices in affine general position. Next, we perform an affine change of coordinates so that one of the vertices of S is at the origin and each other vertex is along a unique coordinate axis. Then we can use the conditions $(\mathbf{p}_k - \mathbf{p}_l)^t Q(\mathbf{p}_k - \mathbf{p}_l) = 0$ to see that Q is skew-symmetric. Since Q is also assumed to be symmetric, it must be zero. ☐

DEFINITION A.2. *We say that an infinitesimal flex $\mathbf{p}'$ of $(G, \mathbf{p})$ is* deforming *if there is some nonedge pair $\{k, l\}$ with $(\mathbf{p}_k - \mathbf{p}_l) \cdot (\mathbf{p}'_k - \mathbf{p}'_l) \neq 0$.*

LEMMA A.3. *Suppose a framework $(G, \mathbf{p})$ with a d-dimensional span in $\mathbb{R}^{d'}$ has an infinitesimal flex $\mathbf{p}'$ such that each $\mathbf{p}'_i$ lies in $\langle \mathbf{p} \rangle$. If $\mathbf{p}'$ is nondeforming, then it is trivial.*

Proof. Without loss of generality we can place both $\mathbf{p}$ and $\mathbf{p}'$ in $\mathbb{R}^d$. For any subset S of $d+1$ vertices in affine general position, the restriction of $\mathbf{p}'$ and $\mathbf{p}$ to S must satisfy $\mathbf{p}'_i = A_S \mathbf{p}_i + \mathbf{t}_S$ for some unique d-by-d matrix A_S and d-vector $\mathbf{t}_S$.

By assumption, for all pairs of vertices $\{k, l\}$ in S we have

$$\begin{aligned} 0 &= (\mathbf{p}_k - \mathbf{p}_l) \cdot (\mathbf{p}'_k - \mathbf{p}'_l) \\ &= (\mathbf{p}_k - \mathbf{p}_l)^t A_S (\mathbf{p}_k - \mathbf{p}_l). \end{aligned}$$

As in the proof of Lemma A.1, this forces A_S to be skew-symmetric.

Let T be another such subset that shares d vertices with S, where we have $\mathbf{p}'_i = A_T \mathbf{p}_i + \mathbf{t}_T$. Since T and S share d vertices in general affine position, this means $A_T = A_S$ and $\mathbf{t}_T = \mathbf{t}_S$. (One way to see this is to work in a coordinate system such that for each of the shared d vertices of S and T, the last coordinate of the associated $\mathbf{p}_i$ vanishes. Then we note that a d-by-d skew-symmetric matrix is fully determined by its first $d-1$ columns.)

This process can be reapplied as needed to show that all of $\mathbf{p}'$ represents a trivial infinitesimal flex. ☐

DEFINITION A.4. *We say that an infinitesimal flex $\mathbf{p}'$ of $(G, \mathbf{p})$ is* affine *if $\mathbf{p}'_i = M\mathbf{p}_i + \mathbf{t}$ for some d'-by-d' matrix M and some d'-vector $\mathbf{t}$.*

LEMMA A.5. *Suppose a framework $(G, \mathbf{p})$ with a d-dimensional span in $\mathbb{R}^{d'}$ has an affine infinitesimal flex $\mathbf{p}'$, where each $\mathbf{p}'_i$ is orthogonal to $\langle \mathbf{p} \rangle$; then $\mathbf{p}'$ is a trivial infinitesimal flex.*

Proof. Without loss of generality, using an orthonormal change of coordinates in $\mathbb{R}^{d'}$, we can assume that the affine span of this framework $(G, \mathbf{p})$ agrees with the first d dimensions of $\mathbb{R}^{d'}$; thus for each $\mathbf{p}_i$, its last $d' - d$ coordinates are zero, and for $\mathbf{p}'_i$, its first d coordinates are zero.

Thus M must have the block form

$$M = \begin{bmatrix} 0 & X \\ Y^t & Z \end{bmatrix},$$

where X is d-by-$(d' - d)$, Y^t is $(d' - d)$-by-d, and Z is $(d' - d)$-by-$(d' - d)$. (Also, the first d coordinates of $\mathbf{t}$ must vanish.)

Since the last $d' - d$ coordinates of each $\mathbf{p}_i$ are zero, we can replace M with

$$A := \begin{bmatrix} 0 & -Y \\ Y^t & 0 \end{bmatrix}$$

and still obtain the same flex, $\mathbf{p}'_i = A\mathbf{p}_i$. As A is skew-symmetric, this proves that $\mathbf{p}'$ is trivial. ∎

LEMMA A.6. *If a framework $(G, \mathbf{p})$ has a nontrivial affine infinitesimal flex $\mathbf{p}'$, then the $\mathbf{p}'$ is deforming.*

Proof. From Lemma A.5 the component of $\mathbf{p}'$ that is orthogonal to $\langle \mathbf{p} \rangle$ is trivial and thus can be subtracted from $\mathbf{p}'$ without changing its nontriviality. From Lemma A.3, this remaining flex must be deforming. The trivial orthogonal component can be added back without changing its deforming property; thus the original $\mathbf{p}'$ must be deforming. ∎

LEMMA A.7. *If a framework $(G, \mathbf{p})$ has a deforming affine infinitesimal flex $\mathbf{p}'$, then the framework has its edge directions at a conic at infinity for $\langle \mathbf{p} \rangle$.*

Proof. From the affine assumption, we have $\mathbf{p}' = M\mathbf{p} + \mathbf{t}$, where M is some d'-by-d' matrix. We can write $M = A + Q$, where A is skew-symmetric and Q is symmetric. By removing from $\mathbf{p}'$ the trivial infinitesimal flex generated by $A\mathbf{p}_i + \mathbf{t}$, the remaining infinitesimal flex generated by $Q\mathbf{p}_i$ must still be deforming.

