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SECOND-ORDER RIGIDITY AND PRESTRESS STABILITY FOR TENSEGRITY FRAMEWORKS*

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Abstract. This paper defines two concepts of rigidity for tensegrity frameworks (frameworks with cables, bars, and struts): prestress stability and second-order rigidity. We demonstrate a hierarchy of rigidity—first-order rigidity implies prestress stability implies second-order rigidity implies rigidity—for any framework. Examples show that none of these implications are reversible, even for bar frameworks. Other examples illustrate how these results can be used to create rigid tensegrity frameworks.

This paper also develops a duality for second-order rigidity, leading to a test which combines information on the self stresses and the first-order flexes of a framework to detect second-order rigidity. Using this test, the following conjecture of Roth is proven: a plane tensegrity framework, in which the vertices and bars form a strictly convex polygon with additional cables across the interior, is rigid if and only if it is first-order rigid.

Key words. tensegrity frameworks, rigid and flexible frameworks, stability of frameworks, static stress, first-order motion, second-order motion

AMS subject classifications. Primary, 52C25; Secondary, 70B15, 70C20

1. Introduction.

A fundamental problem in geometry is to determine when selected distance constraints, on a finite number of points, fix these points up to congruence, at least for small perturbations. We rephrase this as a problem in the rigidity of frameworks. From this point of view, we do the following:

(i) provide methods for recognizing when a given framework is rigid;
(ii) find ways of generating rigid frameworks;
(iii) explore the relationship among these methods;
(iv) solve a conjecture of Roth, in [29], concerning the rigidity of a specific class of frameworks in the plane.

Many of the concepts here are inspired by techniques used in structural engineering, such as the principle of least work and energy, but our treatment is independent of any such concepts. Broadly speaking our objective in this paper is to investigate the geometric properties of configurations of points in Euclidean space. However, our results clarify and justify mathematically some of the techniques used by structural engineers to analyze certain tensegrity structures. (See [5, 27, 28] for example.) Our techniques extend those of [21] and are related to the questions posed by Tarnai [31]. A summary of these results, for engineers, was presented in [13].

1.1. Terminology.

A *tensegrity framework* is an ordered finite collection of points in Euclidean space, called a *configuration*, with certain pairs of these points,

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called *cables*, constrained not to get further apart; certain pairs, called *struts*, constrained not to get closer together; and certain pairs, called *bars*, constrained to stay the same distance apart. Together, struts, cables, and bars are called *members*. If each continuous motion of the points satisfying all the constraints is the restriction of a rigid motion of the ambient Euclidean space, then we say the tensegrity framework is *rigid*. See [9] and [29].

For the recognition problem (i) there has been much work done using the concept of first-order rigidity. A tensegrity framework is *first-order rigid* (or *infinitesimally rigid*) if the only smooth motion of the vertices, such that the first derivative of each member length is consistent with the constraints, has its derivative at time zero equal to that of the restriction of a congruent motion of Euclidean space. See §2.2 for more details. An equivalent dual concept says that a tensegrity framework is *statically rigid* if every equilibrium load can be resolved. See [10] or [29] for more details and a precise definition. Our working definition will be that of first-order rigidity as above.

A *stress* in a tensegrity framework is an assignment of a scalar to each member. It is called a *self stress* if the vector sum of the scalar times the corresponding member vector is zero at each vertex. It is called *proper* if the cable stresses are nonnegative and the strut stresses are nonpositive (with no condition on the bars). It is called *strict* if the stress in each cable and strut is nonzero.

1.2. First-order duality. The interplay between first-order motions and self stresses yields a test for first-order rigidity. Every first-order rigid tensegrity framework has a strict proper self stress, by a result of Roth and Whiteley [29]. We state the first-order stress test as follows: *There is a first-order flex of a framework which strictly changes the length of a strut or cable if and only if every proper self stress is zero on the given member.*

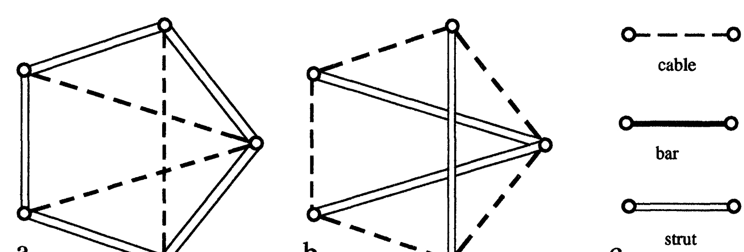

FIG. 1. *Two first-order rigid tensegrity frameworks* (a, b) *where cables and struts are reversed.*

In addition, if a first-order rigid bar framework has any nonzero self stress at all, one can change these members to cables or struts following the sign of the self stress to get another statically rigid tensegrity framework. In the spirit of (ii), this is a first-order method for generating examples of statically rigid tensegrity frameworks. See [29] and [35] for examples. Note that any first-order rigid tensegrity framework can have its cables and struts reversed to struts and cables, respectively, and it will remain first-order rigid. Figure 1 shows a pair of frameworks which are infinitesimally rigid in the plane.

1.3. Prestress stability and second-order rigidity. In this paper, we define two other classes of frameworks, those that are prestress stable and those that are second-order rigid. We call a tensegrity framework *prestress stable* if it has a proper strict self stress such that a certain energy function, defined in terms of the stress and defined for all configurations, has a local minimum at the given configuration, and this minimum is a strict local minimum up to congruence of the whole framework. (See §3.3 for the precise formula for this energy function.)

Prestress stability is a concept we have borrowed from structural engineering. The "principle of least work" is the motivation behind the definition of our energy functions. If a certain configuration of a framework corresponds to a local minimum (modulo rigid motions) of an energy function, which is the sum of the energies of all the members, then it is clear that the framework is rigid. When the usual second derivative test detects such a minimum, this corresponds to prestress stability. Pellegrino and Calladine [28] describe certain matrix rank conditions that are necessary but not sufficient for prestress stability. However, their condition essentially ignores the basic positive definite conditions. See [5] for an improved version, though. For engineering calculations, the stress-strain relation in each member is given, and this information determines the corresponding energy function. On the other hand, for the simpler mathematical recognition problem (i), one is free to choose the member energy functions at will.

A tensegrity framework is *second-order rigid* if every smooth motion of the vertices, which does not violate any member constraint in the first and second derivative, has its first derivative trivial; i.e., its first derivative is the derivative of a one parameter family of congruent motions.

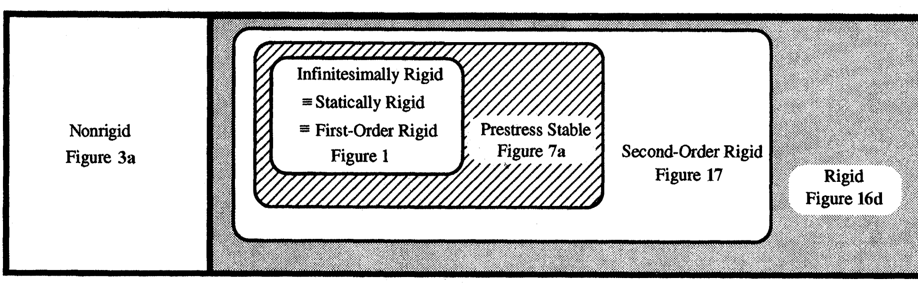

FIG. 2. *A diagram of the hierarchy of "rigidity" (numbers refer to illustrative examples).*

A series of basic results shows that for any tensegrity framework, first-order rigidity (i.e., infinitesimal rigidity) implies prestress stability, which implies second-order rigidity, which implies rigidity, and none of these implications can be reversed. See Figure 2, where the figure numbers refer to examples seen later in this paper that lie only in that region of the diagram. This extends the second-order rigidity results of [6] for bar frameworks and places prestress stability between first-order and second-order rigidity.

1.4. The second-order stress test. Information about a framework, or a class of frameworks, may come in various forms, and it can be useful to relate these different forms for the situation at hand. For example, to test the first-order rigidity of a tensegrity framework we may use both the self stresses and the first-order flexes, as in the first-order stress test.

We extend this first-order duality to the second-order situation. Regard any stress as the constant coefficients of a quadratic form on the space of all configurations as well as the space of first-order flexes. This is a "homogeneous" energy function. Suppose we have a fixed first-order flex of a given framework, and we wish to know when that first-order flex extends to some second-order flex. Our second-order stress test states the following: *A second-order flex exists if and only if for every proper self stress of the framework the quadratic form it defines is nonpositive when evaluated at the given* *first-order flex.* Thus information about proper self stresses of a framework, as well as first-order flexes, can provide information about second-order rigidity. The proof amounts to observing that the (inequality and equality) constraints of second-order rigidity and our dual stress condition is a special case of the "Farkas alternative" (as used in linear programming duality).

It is also possible to sharpen the second-order stress test to provide necessary and sufficient conditions to detect when the second-order inequalities are strict. This sharpening is a generalization of the first-order stress test. The sharpened second-order stress test can be helpful not only in detecting second-order rigidity but also quite often in detecting when there is an actual continuous flex that has the cable and strut conditions slacken at the second-order.

1.5. Roth's conjecture. As an application of these methods we verify a conjecture of Roth about polygons in the plane in [29]. In their *Lectures on Lost Mathematics* [18, 19], Grünbaum and Shepard conjectured that if one has a framework $G(\mathbf{p})$ in the plane with the points as the vertices of a convex polygon, bars on the edges, and cables inside connecting certain pairs of the vertices (Figure 3a, c) in such a way that the framework is rigid in the plane (Figure 1a), then reversing the cables and bars to get $\hat{G}(\mathbf{p})$ (Figure 1b) preserves rigidity. (They also observed that starting with cables on the outside and bars inside does not necessarily preserve rigidity (Figure 3b, a).)

If Grünbaum and Shephard's polygonal frameworks are rigid because they are infinitesimally rigid, then it follows that the reversed framework is also infinitesimally rigid and therefore rigid. Roth's conjecture was that all rigid convex polygons with cables on the inside were indeed infinitesimally rigid. For example, Figure 3c shows a regular octagon with bars on the edges and fourteen cables on the inside. It is easy to check that this framework is not infinitesimally rigid. Thus Roth's conjecture implies that this framework is not rigid since if it were rigid, it would be infinitesimally rigid. The reader is invited to find the motion of the vertices in the plane directly. (See Remark 6.2.1.)

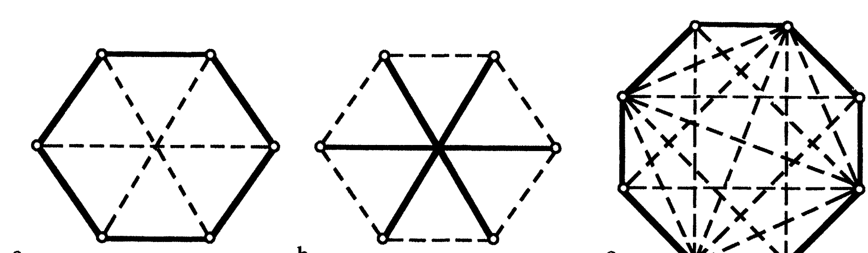

FIG. 3. *Framework* (b) *is rigid, but framework* (a), *where cables and struts are reversed, is not. Framework* (c) *is not first-order rigid and is not rigid, illustrating Roth's conjecture.*

Meanwhile, Connelly [7] showed that any proper self stress coming from one of Grünbaum and Shepard's polygonal frameworks $G(\mathbf{p})$ had an associated negative semidefinite quadratic form with nullity three. Equivalently the reversed framework $\hat{G}(\mathbf{p})$ had a positive semidefinite quadratic form with nullity three. Then it is easy to show that $\hat{G}(\mathbf{p})$ is rigid by showing that (globally) there is no other noncongruent configuration satisfying the bar and cable constraints. This global type of rigidity is somewhat different from infinitesimal rigidity. Neither infinitesimal rigidity nor global rigidity implies the other. However, the energy functions used to prove the global rigidity also imply prestress stability. Thus Grünbaum's conjecture was proved and generalized in one direction, but Roth's conjecture, a generalization in another direction, was not proved.

1.6. The proof of Roth's conjecture. The idea behind our proof of Roth's conjecture is the following. We observe that the conditions for a strict second-order flex, in the second-order stress test, are satisfied by any one of Grünbaum and Shepard's frameworks $G(\mathbf{p})$, since any proper self stress defines a negative semidefinite quadratic form which is negative definite on any space of nontrivial first-order flexes. Thus if there is any nontrivial first-order flex, it will extend to a strict second-order flex which in turn implies that there will be a nontrivial continuous flex of the framework. Thus (the contrapositive of) Roth's conjecture is verified: if $G(\mathbf{p})$ is not infinitesimally rigid, then $G(\mathbf{p})$ is not rigid. In particular, if any one of Grünbaum and Shepard's frameworks $G(\mathbf{p})$ is rigid, the reversed framework $\hat{G}(\mathbf{p})$ is both infinitesimally rigid and globally rigid.

In an appendix we summarize a series of "replacement principles" which describe when and how one can switch between bars and cables or struts and preserve the various levels of rigidity or flexibility.

2. Review of tensegrity frameworks. Throughout this paper the word tensegrity is used to describe any framework with *cables*—each cable determines a maximum distance between two points, *struts*—each strut determines a minimum distance between two points, and *bars*—each bar determines a fixed distance between two points. Statically, cables can only apply tension and struts can only apply compression. We partition the edges of our graph into three disjoint classes — E_- for cables, E_0 for bars, and E_+ for struts, creating a *signed graph* $G = (V; E_-, E_0, E_+)$. In our figures, cables are indicated by dashed lines, struts by double thin lines, and bars by single thick lines (see Figure 1). General references for this chapter are [8, 9, 29, 35].

2.1. Rigidity.

DEFINITION 2.1.1. *A* tensegrity framework *in d-space $G(\mathbf{p})$ is a signed graph $(V; E_-, E_0, E_+)$, and an assignment $\mathbf{p} \in \mathbb{R}^{dv}$ such that each $\mathbf{p}_i \in \mathbb{R}^d$ corresponds to a vertex of G, where $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_v)$ is a* configuration. *The members in E_- are cables, the members in E_0 are bars, and the members in E_+ are struts. A* bar framework *is a tensegrity framework with no cables or struts*; *i.e., $E = E_0$.*

DEFINITION 2.1.2. *A tensegrity framework $G(\mathbf{p})$* dominates *the tensegrity framework $G(\mathbf{q})$, written $G(\mathbf{p}) \geq G(\mathbf{q})$, if*

$$\begin{aligned} |\mathbf{p}_i - \mathbf{p}_j| &\geq |\mathbf{q}_i - \mathbf{q}_j| \quad \text{when} \quad \{i,j\} \in E_-, \\ |\mathbf{p}_i - \mathbf{p}_j| &= |\mathbf{q}_i - \mathbf{q}_j| \quad \text{when} \quad \{i,j\} \in E_0, \\ |\mathbf{p}_i - \mathbf{p}_j| &\leq |\mathbf{q}_i - \mathbf{q}_j| \quad \text{when} \quad \{i,j\} \in E_+. \end{aligned}$$

A tensegrity framework $G(\mathbf{p})$ is rigid *in $\mathbb{R}^d$ if any of the following three equivalent conditions holds* [9] *or* [29]*:*

(a) *there is an $\varepsilon > 0$ such that if $G(\mathbf{p}) \geq G(\mathbf{q})$ and $|\mathbf{p} - \mathbf{q}| < \varepsilon$ then $\mathbf{p}$ is congruent to $\mathbf{q}$*; *or*

(b) *for every continuous path, or* continuous flex, *$\mathbf{p}(t) \in \mathbb{R}^{vd}$, $\mathbf{p}(0) = \mathbf{p}$, such that $G(\mathbf{p}) \geq G(\mathbf{p}(t))$ for all $0 \leq t \leq 1$, then $\mathbf{p}$ is congruent to $\mathbf{p}(t)$ for all $0 \leq t \leq 1$*; *or*

(c) *for every analytic path, or* analytic flex, *$\mathbf{p}(t) \in \mathbb{R}^{vd}$, $\mathbf{p}(0) = \mathbf{p}$, such that $G(\mathbf{p}) \geq G(\mathbf{p}(t))$ for all $0 \leq t \leq 1$, then $\mathbf{p}$ is congruent to $\mathbf{p}(t)$ for all $0 \leq t \leq 1$.*

2.2. First-order rigidity.

DEFINITION 2.2.1. *A* first-order flex, *or an* infinitesimal flex, *of a tensegrity framework* $G(\mathbf{p})$ *is an assignment* $\mathbf{p}' : V \to \mathbb{R}^n$, $\mathbf{p}'(v_i) = \mathbf{p}_i'$, *such that for each edge* $\{i, j\} \in E$ *(Figure 4),*

$$
\begin{array}{lll}
(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j' - \mathbf{p}_i') \leq 0 & \textit{for cables} & \{i,j\} \in E_-, \\
(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j' - \mathbf{p}_i') = 0 & \textit{for bars} & \{i,j\} \in E_0, \\
(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j' - \mathbf{p}_i') \geq 0 & \textit{for struts} & \{i,j\} \in E_+.
\end{array}
$$

The dot product of two vectors X, Y is indicated by XY or $X \cdot Y$.

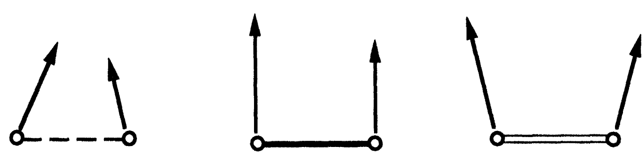

FIG. 4. *Some first-order motions (velocities) permitted by cables, bars, and struts.*

DEFINITION 2.2.2. *A first-order flex* $\mathbf{p}'$ *of a tensegrity framework* $G(\mathbf{p})$ *is* trivial *if there is a skew symmetric matrix* S *and a vector* $\mathbf{t}$ *such that* $\mathbf{p}_i' = S\mathbf{p}_i + \mathbf{t}$ *for all vertices* i.

DEFINITION 2.2.3. *A tensegrity framework* $G(\mathbf{p})$ *is* first-order rigid *(or* infinitesimally rigid*) if every first-order flex is trivial and* first-order flexible *otherwise.*

Let

$$
\mathbf{p}^* = \begin{bmatrix} \mathbf{p}_1^* \\ \vdots \\ \mathbf{p}_v^* \end{bmatrix}
$$

be regarded as a column vector in $\mathbb{R}^{dv}$, where each $\mathbf{p}_i^* \in \mathbb{R}^d$, $i = 1, \ldots, v$. Then $R(\mathbf{p})$ is the e-by-dv matrix defined by

$$
R(\mathbf{p})\mathbf{p}^* = \begin{bmatrix} \vdots \\ (\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*) \\ \vdots \end{bmatrix}.
$$

See [9] or [29]. $R(\mathbf{p})$ is called the *rigidity matrix* for the framework $G(\mathbf{p})$. Notice that a first-order flex of a bar framework is a solution to the linear equations

$$
R(\mathbf{p})\mathbf{p}^* = 0.
$$

Remark 2.2.1. A basic theorem of the subject says that *first-order rigidity for a tensegrity framework implies rigidity for the framework.* (See [9, 29].)

DEFINITION 2.2.4. *A first-order flex* $\mathbf{p}'$ *is an* equilibrium flex *if* $\mathbf{p}' \cdot \mathbf{q}' = 0$ *for all trivial first-order flexes* $\mathbf{q}'$.

Physically, a first-order flex is a velocity vector field associated with the configuration, and it turns out that an equilibrium flex is a vector field such that the linear and angular momentum is preserved (Figure 5).

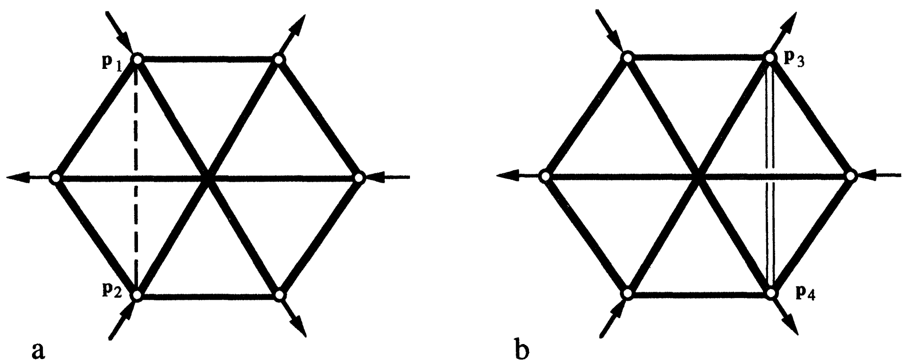

FIG. 5. *The cable* $(\mathbf{p}_1, \mathbf{p}_2)$ *in example* (a) *is unstressed because it can be shortened, while the strut* $(\mathbf{p}_3, \mathbf{p}_4)$ *in* (b) *is unstressed because it can be lengthened.*

2.3. Stresses.

DEFINITION 2.3.1. *A stress* ω *on a tensegrity framework* $G(\mathbf{p})$ *is an assignment of scalars* $\omega_{ij} = \omega_{ji}$ *to the edges of* G, *where* $\omega = (\dots, \omega_{ij}, \dots) \in \mathbb{R}^e$, *and* e *is the number of edges of* G.

A stress ω *on a tensegrity framework is a* self stress *if the following equilibrium condition holds at each vertex* i:

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0,$$

where the sum is taken over all j *with* $\{i,j\} \in E$.

A self stress ω *is called a* proper self stress *if* (a) $\omega_{ij} \geq 0$ *for cables* $\{i,j\} \in E_-$ *and* (b) $\omega_{ij} \leq 0$ *for struts* $\{i,j\} \in E_+$.
There is no condition for a bar.

A proper self stress ω *is* strict *if the inequalities in* (a) *and* (b) *are strict.*

With this notation a self stress ω is a solution to the linear equations $\omega R(\mathbf{p}) = \mathbf{0}$, and ω is a proper self stress if each of the ω_{ij} corresponding to cables and struts have the proper sign. The following is shown in [35].

THEOREM 2.3.2 (first-order stress test). *Let* $G(\mathbf{p})$ *be a tensegrity framework, where* $\{i,j\}$ *is a fixed cable or strut. There is a first-order flex* $\mathbf{p}'$ *for* $G(\mathbf{p})$ *such that*

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \neq 0,$$

which means that a cable $\{\mathbf{p}_i, \mathbf{p}_j\}$ *is shortened or a strut* $\{\mathbf{p}_i, \mathbf{p}_j\}$ *is lengthened, to first-order, if and only if for every proper self stress* ω *for* $G(\mathbf{p})$ *has* $\omega_{ij} = 0$.