As an infinitesimal flex, for all edges $\{i, j\}$, we have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (Q\mathbf{p}_i - Q\mathbf{p}_j) = 0$, which gives us $(\mathbf{p}_i - \mathbf{p}_j)^t Q(\mathbf{p}_i - \mathbf{p}_j) = 0$. Since the flex generated by $Q\mathbf{p}_i$ is deforming, we have some nonedge pair $\{k, l\}$ with $(\mathbf{p}_k - \mathbf{p}_l)^t Q(\mathbf{p}_k - \mathbf{p}_l) \neq 0$. This gives us our desired conic at infinity. ∎

We can now prove one direction of Theorem 2.12: If $(G, \mathbf{p})$ is super stable, it must have a PSD equilibrium stress matrix Ω of rank $n - d - 1$. From Proposition 2.10, the corresponding stress energy must be positive on any nontrivial infinitesimal flex unless it is an affine infinitesimal flex. From Lemmas A.6 and A.7, such a nontrivial affine infinitesimal flex can exist only if the edge directions of $(G, \mathbf{p})$ are on a conic at infinity for $\langle \mathbf{p} \rangle$. But our assumption of super stability also rules this out. Thus $(G, \mathbf{p})$ must be prestress stable in $\mathbb{R}^{d'}$.

To prove that prestress stability in $\mathbb{R}^{d'}$ implies super stability, our main step is to show that, in any dimension $d' > d$, *any* one-dimensional configuration that is not an affine image of $\mathbf{p}$ can be used to generate a nontrivial infinitesimal flex for $(G, \mathbf{p})$. Thus the blocking equilibrium stress, assumed by prestress stability, must be PSD and of rank $n - d - 1$.

LEMMA A.8. *If $(G, \mathbf{p})$, a framework with a d-dimensional affine span in $\mathbb{R}^{d+1}$, is prestress stable, then it must have a PSD equilibrium stress matrix of rank $n - d - 1$.*

Proof. Using an orthonormal change of coordinates in $\mathbb{R}^{d+1}$, we can assume that the affine span of this framework $(G, \mathbf{p})$ agrees with the first d dimensions of $\mathbb{R}^{d+1}$; thus for each $\mathbf{p}_i$, its last coordinate is zero.

Let $\mathbf{p}' \in \mathbb{R}^{n(d+1)}$ be any "configuration" such that each $\mathbf{p}'_i$ has zero values for all but its last coordinate. Since $\mathbf{p}_i$ has zero values in its last coordinate, then $\mathbf{p}'$ must always be an infinitesimal flex for $(G, \mathbf{p})$. Let v_i be the last coordinate of $\mathbf{p}'_i$, with $\mathbf{v}$ a vector in $\mathbb{R}^n$. This infinitesimal flex, $\mathbf{p}'$, will be nontrivial unless $\mathbf{v}$ arises as a skew-symmetric affine image of $\mathbf{p}$ and thus can be written as $v_i = \mathbf{a}^t\mathbf{p}_i + t$ for a fixed d-vector $\mathbf{a}$ and scalar t. Thus we can generate a space of nontrivial infinitesimal flexes of dimension $n - d - 1$.

From prestress stability in $\mathbb{R}^{d+1}$, $(G, \mathbf{p})$ must have an equilibrium stress matrix Ω with $\mathbf{v}^t\Omega\mathbf{v} > 0$ when $\mathbf{p}'$ is nontrivial. Thus Ω must have positive energy on a linear space of dimension $n-d-1$. As this is the maximum possible rank for an equilibrium stress matrix (Proposition 2.10) it must in fact be PSD and of rank $n - d - 1$. ∎

LEMMA A.9. *If $(G, \mathbf{p})$, a framework with a d-dimensional affine span in $\mathbb{R}^{d+1}$, is prestress stable, then it cannot have its edge directions on a conic at infinity of $\langle \mathbf{p} \rangle$.*

Proof. Since $(G, \mathbf{p})$ is prestress stable, it is rigid. A rigid framework cannot have its edge directions on a conic at infinity [12]. ∎

This gives us the other direction of Theorem 2.12: If $(G, \mathbf{p})$ is prestress stable in $\mathbb{R}^{d'}$, it is certainly prestress stable when thought of as a framework in $\mathbb{R}^{d+1}$. We now simply combine Lemmas A.8 and A.9 to conclude that $(G, \mathbf{p})$ is super stable.

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