It is helpful to change the presentation of a stress. Let $\omega = (\dots, \omega_{ij}, \dots) \in \mathbb{R}^e$ be a stress for $G(\mathbf{p})$. Define a v-by-v symmetric matrix, the *reduced stress matrix* $\overline{\Omega}$, by setting the (i,j) entry to be

$$\overline{\Omega}_{ij} = \begin{cases} -\omega_{ij} & \text{if} \quad i \neq j, \\ \sum_k \omega_{ik} & \text{if} \quad i = j \; . \end{cases}$$

Denoting by p_{ik} the kth coordinate of $\mathbf{p}_i \in \mathbb{R}^d$, $k = 1, \dots, d$, the expression

$$\sum_{k=1}^{d} \Big[p_{1k}, \dots, p_{vk}\Big] \overline{\Omega} \begin{bmatrix} p_{1k} \\ \vdots \\ p_{vk} \end{bmatrix}$$

is a quadratic form on the vd-dimensional space of coordinates of all points of the configuration. Reordering the coordinates by the order of the points yields

$$\left[\mathbf{p}_1^T, \ldots, \mathbf{p}_v^T\right] \Omega \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_v \end{bmatrix},$$

where $(\ )^T$ represents the transpose operation. This defines the *stress matrix* Ω, which is, up to permutation of the coordinates of $\mathbf{p}$, just k "copies" of $\overline{\Omega}$. It is easy to check that if $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{vd}$, $\omega \in \mathbb{R}^e$, then

$$\omega R(\mathbf{p})\mathbf{q} = \mathbf{p}^T \Omega \mathbf{q} = \sum_{ij} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i - \mathbf{q}_j). \tag{1}$$

Thus Ω is just the matrix of a bilinear form in the coordinates of $\mathbf{p}$ and $\mathbf{q}$, and it turns out that ω is a self stress if $\mathbf{p}^T\Omega = 0$.

3. Prestress stability.

3.1. The energy principle. If a cable is stretched, the energy in the cable increases. Similarly, if a strut is shortened, or a bar is changed in length, the energy increases. We put these together in the following energy function:

$$H^*(\mathbf{q}) = \sum_{ij} f_{ij}\left(|\mathbf{q}_j - \mathbf{q}_i|^2\right), \tag{2}$$

where

$$\begin{array}{llll} \text{for each cable} & \{i,j\}, & f_{ij} & \text{is strictly monotone increasing,} \\ \text{for each strut} & \{i,j\}, & f_{ij} & \text{is strictly monotone decreasing,} \\ \text{for each bar} & \{i,j\}, & f_{ij} & \text{has a strict minimum at } |\mathbf{p}_j - \mathbf{p}_i|^2. \end{array} \tag{3}$$

See Figure 6.

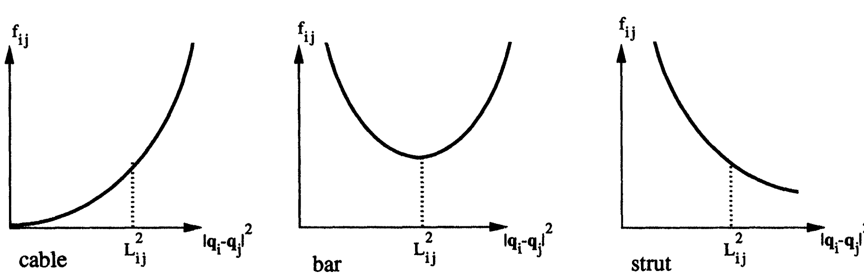

FIG. 6. *At the given configuration, the energy function is increasing on cables, a local minimum on bars, and decreasing on struts.*

We have the following Theorem 3.1.1.

THEOREM 3.1.1 (energy principle). *If such an H^* has a local minimum at $\mathbf{p}$ which is strict up to congruence in some neighborhood of $\mathbf{p}$ in $\mathbb{R}^{dv}$, then the framework $G(\mathbf{p})$ is rigid.*

Proof. Any nearby $\mathbf{q}$ with $G(\mathbf{q}) \leq G(\mathbf{p})$ will have $f_{ij}\left(|\mathbf{q}_j - \mathbf{q}_i|^2\right) \leq f_{ij}\left(|\mathbf{p}_j - \mathbf{p}_i|^2\right)$ for all members. Since this makes $H^*(\mathbf{q}) \leq H^*(\mathbf{p})$, we conclude that $\mathbf{q}$ is congruent to $\mathbf{p}$. This makes $G(\mathbf{p})$ rigid, by Definition 2.1.2(a) of rigidity. $\square$

Remark 3.1.1. It turns out that any rigid tensegrity framework $G(\mathbf{p})$ will have "some" energy functions that make H^* a minimum at $\mathbf{p}$, but they would only satisfy certain "relaxed" conditions (2). Later it will be necessary to use conditions (2) as they stand.

3.2. Stiffness matrix, stress matrix decomposition. We apply the energy principle for functions $H^*(\mathbf{p})$ which have their minimum by the second derivative test. Let equation (2) define an energy function, where each f_{ij} is twice continuously differentiable and chosen so that the first derivative $f'_{ij}(|\mathbf{p}_j - \mathbf{p}_i|^2) = \omega_{ij}$ for each member, $\omega_{ij} \neq 0$ are scalars for all cables and struts, and the second derivative $f''(|\mathbf{p}_j - \mathbf{p}_i|^2) = c_{ij} > 0$ for all members. Note that (3) insures that ω is a strict and proper stress. We suppose that $\mathbf{p}$ is a fixed particular configuration.

To find a local minimum, the first step is to find a critical point. Note that $\mathbf{p}$ is a critical point for H^* if and only if the directional derivative at $\mathbf{p}$ is zero for all directions $\mathbf{p}^*$. Hence, we compute the directional derivative of this energy function in the direction $\mathbf{p}^*$ starting from $\mathbf{p}$.

Let $\mathbf{p}(t) = \mathbf{p} + t\mathbf{p}^*$. So $D_t(\mathbf{p}(t)) = \mathbf{p}^*$, where D_t represents differentiation with respect to t. We compute the derivative of $H^*(\mathbf{p}(t))$ with respect to t,

$$D_t\big(H^*(\mathbf{p}(t))\big) = \sum_{ij} f'_{ij}\big(|\mathbf{p}_j(t)|^2\big)\big[2(\mathbf{p}_j(t) - \mathbf{p}_i(t)) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)\big].$$

The directional derivative is then the above function evaluated at $t = 0$.

$$D_t\big(H^*(\mathbf{p}(t))\big)\Big|_{t=0} = \sum_{ij} f'_{ij}\big(|\mathbf{p}_j - \mathbf{p}_i|^2\big)\big[2(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)\big].$$

Since $f'_{ij}(|\mathbf{p}_j - \mathbf{p}_i|^2) = \omega_{ij}$, using (1) we have

$$D_t\big(H^*(\mathbf{p}(t))\big)\Big|_{t=0} = 2\sum_{ij} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*) = 2\omega R(\mathbf{p})\mathbf{p}^*.$$

By the above calculation $\mathbf{p}$ is a critical point for H^* if and only if $2\omega R(\mathbf{p})\mathbf{p}^* = 0$ for all $\mathbf{p}^*$ if and only if $\omega R(\mathbf{p}) = 0$ if and only if ω is a self stress for $G(\mathbf{p})$.

If ω is a self stress on $G(\mathbf{p})$, we use the second derivative test to check whether H^* has a strict minimum at $\mathbf{p}$, up to congruences. For each direction $\mathbf{p}^*$ we calculate the second derivative along the path $\mathbf{p}(t)$ and then evaluate when $t = 0$ and $\mathbf{p}(0) = \mathbf{p}$.

$$\begin{aligned} D_t^2[H^*(\mathbf{p}(t))]\Big|_{t=0} &= \sum_{ij} f''_{ij}(|\mathbf{p}_j - \mathbf{p}_i|^2)[2(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)]^2 \\ &\quad + \sum_{ij} f'_{ij}(|\mathbf{p}_j - \mathbf{p}_i|^2)2|\mathbf{p}_j^* - \mathbf{p}_i^*|^2 \\ &= \sum_{ij} 4c_{ij}[(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)]^2 \\ &\quad + \sum_{ij} 2\omega_{ij}(\mathbf{p}_j^* - \mathbf{p}_i^*) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*). \end{aligned}$$

The constant c_{ij} is often called the *stiffness coefficient* for the member $\{i, j\}$ and in physics it is normally a function of the Young modulus of the material forming the member, the rest length of the member, and the cross-sectional area of the member.

The rigid congruences of $\mathbf{p}$ form a submanifold in the space of all configurations, and it is clear that H^* is constant on this set. Thus when $\mathbf{p}^*$ is a trivial infinitesimal flex of $G(\mathbf{p})$ it is easy to see that both the first and second derivative of H^* along a path in the direction of $\mathbf{p}^*$ are zero. (This can also be seen by a direct calculation.) We conclude with Proposition 3.2.1.

PROPOSITION 3.2.1. *The energy function H^* has a strict local minimum, up to congruence, if the quadratic form*

$$H(\mathbf{p}^*) = \sum_{ij} 2\omega_{ij}(\mathbf{p}_j^* - \mathbf{p}_i^*) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*) + \sum_{ij} 4c_{ij}[(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)]^2,$$

regarded as a function of the coordinates of $\mathbf{p}^*$, satisfies $H(\mathbf{p}^*) \geq 0$ for all $\mathbf{p}^*$, and $H(\mathbf{p}^*) = 0$ if and only if $\mathbf{p}^*$ is a trivial infinitesimal flex of $G(\mathbf{p})$. In other words H is positive definite on any complement of the trivial infinitesimal flexes.

Note that each f_{ij} for each cable and strut is a monotone function with nonzero derivative. For that reason we know that the self stress is strict and proper. In other words the self stress ω is nonzero with the correct sign on each cable and strut.

What we have done is calculate the Hessian H as a quadratic form for the function H^*. Note, however, that H itself is not the sum of energy functionals of the members of G. Thus the energy principle does not apply directly to H but applies only because H^* is locally approximated by H.

We now rewrite the formula for H in terms of the rigidity matrix. Let C denote the dv-by-dv diagonal matrix with entries c_{ij}, where its rows and columns correspond to the members of G. If $\{i, j\}$ is a member of G, then the corresponding diagonal entry is c_{ij}.

$$\begin{aligned} H &= 2\omega R(\mathbf{p}^*)\mathbf{p}^* + 4(\mathbf{p}^*)^T R(\mathbf{p})^T C R(\mathbf{p})\mathbf{p}^* \\ &= 2(\mathbf{p}^*)^T \Omega \mathbf{p}^* + 4(\mathbf{p}^*)^T R(\mathbf{p})^T C R(\mathbf{p})\mathbf{p}^* \\ &= (\mathbf{p}^*)^T [2\Omega + 4R(\mathbf{p})^T C R(\mathbf{p})]\mathbf{p}^*. \end{aligned}$$

In structural engineering, the matrix $R(\mathbf{p})^T C R(\mathbf{p})$ is called the *stiffness matrix* of the framework, Ω is the *geometric stiffness matrix* or the *stress matrix*, and $2[\Omega + 2R(\mathbf{p})^T C R(\mathbf{p})]$ is the *tangential stiffness matrix.* The matrix $R(\mathbf{p})^T C R(\mathbf{p})$ is clearly positive semidefinite with the first-order flexes in its kernel. If the framework is infinitesimally rigid, then $\Omega = 0$ can be used in the above. The interesting cases for us occur when there are some nontrivial infinitesimal flexes and some nonzero self stresses.

Remark 3.2.1. The condition of Proposition 3.2.1 is a particular kind of stability which corresponds to the engineer's concept of first-order stiffness [30]. If we take gradients of this energy function, we find that if the force at the ith joint is $\mathbf{F}_i$, then this set of forces is resolved, at first-order, by a displacement $\mathbf{p}^*$ of the joints, where

$$\mathbf{F}_i = \Delta H = 2\sum_j \omega_{ij}(\mathbf{p}_j^* - \mathbf{p}_i^*) + 4\sum_j c_{ij}\left[(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_j^* - \mathbf{p}_i^*)\right](\mathbf{p}_j - \mathbf{p}_i).$$

Regarding the forces as one column vector $\mathbf{F} = (\mathbf{F}_1^T, \ldots, \mathbf{F}_v^T)^T$, we get

$$\mathbf{F} = \left[2\Omega + 4R(\mathbf{p})^T C R(\mathbf{p})\right]\mathbf{p}^*.$$

All equilibrium loads are resolved if and only if the matrix $[2\Omega + 4R(\mathbf{p})^T C R(\mathbf{p})]$ is invertible when restricted to the orthogonal complement of the trivial motions. In this case, the deformation $\mathbf{p}^*$ resolves the load $[2\Omega + 4R(\mathbf{p})^T CR(\mathbf{p})]\mathbf{p}^*$. This is a feasible physical response of the structure, corresponding to positive work by the force, if and only if $\mathbf{p}^*$ is in the same direction as $\mathbf{F}$ or $\mathbf{p}^* \cdot \mathbf{F} \geq 0$, with equality only if $\mathbf{F} = 0$. This is a restatement of the fact that H is positive definite on a complement of the trivial motions.

If H is only positive semidefinite on a complement of the trivial infinitesimal motions, then there is a direction $\mathbf{p}^*$ for which there is no change in the energy up to the second derivative. It may turn out that there will still be third- or higher-order effects of a real energy which produce rigidity. However, if H is indefinite there is a direction $\mathbf{p}^*$ in which the energy strictly decreases.

Remark 3.2.2. Looking at this discussion in terms of the physics we can also understand the rule of the energy functions. If H strictly decreases in the direction $\mathbf{p}^*$, then any smooth energy function $\mathbf{H}^*$ with the equilibrium stresses ω_{ij} as the first derivatives and the c_{ij} as the second derivatives for each member will have a local maximum at $\mathbf{p}$ along the line $\mathbf{p} + t\mathbf{p}^*$. If released with this energy in the direction of $\mathbf{p}^*$, the framework will continue to move while seeking a smaller overall energy. It is also possible to interpret the behavior of the framework in terms of Lagrange multipliers.

3.3. Definition of prestress stability. Recall that if Q is a quadratic form on a finite-dimensional vector space, then there is a symmetric matrix A such that $Q(\mathbf{p}) = \mathbf{p}^T A\mathbf{p}$, where $\mathbf{p}$ is a vector written as coordinates with respect to some basis of the vector space. If $Q(\mathbf{p}) \geq 0$ for all vectors $\mathbf{p}$, then Q (or A) is called *positive semidefinite.* The *zero set* of Q is the set of vectors $\mathbf{p}$ such that $Q(\mathbf{p}) = 0$. If Q is positive semidefinite and the zero set consists of just the zero vector, then Q is called *positive definite.*

DEFINITION 3.3.1. *We say a tensegrity framework $G(\mathbf{p})$ is* prestress stable *if there is a proper self stress ω and nonnegative scalars c_{ij} (where $\{i,j\}$ is a member of G) such that the energy function regarded as quadratic form in the coordinates of* $\mathbf{p}^*$

$$H = \sum_{ij} \omega_{ij}(\mathbf{p}_i^* - \mathbf{p}_j^*)^2 + \sum_{ij} c_{ij}\left[(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^* - \mathbf{p}_j^*)\right]^2$$

is positive semidefinite, $c_{ij} = 0$ when $\omega_{ij} = 0$ and $\{i,j\}$ is a cable or strut, and only the trivial infinitesimal flexes of $\mathbf{p}$ are in the kernel of H. In this case we say ω stabilizes *$G(\mathbf{p})$. The c_{ij} are called the stiffness coefficients as in* §3.2. *We have dropped the constants* 2 *and* 4 *that appeared in* §3.2 *for simplicity.*

Remark 3.3.1. If, for some member $\{i,j\}$, H has $\omega_{ij} = 0$ but $c_{ij} > 0$, then for the energy principle to apply, the member must be a bar. Imagine a single cable, which has no nonzero self stress. It is certainly not rigid, but it would be prestress stable with the zero self stress if we did not insist that $c_{ij} = 0$ when $\omega_{ij} = 0$. This is why the definition insists that each cable or strut which appears with a zero stress does not appear in the formula at all.

Thus if any $G(\mathbf{p})$ is prestress stable, stabilized at ω, then we might as well change $G(\mathbf{p})$ to have struts only for those members where $\omega_{ij} < 0$ and cables only for those members where $\omega_{ij} > 0$, deleting any cables or struts with a zero stress.

Note also that if we regard unstressed members as not being in G, then increasing any c_{ij} keeps H positive definite (if it already was) and so we may assume they are all equal to each other, assuming that we are only interested in recognizing the rigidity of the framework. Thus if ω stabilizes $G(\mathbf{p})$ with all cables and struts stressed, with stiffness coefficients c_{ij}, then $\frac{\omega}{\max\{c_{ij}\}}$ stabilizes $G(\mathbf{p})$ with all the stiffness coefficients equal to one.

PROPOSITION 3.3.2. *If a tensegrity framework $G(\mathbf{p})$ is prestress stable for a self stress ω, then $G(\mathbf{p})$ is rigid.*

Proof. The positive definite property of H guarantees a strict local minimum modulo trivial first-order flexes of $\mathbf{p}$. Since $\omega_{ij} \neq 0$ for all the cables and struts of G that appear in H^*, we have the strictly monotone property of their energy functions, near $\mathbf{p}$. Thus the energy principle applies to show $G(\mathbf{p})$ is rigid. □

Remark 3.3.2. We will see, by Theorem 4.4.1, that if $G(\mathbf{p})$ is prestress stable, then $G(\mathbf{p})$ is second-order rigid and, by Theorem 4.3.1, $G(\mathbf{p})$ is rigid. However, the present proof is much simpler since it is a direct application of the energy principle. □

PROPOSITION 3.3.3. *If a tensegrity framework $G(\mathbf{p})$ is first-order rigid, then it is prestress stable.*

Proof. The underlying bar framework $\overline{G}(\mathbf{p})$ is certainly first-order rigid; thus $R(\mathbf{p})^T R(\mathbf{p})$ has only the trivial infinitesimal flexes in its kernel. In other words $R(\mathbf{p})^T R(\mathbf{p})$ is positive definite on the equilibrium infinitesimal motions perpendicular to the trivial infinitesimal motions in $\mathbb{R}^{vd}$. By [29] there is a proper self stress ω which is nonzero on each cable and strut. By choosing ω sufficiently small, then $\Omega + R(\mathbf{p})^T R(\mathbf{p})$ also will be positive definite on the equilibrium infinitesimal flexes restricted to the compact unit sphere. Thus $G(\mathbf{p})$ is prestress stable. □

Often it is convenient to assume that a proper stress ω for $G(\mathbf{p})$ is *strict*; i.e., $\omega_{ij} \neq 0$ for every cable or strut. If we are willing to consider subframeworks of $G(\mathbf{p})$, we need only consider strict self stresses for prestress stability.

Remark 3.3.3. Pellegrino and Calladine [28] use a different analysis of the rigidifying effect of a prestress. (See also [4].) Given a framework $G(\mathbf{p})$ with a self stress ω and a set of generators $\mathbf{p}'_1, \ldots, \mathbf{p}'_k$ for a complementary space of nontrivial first-order flexes, they add k new rows to the rigidity matrix $\omega R(\mathbf{p}'_1), \omega R(\mathbf{p}'_2), \ldots, \omega R(\mathbf{p}'_k)$. If this extended matrix $R^*(\mathbf{p}, \omega)$ has rank $vd - \frac{d(d+1)}{2}$, they say that the prestress ω "stiffens" the framework, modulo the positive definiteness of an unspecified matrix.

If $R^*(\mathbf{p}, \omega)$ does not have rank $vd - d(d+1)/2$ (assuming the vertices span $\mathbb{R}^d$), then there is a nontrivial first-order flex $\mathbf{p}' = \sum \alpha_i \mathbf{p}_j$ satisfying $\omega R(\mathbf{p}'_i)\mathbf{p}' = 0$ for all $i = 1, \ldots, k$. Thus $\omega R(\mathbf{p}')\mathbf{p}' = 0$ and $G(\mathbf{p})$ is certainly not prestress stable for this self stress ω. In fact, it is easy to see that their condition for stiffening is equivalent to requiring that the rank of $\alpha^2\Omega + R(\mathbf{p})^T R(\mathbf{p})$ be $vd - \frac{d(d+1)}{2}$ for all sufficiently small α (assuming that the affine span of $\mathbf{p}$ is at least $(d-1)$-dimensional). The matrix that they have in mind, which must be positive definite, must be equivalent to $\Omega + R(\mathbf{p})^T R(\mathbf{p})$ restricted to some space complementary to the space of trivial first-order flexes. If no positive definiteness is required, many mechanisms, such as collinear parallelograms, would be declared "stiff."

On the other hand, if there is a one-dimensional space of equilibrium first-order flexes, then we will see that prestress stability, the rank of $R^*(\mathbf{p}) = dv - \frac{d(d+1)}{2}$, and second-order rigidity will all coincide. It is interesting that in the paper [28], most examples have a one-dimensional space of equilibrium flexes. See [5] for corrections, as well as [22–26] for a discussion of the problem of how to do the second-order analysis.

3.4. Interpretation in terms of the stress matrix and quadratic forms. We now present some simple facts about quadratic forms that we will find useful later.

LEMMA 3.4.1. *Let Q_1 and Q_2 be two quadratic forms on a finite-dimensional real (inner product) vector space. Suppose that Q_2 is positive semidefinite with zero* *set K, and Q_1 is positive definite on K. Then there is a positive real number α such that $Q_1 + \alpha Q_2$ is positive definite.*

Proof. Let X denote the compact set (a sphere) of unit vectors in the inner product space.

$$X = \{\mathbf{p} \mid \mathbf{p} \cdot \mathbf{p} = 1\}.$$

Recall that the zero set of Q_2 is

$$K = \{\mathbf{p} \mid Q_2(\mathbf{p}) = 0\}.$$

Let $K \cap X \subset N \subset X$ be an open neighborhood of K in X such that

$$Q_1(\mathbf{p}) > 0 \qquad \text{for all} \qquad \mathbf{p} \in N.$$

Such an open set N exists since $K \cap X$ is compact, Q_1 is positive on $K \setminus \{0\} \supset K \cap X$, and thus Q_1 restricted to $K \cup X$ must have a positive minimum m. Then take $N = \{\mathbf{p} \in X | Q_1(\mathbf{p}) > \frac{m}{2}\}$. For similar reasons there are real constants c_1, c_2 such that

$$Q_1(\mathbf{p}) > c_1 \qquad \text{for all} \qquad \mathbf{p} \in X,$$
$$Q_2(\mathbf{p}) \geq c_2 > 0 \qquad \text{for all} \qquad \mathbf{p} \in X \setminus N.$$

Then we define $\alpha = \frac{|c_1|}{c_2}$. We calculate for $\mathbf{p} \in N \cap X$,

$$Q_1(\mathbf{p}) + \alpha Q_2(\mathbf{p}) \geq Q_1(\mathbf{p}) > 0.$$

For $\mathbf{p} \in X \setminus N$,

$$Q_1(\mathbf{p}) + \alpha Q_2(\mathbf{p}) = Q_1(\mathbf{p}) + \frac{|c_1|}{c_2} Q_2(\mathbf{p}) > c_1 + |c_1| \geq 0.$$

Thus $Q_1 + \alpha Q_2$ is positive on all of X, and hence it is positive definite. □

Remark 3.4.1. Note that Q_1 is allowed to be an indeterminate term on the whole vector space. It is only required to be positive definite on K. Note also, for any quadratic form given by a symmetric matrix A, where $Q(\mathbf{p}) = \mathbf{p}^T A \mathbf{p}$, certainly the kernel of A is contained in the zero set of Q. If $A\mathbf{p} = \mathbf{0}$, then $Q(\mathbf{p}) = \mathbf{p}^T A \mathbf{p} = 0$. However, the converse is not true unless A is positive semidefinite. In particular the converse is true when $A = R(\mathbf{p})^T C R(\mathbf{p})$, the stiffness matrix. Then

$$(\mathbf{p}^*)^T R(\mathbf{p})^T C R(\mathbf{p}) \mathbf{p}^* = \sum_{ij} c_{ij} \left[(\mathbf{p}_j^* - \mathbf{p}_i^*) \cdot (\mathbf{p}_j - \mathbf{p}_i)\right]^2 \geq 0,$$

and the kernel of A, assuming all the $c_{ij} > 0$, is the same as the zero set of its quadratic form. It is also clear from the above that the kernel of A is precisely the space of all first-order flexes of the corresponding bar framework.

For any tensegrity framework $G(\mathbf{p})$ we recall that $\overline{G}(\mathbf{p})$ is the corresponding bar framework with all members converted to bars. We denote

$$\overline{I} = I(\overline{G}(\mathbf{p})) = \left\{\mathbf{p}' \in \mathbb{R}^{vd} \mid (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i' - \mathbf{p}_j') = 0, \quad \{i, j\} \quad \text{a member of} \quad G\right\}.$$

In other words $\overline{I}$ is the space of first-order flexes of $\overline{G}(\mathbf{p})$, a linear subspace of $\mathbb{R}^{vd}$. Recall also that $T_\mathbf{p}$ is the space of trivial first-order flexes at $\mathbf{p}$. (So $T_\mathbf{p} \subset \overline{I}$.) Note that if $G(\mathbf{p})$ has a strict proper self stress, then the first-order stress test implies that $I = \overline{I}$.

We now give a way of checking the prestress stability of a tensegrity framework which is useful for calculations later. Recall a stress ω is strict if it is nonzero on every cable and strut.

PROPOSITION 3.4.2. *A tensegrity framework $G(\mathbf{p})$ is prestress stable for the strict proper self stress ω if and only if the associated stress matrix Ω is positive definite on any subspace $K \subset \overline{I}$ complementary to $T_{\mathbf{p}}$.*

Proof. Assume that $[\Omega + R(\mathbf{p})^T CR(\mathbf{p})]$ is positive semidefinite with only $T_{\mathbf{p}}$ as the kernel, where C is a diagonal matrix with positive stiffness coefficients. Let $\mathbf{p}' \in \overline{I}$. Then $R(\mathbf{p})\mathbf{p}' = 0$ and so

$$0 \leq (\mathbf{p}')^T\big[\Omega + R(\mathbf{p})^T CR(\mathbf{p})\big]\mathbf{p}' = (\mathbf{p}')^T\Omega\mathbf{p}',$$

with equality if and only if $\mathbf{p}' \in T_{\mathbf{p}}$. Thus on K, Ω is positive definite.

Before proving the converse we remark that if $\mathbf{p}'$ is any trivial first-order flex at $\mathbf{p}$, then by Definition 2.2.2 there is a d-by-d (skew symmetric) matrix S and a vector $\mathbf{t} \in \mathbb{R}^d$ such that $\mathbf{p}'_i = S\mathbf{p}_i + \mathbf{t}$, $i = 1, \ldots, v$. Thus

$$\Omega\mathbf{p}' = \begin{bmatrix} \vdots \\ \sum_j \omega_{ij}(\mathbf{p}'_i - \mathbf{p}'_j) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \sum_j \omega_{ij}(S\mathbf{p}'_i - S\mathbf{p}'_j) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ S\sum_j \omega_{ij}(\mathbf{p}'_i - \mathbf{p}'_j) \\ \vdots \end{bmatrix} = 0.$$

Now suppose Ω is positive definite on K. Let $\mathbf{p}' \in \mathbb{R}^{vd}$ be arbitrary. Write $\mathbf{p}' = \mathbf{p}_T + \mathbf{p}_K + \mathbf{p}_E$, where $\mathbf{p}_T \in T_{\mathbf{p}}$, $\mathbf{p}_K \in K$, and $\mathbf{p}_e \in E$, the (orthogonal) complement of $\overline{I}$ in $\mathbb{R}^{vd}$. Since $\Omega\mathbf{p}_T = 0 = R(\mathbf{p})\mathbf{p}_T$, we have

$$(\mathbf{p}')^T\big[\Omega + R(\mathbf{p})^T CR(\mathbf{p})\big]\mathbf{p}' = (\mathbf{p}_K + \mathbf{p}_E)^T\big[\Omega + R(\mathbf{p})^T CR(\mathbf{p})\big](\mathbf{p}_K + \mathbf{p}_E). \tag{4}$$

By Remark 3.4.1, the kernel of $R(\mathbf{p})^T CR(\mathbf{p})$ is $\overline{I} = T_{\mathbf{p}} + K$. Now apply Lemma 3.4.1 to the inner product space $K + E$, where Q_1 is the quadratic form corresponding to Ω, and Q_2 is the quadratic form corresponding to $R(\mathbf{p})^T CR(\mathbf{p})$. The kernel of Q_2 is precisely K, and we have assumed that Q_1 is positive definite on K. Thus by possibly multiplying C by a positive constant we can assume that $\Omega + R(\mathbf{p})^T CR(\mathbf{p})$ is positive definite on $K + E$. By (4), this implies that $\Omega + R(\mathbf{p})^T CR(\mathbf{p})$ is positive semidefinite with kernel $T_{\mathbf{p}}$, and $G(\mathbf{p})$ is prestress stable. □

3.5. Examples of prestress stable frameworks. The following are examples of tensegrity frameworks that are prestress stable, but not first-order rigid. Thus the converse of Proposition 3.3.3 is false.

In Figure 7a there is a self stress such that the outside members have a positive self stress. A nontrivial first-order flex is given so that one can apply Proposition 3.4.2. Figure (7b) is stable by a result in [7] concerning spider webs. In Figure 7b it is the inside members we can choose to be positive. In both of these examples there is a strict proper self stress such that the indicated first-order flex is nonzero only on vertices of members that have a positive self stress, and the given first-order flex generates a complementary space to the trivial flexes in the space of all first-order

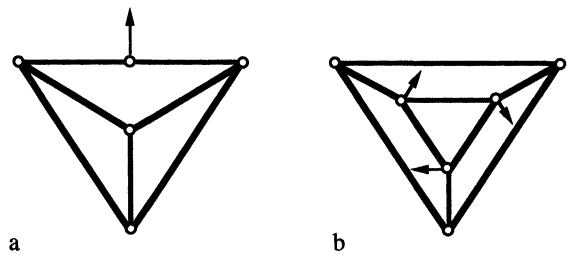

FIG. 7. *Examples of prestress stable bar frameworks that are not first-order rigid. (Nontrivial first-order motions are indicated).*

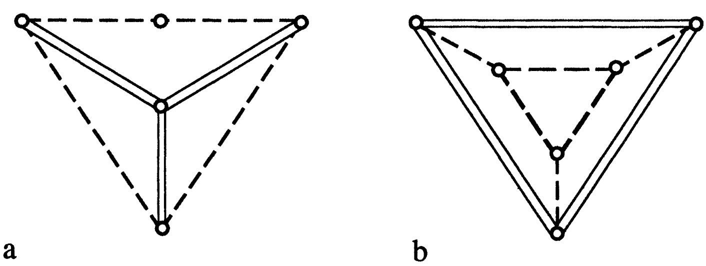

FIG. 8. *Corresponding examples of prestress stable tensegrity frameworks, where the bars have been replaced by cables and struts following the stabilizing self stress.*

flexes. Hence the stress matrix on this space is positive definite and the framework is prestress stable.

Following Remark 3.3.1, we can change appropriate members to be cables or struts and preserve prestress stability, as in Figure 8.

Can a framework be rigid but not prestress stable? Consider the next two examples. Note that any first-order rigid bar framework with no self stress at all must have **0** as its stabilizing self stress. But the example in Figure 9a has a self stress on part of the framework, and the bar can have no stress other than 0. It still is prestress stable.

However, the example of Figure 9b is not prestress stable, because the short horizontal cable and the horizontal strut are unstressed and the framework becomes nonrigid upon their removal. Nevertheless the framework is clearly still rigid. In fact, built with all bars, it is prestress stable.

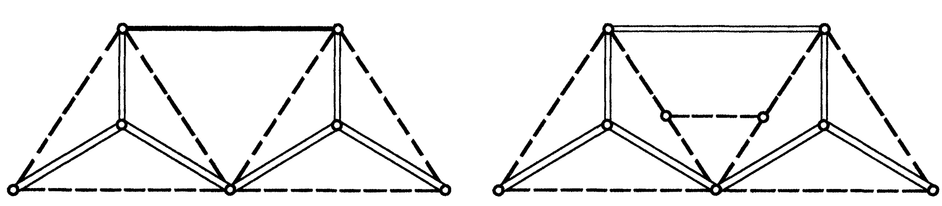

FIG. 9. *Example* (a) *is first-order rigid, and thus prestress stable. Example* (b) *is rigid but not prestress stable.*

Suppose we fix (or pin) certain vertices. For our purposes this can be accomplished by adding some first-order framework that contains these vertices and none of the other original vertices. If the original framework has only cables with a proper self stress (where the equilibrium condition only holds at the vertices that are not fixed), then we say that the framework is a *spider web*. There is a discussion of this in [9] and [36] as well as [7]. It is clear that spider webs are prestress stable. See Figure 10.

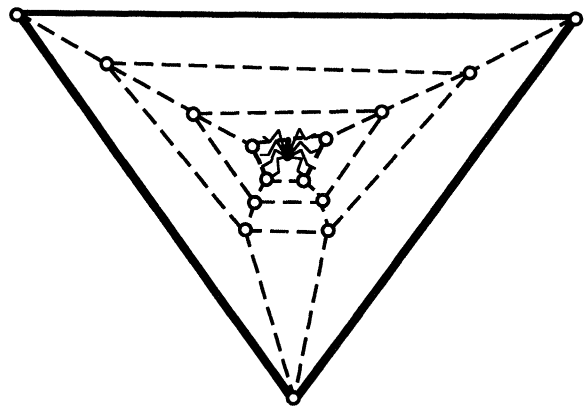

FIG. 10. *This example is a spider web and thus prestress stable.*

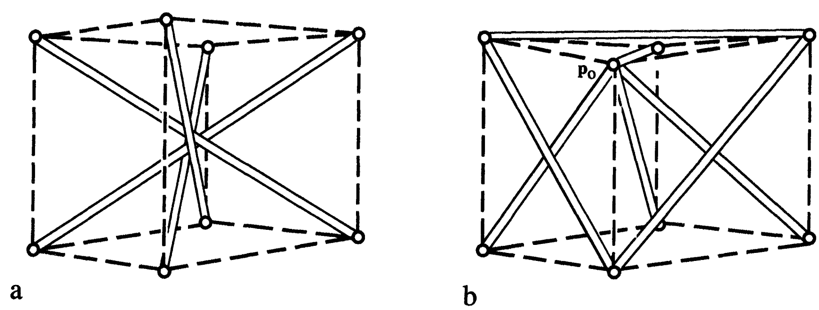

FIG. 11. *Two prestress stable tensegrity frameworks in three-space.*

In three-space there are many examples of prestress stable but not necessarily infinitesimally rigid tensegrity frameworks, such as in Figure 11.

Figure 11a is a regular cube with its main diagonals as struts and its edges as cables. Examples such as in 11b can be obtained by taking any convex polyhedron with a triangular face (in this case a cube with its near upper corner truncated), choosing a point $\mathbf{p}_0$ close to that face, and joining $\mathbf{p}_0$ to all the other vertices of the polyhedron with struts and making all the edges of the polyhedron cables except the triangle which is composed of struts. Again it turns out that there is a strict proper self stress ω, and Ω is positive semidefinite with only the affine motions in the kernel. This example is closely related to three-dimensional spider webs. (See [36].)

Another three-dimensional example can be obtained by taking a tetrahedron and putting struts on each of the six edges and some prestressed spider web on the inside of each triangular face, as in Figure 12.

Each face is prestress stable even in $\mathbb{R}^3$, so the sum of the energy functions is positive semidefinite with only the first-order flexes that are trivial on each face in the kernel. But since the tetrahedron itself is first-order rigid, flexes which are trivial on each face are trivial on the whole framework. The framework is prestress stable.

3.6. Roth's conjecture. Suppose $G(\mathbf{p})$ has its points as the vertices of a convex polygon in the plane. If the exterior edges of G are cables, and all of the other members are struts, say, then we call it a *cable-strut polygon* or a *c-s polygon.*

It follows from [7] that any *c-s* polygon that has a proper self stress $\omega \neq 0$ has Ω as positive semidefinite with only the affine motions in the kernel. It turns out that the affine motions are never a first-order flex of such a framework [36]. Thus such an

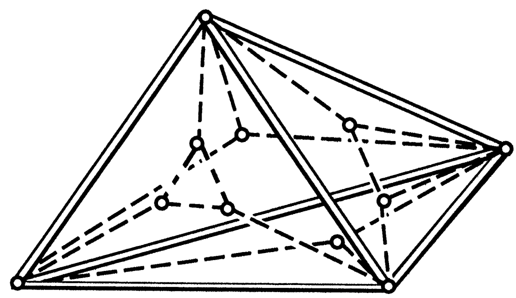

FIG. 12. *A prestress stable tetrahedron with cables in the faces and struts on the edges.*

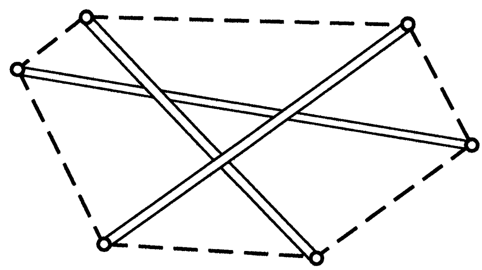

FIG. 13. *A cable-strut (c-s) polygon.*

ω also stabilizes $G(\mathbf{p})$, and thus $G(\mathbf{p})$ is prestress rigid. Note that such a *c-s* polygon need not be first-order rigid. In the case of Figure 13, the six vertices lie on an ellipse, and by a classical result (see [3] and [34]), the framework has a strict proper self stress and a nontrivial first-order flex. So this framework is prestress stable, but it is not first-order rigid.

On the other hand Roth conjectured that any rigid *b-c* polygon (bars on the outside, cables inside) was first-order rigid. In §6.2 we show that this is true. In Figure 14 we show three examples of nonrigid *b-c* hexagons with first-order flex indicated.

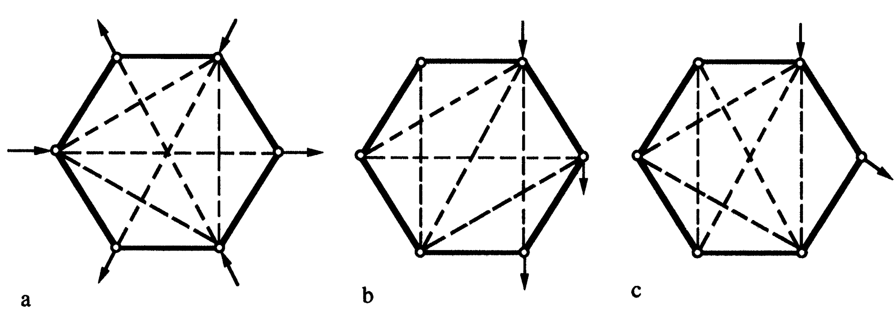

FIG. 14. *Examples of bar-cable (b-c) polygons, with nontrivial first-order motions indicated.*

For the three frameworks in Figure 14, the six vertices of each configuration form a regular hexagon. For the first two cases, there are certain other configurations for the same tensegrity graph (but still a convex *b-c* polygon) such that the framework is rigid. This is not true for the last case, though. The reader is invited to find the continuous nontrivial flex of each of these frameworks. But see §6.2 for a proof that the flex exists.

Following [29] we see that there are many cabling schemes that guarantee first-order rigidity, as in Figure 15.

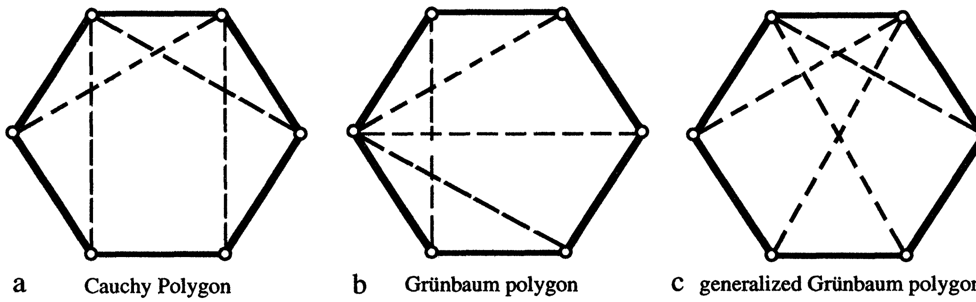

FIG. 15. *Examples of first-order rigid bar-cable polygons.*

Consider a *b-c* hexagon with four cables. One can see that either it contains one of the examples of Figure 15 and thus is always first-order rigid, or it is contained in one of the examples of Figure 14 and thus, at least for some convex configurations, it is not rigid.

4. Second-order rigidity for tensegrity frameworks.

4.1. The definition of second-order rigidity. Our definition of second-order rigidity for tensegrity frameworks comes from differentiating the equation $|\mathbf{p}_j(t) - \mathbf{p}_i(t)|^2 = L_{ij}^2$ twice. This generalizes the previous definition of second-order rigidity for bar frameworks in [7].

DEFINITION 4.1.1. *A* second-order flex $(\mathbf{p}', \mathbf{p}'')$ *for a tensegrity framework* $G(\mathbf{p})$ *is a solution to the following constraints, where* $\mathbf{p}'$ *and* $\mathbf{p}''$ *are configurations in* $\mathbb{R}^d$ *(each regarded as an associated pair of vectors* $\mathbf{p}_i'$ *and* $\mathbf{p}_i''$ *to each point* $\mathbf{p}_i$*).*

(a) *For* $\{i,j\}$ *a bar,* $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i'-\mathbf{p}_j') = 0$ *and* $|\mathbf{p}_i'-\mathbf{p}_j'|^2+(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i''-\mathbf{p}_j'') = 0$.

(b) *For* $\{i,j\}$ *a cable, either* $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i'-\mathbf{p}_j') < 0$ *or* $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i'-\mathbf{p}_j') = 0$ *and* $|\mathbf{p}_i'-\mathbf{p}_j'|^2+(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i''-\mathbf{p}_j'') \leq 0$.

(c) *For* $\{i,j\}$ *a strut, either* $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i'-\mathbf{p}_j') > 0$ *or* $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i'-\mathbf{p}_j') = 0$ *and* $|\mathbf{p}_i'-\mathbf{p}_j'|^2+(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i''-\mathbf{p}_j'') \geq 0$.

A tensegrity framework is second-order rigid *if all second-order flexes* $(\mathbf{p}', \mathbf{p}'')$ *have* $\mathbf{p}'$ *as a trivial first-order flex. Otherwise* $G(\mathbf{p})$ *is* second-order flexible.

Figure 16 shows second-order flexes of some tensegrity frameworks (double arrows for $\mathbf{p}''$, single arrow for $\mathbf{p}'$). The flex in Figure 16a is nontrivial for $\mathbf{p}'$. The flex in Figure 16b is trivial for $\mathbf{p}'$, but $(\mathbf{p}', \mathbf{p}'')$ is not the first and second derivative of a rigid motion of $\mathbf{p}$. The flex in Figure 16c is the derivative of a rigid motion of $\mathbf{p}$. Figure 16d shows a nontrivial second-order flex in a framework which is still rigid.

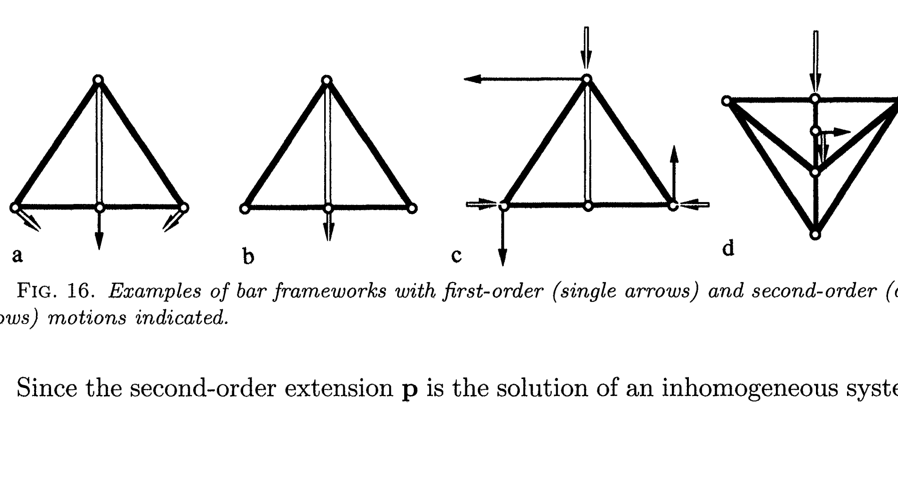

FIG. 16. *Examples of bar frameworks with first-order (single arrows) and second-order (double arrows) motions indicated.*

Since the second-order extension $\mathbf{p}$ is the solution of an inhomogeneous system of equations and inequalities, we can add any solution $\mathbf{q}'$ of the corresponding homogeneous system to $\mathbf{p}''$.

PROPOSITION 4.1.2. *If $(\mathbf{p}', \mathbf{p}'')$ is a second-order flex of a tensegrity framework $G(\mathbf{p})$ and $\mathbf{q}'$ is any first-order flex of $G(\mathbf{p})$, then $(\mathbf{p}', \mathbf{p}'' + \mathbf{q}')$ is a second-order flex of $G(\mathbf{p})$.*

Proof. Assume that for each cable $\{i, j\}$ (respectively, each bar, strut) with $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$,

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) \leq 0 \quad (\text{respectively}, \ = 0, \ \geq 0)$$

and

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}'_i - \mathbf{q}'_j) \leq 0 \quad (\text{respectively}, \ = 0, \ \geq 0).$$

Therefore by adding these inequalities we obtain

$$(\mathbf{p}'_i - \mathbf{p}'_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + (\mathbf{p}_i - \mathbf{p}_j) \cdot [(\mathbf{p}''_i + \mathbf{q}'_i) - (\mathbf{p}''_j + \mathbf{q}'_j)] \leq 0 \quad (\text{respectively}, \ = 0, \ \geq 0).$$

These are the inequalities required for Definition 4.1.1. □

If we add a multiple of $\mathbf{p}'$ itself to any second-order extension $\mathbf{p}$ we can make the second-order extension also satisfy the second-order inequalities even for those members with $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \neq 0$.

4.2. Trivial higher-order motions. In the following $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots)$, $0 \leq t \leq 1$ will be an analytic path in the configuration space, so $\mathbf{p}_i(t) \in \mathbb{R}^d$, $i = 1, \ldots, v$. Following [6] we say that $\mathbf{p}(t)$ is a trivial flex if

$$\mathbf{p}(t) = T(t)\mathbf{p}(0) = (T(t)\mathbf{p}_1(0), T(t)\mathbf{p}_2(0), \ldots, T(t)\mathbf{p}_v(0)),$$

where $T(0) = I$, $T(t)$ is a rigid motion of $\mathbb{R}^d$, and $T(t)$ is an analytic function of t. In particular, this means that we can write $T(t)\mathbf{p}_i = A(t)\mathbf{p}_i + \mathbf{b}(t)$, $i = 1, \ldots, v$, where $A(t)$ is an orthogonal matrix, $\mathbf{b}(t) \in \mathbb{R}^d$, and all the coordinates are real analytic functions of t.

We next say that $\mathbf{p}', \mathbf{p}'', \ldots, \mathbf{p}^{(k)}$, each $\mathbf{p}^{(j)} \in \mathbb{R}^{dv}$ for $j = 1, \ldots, k$, is a *k-trivial* flex of $\mathbf{p}$ if there is a trivial flex $\mathbf{p}(t)$ such that

$$D_t^j \mathbf{p}(t)\Big|_{t=0} = \mathbf{p}^{(j)} \qquad \text{for } \ j = 1, 2, \ldots, k.$$

Recall that D_t^j represents the jth derivative with respect to t.

It is easy to check that if $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is a k-trivial flex of any framework $G(\mathbf{p})$ in $\mathbb{R}^d$, then the analogue of equations (a) in Definition 4.1.1 holds for $j = 1, \ldots, k$, since clearly edge lengths are preserved up to any order k.

In fact we will give a fairly explicit description of k-trivial flexes. Although this description is long, it seems important to be precise, given the long history of confusion in this area.

We already know that 1-trivial flexes $\mathbf{p}'$ are given by

$$\mathbf{p}'_i = S\mathbf{p}_i + \mathbf{b}', \quad i = 1, \ldots, v,$$

where $S = -S^T$ is a d-by-d skew symmetric matrix and $\mathbf{b}' \in \mathbb{R}^d$. See [6] or [9]. In fact every orthogonal matrix A sufficiently close to the identity matrix I can be written as

$$A = e^S = 1 + S = \frac{1}{2}S^2 + \frac{1}{2 \cdot 3}S^3 + \ldots,$$

where S is a skew symmetric matrix, and the above infinite series converges. It is well known that the exponential map

$$S \mapsto e^S$$

takes the tangent space of the Lie group to orthogonal matrices, which is the Lie algebra of the Lie group, into the Lie group itself. This exponential map is a local analytic diffeomorphism near I, the identity. Thus any analytic path $A(t)$ with $A(0) = I$ pulls back to a path $S(t)$ in the Lie algebra, which is itself analytic. Thus

$$A(t) = e^{S(t)}.$$

On the other hand since $e^0 = I$ and thus $S(0) = 0$ we can write

$$S(t) = tS_1 + \frac{t^2}{2}S_2 + \frac{t^3}{2 \cdot 3}S_3 + \cdots,$$

where each $S_1, S_2, \dots$ is skew symmetric. Thus

$$A(t) = I + \left(tS_1 + \frac{t^2}{2}S_2 + \frac{t^3}{2 \cdot 3}S_3 + \dots\right) + \frac{1}{2}\left(tS_1 + \frac{t^2}{2}S_2 + \frac{t^3}{2 \cdot 3}S_3 + \dots\right)^2 + \dots.$$

Rearranging terms, which is possible since we have an absolutely convergent power series, we get

$$A(t) = I + tS_1 + \frac{t^2}{2}(S_2 + S_1^2) + \frac{t^3}{2 \cdot 3}\left(S_3 + \frac{3}{2}S_1S_2 + \frac{3}{2}S_2S_1 + S_1^3\right) + \dots. \tag{5}$$

Thus each of the matrix coefficients of $\frac{t^j}{j!}$ gives a parametric description of the jth derivative of $A(t)$, and thus a description of a k-trivial flex of $\mathbf{p}$. In particular $\mathbf{p}'$, $\mathbf{p}''$ is a 2-trivial flex of $\mathbf{p}$ if and only if there are skew symmetric matrices S_1, S_2 and $\mathbf{b}', \mathbf{b}'' \in \mathbb{R}^d$ such that

$$\mathbf{p}' = S_1\mathbf{p} + (\mathbf{b}', \dots, \mathbf{b}'),$$
$$\mathbf{p}'' = (S_2 + S_1^2)\mathbf{p} + (\mathbf{b}'', \dots, \mathbf{b}'').$$

Later it will be convenient to be able to "cancel" initial parts of a kth order flex, with a k-trivial flex. Thus we state the following. See [6].

PROPOSITION 4.2.1. *Let $\mathbf{p}(t)$, $\mathbf{p}(0) = \mathbf{p}$, be an analytic path in configuration space such that*

$$\mathbf{p}^{(j)} = D_t^j[\mathbf{p}(T)]\Big|_{t=0}, \quad j = 0, 1, \dots, k$$

is k-trivial. Then there is a rigid motion $T(t)$ of $\mathbb{R}^d$, analytic in t, such that $T(0) = I$ and

$$D_t^j[T(t)\mathbf{p}(t)]\Big|_{t=0} = 0, \qquad \text{for } j = 1, \dots, k.$$

Proof. We proceed by induction on k. For $k = 0$, there is nothing to prove. So we assume

$$D_t^j[\mathbf{p}(t)]\Big|_{t=0} = \begin{cases} \mathbf{p} & \text{if } j = 0, \\ 0 & \text{if } j = 1, \dots, k, \end{cases}$$

and we wish to find $T(t)$ such that the $(k+1)$st derivative is 0 as well, assuming that the first $k+1$ derivatives are k-trivial.

We restrict to the space spanned by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_v$. By adding a translation we have, using (5),

$$\mathbf{b}^{(j)} = 0, \quad j = 1, \ldots, k+1,$$
$$S_j = 0, \quad j = 1, \ldots, k.$$

Note then that $D_t^{k+1}\mathbf{p}(t)|_{t=0} = S_{k+1}\mathbf{p}$. Define $T(t)$ by

$$T(t) = e^{\frac{-t^{k+1}}{(k+1)!}S_{k+1}} = I - \frac{t^{k+1}}{(k+1)!}S_{k+1} + \ldots .$$

We then observe, for all $j = 1, 2, \ldots$,

$$D_t^j[T(t)\mathbf{p}(t)] = \sum_{\ell=0}^{j} \binom{j}{\ell} D_t^\ell[T(t)] \cdot D_t^{j-\ell}[\mathbf{p}(t)]. \tag{6}$$

But

$$D_t^\ell T(t)\Big|_{t=0} = \begin{cases} 0 & \ell = 1, \ldots, k, \\ -S_{k+1} & \ell = k+1. \end{cases}$$

Thus

$$D_t^j[T(t)\mathbf{p}(t)]\Big|_{t=0} = \left\{ \begin{array}{ll} S_{k+1}\mathbf{p} - S_{k+1}\mathbf{p} = 0, & \text{for } j = k+1 \\ 0 & \text{for } j = 1, \ldots, k \end{array} \right\} = 0. \qquad \square$$

Remark 4.2.1. There are several ways of handling the problem of "normalizing" the first few derivatives. One way is the above technique; another way is to use "tie downs" as in [33] and discussed in [9]; a third way is to use the method described by [20]. (See also [9].) This normalizing is a nuisance but it is convenient to have for the argument used to show that second-order rigidity implies rigidity.

The following is an immediate consequence of the definition of k-trivial and the formula (5).

LEMMA 4.2.2. *Let $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ be such that $\mathbf{p}' = \mathbf{p}'' \cdots = \mathbf{p}^{(k-1)} = 0$. Then $\mathbf{p}^{(k)}$ is a 1-trivial flex at $\mathbf{p}$ if and only if $\mathbf{p}', \mathbf{p}'', \ldots, \mathbf{p}^{(k)}$ is k-trivial at $\mathbf{p}$.*

It is also useful to have the following.

LEMMA 4.2.3. *Let $\mathbf{p}(t)$ be any analytic path in configuration space such that $\mathbf{p}' = D_t\mathbf{p}(t)\big|_{t=0}, \ldots, \mathbf{p}^{(k)} = D_t^k\mathbf{p}(t)\big|_{t=0}$. Let $T(t)$ be any rigid motion of $\mathbb{R}^d$, analytic in t; $T(0) = I$. Then $D_t[T(t)\mathbf{p}(t)]\big|_{t=0}, \ldots, D_t^k[T(t)\mathbf{p}(t)]\big|_{t=0}$ is k-trivial at $\mathbf{p}$ if and only if $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is k-trivial at $\mathbf{p}$.*

Proof. Since $T(t)$ is invertible it is enough to show that if $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is k-trivial, then $D_t[T(t)\mathbf{p}(t)]\big|_{t=0}, \ldots, D_t^k[T(t)\mathbf{p}(t)]\big|_{t=0}$ is k-trivial. But then $\mathbf{p}^{(\ell)} = D_t^\ell[\overline{T}(t)]\big|_{t=0}$ for $\ell = 1, \ldots, k$ for some rigid motion $\overline{T}(t)$ of $\mathbb{R}^d$, analytic in t, $\overline{T}(0) = I$. But then clearly for $\ell = 1, \ldots, k$,

$$D_t^\ell[T(t)\mathbf{p}(t)]\Big|_{t=0} = D_t^\ell[T(t)\overline{T}(t)\mathbf{p}(t)]\Big|_{t=0},$$

by expanding both sides by the product rule (6). But $T(t)\overline{T}(t)$ is again a rigid analytic motion of $\mathbb{R}^d$ and we are done. $\square$

4.3. Second-order rigidity implies rigidity. We have generalized the notion of second-order flex (see [6] for bar frameworks) to general tensegrity frameworks. In the next theorem we will show that a nontrivial analytic flex of a tensegrity framework gives rise to a second-order flex $(\mathbf{p}', \mathbf{p}'')$ whose first-order part $\mathbf{p}'$ is nontrivial. The natural idea is to take the first and second derivatives of the analytic flex evaluated at the starting point. Unfortunately, this may not work because the first derivative of the analytic flex may be trivial, and we may have to wait for some higher derivative to be nontrivial. For cables and struts we use the principle that the first nonvanishing derivative of the member length squared has the correct sign.

THEOREM 4.3.1. *If a tensegrity framework $G(\mathbf{p})$ is second-order rigid, then it is rigid.*

Proof. Assume $G(\mathbf{p})$ is not rigid. Then we will show that $G(\mathbf{p})$ is not second-order rigid by finding a second-order flex $(\mathbf{q}', \mathbf{q}'')$ such that $\mathbf{q}'$ is not 1-trivial at $\mathbf{p}$.

Since $G(\mathbf{p})$ is not rigid we know that $G(\mathbf{p})$ has a nonrigid analytic flex $\mathbf{p}(t)$ by Definition 2.1.2 (c). (See [6] or [9].) Define, for $\ell = 1, 2, \ldots,$

$$\mathbf{p}^{(\ell)} = D_T^\ell \mathbf{p}(t)\Big|_{t=0}.$$

Suppose for all $k = 1, 2, \ldots, \mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is k-trivial. Then for any $\{i, j\}$, not just those members of G, for all $k = 1, 2, \ldots,$

$$D_t^k [|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2]\Big|_{t=0} = 0,$$

which implies that $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2$ is constant in t, which implies that $\mathbf{p}(t)$ is a rigid analytic flex, contradicting the choice of $\mathbf{p}(t)$. Thus for some $k \geq 1$, $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is not k-trivial.

Now let k be the smallest positive integer such that $\mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is not k-trivial, fixing k. (If $k = 1$, life is especially easy.) Applying Proposition 4.2.1, we can alter $\mathbf{p}(t)$ so that not only is $\mathbf{p}', \ldots, \mathbf{p}^{(k-1)}$ $(k-1)$-trivial, but $\mathbf{p}' = \cdots = \mathbf{p}^{(k-1)} = 0$. (If $k = 1$ we do nothing.) Note that by Lemma 4.2.3, $0 = \mathbf{p}', \ldots, \mathbf{p}^{(k)}$ is still not k-trivial.

We observe that for any $\{i, j\}$,

$$\begin{aligned} D_t^k [|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2]\Big|_{t=0} &= \sum_{\ell=0}^{k} \binom{k}{\ell} (\mathbf{p}_i^{(\ell)} - \mathbf{p}_i^{(\ell)}) \cdot (\mathbf{p}_i^{(k-\ell)} - \mathbf{p}_j^{(k-\ell)}) \\ &= 2(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^{(k)} - \mathbf{p}_j^{(k)}). \end{aligned}$$

Hence $\mathbf{p}^{(k)}$ is a first-order flex for $G(\mathbf{p})$. By Lemma 4.2.2, $\mathbf{p}^{(k)}$ is not 1-trivial. Now we define

$$\mathbf{q}' = \mathbf{p}^{(k)}.$$

We now proceed to find $\mathbf{q}''$. We still must pay attention to conditions analogous to first-order conditions. Recall that E_0 is the set of bars of G, E_- is the set of cables of G, and E_+ is the set of struts of G. For every $n = k, k+1, \ldots, 2k-1$, define

$$E_n = \left\{ \{i, j\} \in E_0 \cup E_- \cup E_+ \;\middle|\; \begin{array}{l} (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^{(\ell)} - \mathbf{p}_j^{(\ell)}) = 0 \quad \text{for } \ell = 1, \ldots, n-1 \\ (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_j^{(n)} - \mathbf{p}_j^{(n)}) \neq 0 \end{array} \right\}.$$

Note that when $m = 1, \ldots, n$, and $\{i, j\} \in E_n$, or when $\{i, j\}$ is a bar,

$$\begin{aligned} D_t^m [|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2]\Big|_{t=0} &= \sum_{\ell=0}^{m} \binom{m}{\ell} (\mathbf{p}_i^{(\ell)} - \mathbf{p}_j^{(\ell)}) \cdot (\mathbf{p}_i^{(m-\ell)} - \mathbf{p}_j^{(m-\ell)}) \\ &= 2(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^{(m)} - \mathbf{p}_j^{(m)}) \\ &\begin{cases} = 0 & \text{if } \{i, j\} \in E_0 \\ = 0 & \text{if } m = 1, 2, \ldots, n-1 \quad \text{and} \quad \{i, j\} \in E_n \\ < 0 & \text{if } m = n \quad \text{and} \quad \{i, j\} \in E_n \cap E_- \\ > 0 & \text{if } m = n \quad \text{and} \quad \{i, j\} \in E_n \cap E_+ \end{cases}, \end{aligned}$$

since either $\mathbf{p}^{(\ell)} = 0$ or $\mathbf{p}^{(m-\ell)} = 0$ if $\ell = 1, \dots, m-1 \leq 2k-1$. (Note that only cables or struts are in any E_n.) In other words, for just those members in E_n, $\mathbf{p}^{(n)}$ acts as a strict first-order flex of $G(\mathbf{p})$.

We will next find a sequence of real numbers $\varepsilon_1 >> \varepsilon_2 >> \varepsilon_{k-1} > 0$ and define

$$\mathbf{r}' = \mathbf{p}^{(k)} + \varepsilon_1 \mathbf{p}^{(k+1)} + \varepsilon_1 \mathbf{p}^{(k+2)} + \cdots + \varepsilon_{k-1} \mathbf{p}^{(2k-1)},$$

where $\varepsilon_i >> \varepsilon_{i+1}$ means that ε_{i+1} is chosen sufficiently small such that later inequalities will remain satisfied. We see that $\mathbf{r}'$ is also a (nontrivial) first-order flex of $G(\mathbf{p})$. In fact we require that for $\{i,j\}$ a member of G,

$$(7) \qquad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{r}'_i - \mathbf{r}'_j) \left\{ \begin{array}{ll} < 0 & \text{if } \{i,j\} \in (E_k \cup \cdots \cup E_{2k-1}) \cap E_- \\ > 0 & \text{if } \{i,j\} \in (E_k \cup \cdots \cup E_{2k-1}) \cap E_+ \\ = 0 & \text{otherwise} \end{array} \right\}.$$

To see that this is possible we proceed by induction. Define for $n = k, k+1, \dots, 2k+1$,

$$\mathbf{r}'(n) = \mathbf{p}^{(k)} + \varepsilon_1 \mathbf{p}^{(k+1)} + \cdots + \varepsilon_{n-k} \mathbf{p}^{(n)},$$

where $\mathbf{r}'(k) = \mathbf{p}^{(k)}$. We require that for $\{i,j\}$ a member of G,

$$(8) \qquad (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{r}'_i(n) - \mathbf{r}'_j(n)) \left\{ \begin{array}{ll} < 0 & \text{if } \{i,j\} \in (E_k \cup \cdots \cup E_n) \cap E_- \\ > 0 & \text{if } \{i,j\} \in (E_k \cup \cdots \cup E_n) \cap E_+ \\ = 0 & \text{otherwise} \end{array} \right\}.$$

But this is true for $n = k$, and if $\varepsilon_{n+1} > 0$ is chosen small enough we can satisfy (8) for $n+1$, assuming it is true for n.

In other words for every cable and strut of G, either $\mathbf{r}'$ is strict or $\mathbf{r}'$ and $\mathbf{p}', \dots, \mathbf{p}^{(2k-1)}$ act as if $\{i,j\}$ were a bar for the first-order conditions.

We now choose a large real number $B > 0$ and define

$$\mathbf{q}'' = \frac{2}{\binom{2k}{k}} \mathbf{p}^{(2k)} + B\mathbf{r}'.$$

Recalling that $\mathbf{q}' = \mathbf{p}^{(k)}$, we claim that $(\mathbf{q}', \mathbf{q}'')$ is a second-order flex of $G(\mathbf{p})$ for B large enough. For $\{i,j\} \in E_k \cup \cdots \cup E_{2k-1}$ we calculate

$$\begin{aligned} &(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}''_i - \mathbf{q}''_j) + |\mathbf{q}'_i - \mathbf{q}'_j|^2 \\ &\qquad = \frac{2}{\binom{2k}{k}} (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^{(2k)} - \mathbf{p}_j^{(2k)}) + B(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{r}'_i - \mathbf{r}'_j) + |\mathbf{q}'_i - \mathbf{q}'_j|^2 \\ &\qquad \left\{ \begin{array}{ll} > 0 & \text{if } \{i,j\} \in E_- \cap (E_k \cup \cdots \cup E_{2k-1}) \\ < 0 & \text{if } \{i,j\} \in E_+ \cap (E_k \cup \cdots \cup E_{2k-1}) \end{array} \right\} \end{aligned}$$

if B is chosen large enough by (7).

For $\{i,j\}$ a member of G but $\{i,j\} \notin E_k \cup \cdots E_{2k-1}$, then $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{r}'_i - \mathbf{r}'_j) = 0$ and $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_I^{(\ell)} - \mathbf{p}_j^{(\ell)}) = 0$ for $\ell = 1, \dots, 2k-1$. Then

$$\begin{aligned} D_t^{2k}[|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2]\Big|_{t=0} &= \sum_{\ell=0}^{2k} \binom{2k}{\ell} (\mathbf{p}_i^{(\ell)} - \mathbf{p}_j^{(\ell)}) \cdot (\mathbf{p}_i^{(2k-\ell)} - \mathbf{p}_j^{(2k-\ell)}) \\ &= 2(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i^{(2k)} - \mathbf{p}_j^{(2k)}) + \tbinom{2k}{k} |\mathbf{p}_i^{(k)} - \mathbf{p}_j^{(k)}|^2 \\ &= \tbinom{2k}{k} [(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i - \mathbf{q}_j) + |\mathbf{q}'_i - \mathbf{q}'_j|^2] \\ &\left\{ \begin{array}{ll} \leq 0 & \text{if } \{i,j\} \in E_- \setminus (E_k \cup \cdots \cup E_{2k-1}) \\ = 0 & \text{if } \{i,j\} \in E_0 \setminus (E_k \cup \cdots \cup E_{2k-1}) \\ \geq 0 & \text{if } \{i,j\} \in E_+ \setminus (E_k \cup \cdots \cup E_{2k-1}). \end{array} \right. \end{aligned}$$

Thus in either case the second-order conditions are satisfied, $(\mathbf{q}', \mathbf{q}'')$ is a second-order flex of $G(\mathbf{p})$, and $\mathbf{q}'$ is not 1-trivial. □

Remark 4.3.1. The general outline for the above proof is the same as in [6], except that the cables and struts can cause complications. Differentiating the edge length condition allows us to detect any cable or strut, but its occurrence causes an appropriate sign somewhere from the level k to $2k$. The intermediate levels from $k+1$ to $2k-1$ must be introduced into the $(\mathbf{q}', \mathbf{q}'')$ carefully.

4.4. Prestress stability and second-order rigidity. We observe that prestress stability is stronger than second-order rigidity.

THEOREM 4.4.1. *If a tensegrity framework $G(\mathbf{p})$ is prestress stable, then it is second-order rigid.*

Proof. Let ω be the prestress that stabilizes $G(\mathbf{p})$. Then from the comments following the definition of prestress stability, ω also stabilizes that subframework of $G(\mathbf{p})$ consisting of the same bars and all cables and struts with $\omega_{ij} \neq 0$. Thus, without any loss of generality, we can assume that ω is strict. For example, $\omega_{ij} \neq 0$ on all the cables and struts of G.

Suppose $(\mathbf{p}', \mathbf{p}'')$ is a second-order flex of $G(\mathbf{p})$, where $\mathbf{p}'$ is not a trivial first-order flex. We wish to find a contradiction. By the first-order stress test we see that $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ for all members of G (because $\omega_{ij} \neq 0$ on all cables and struts). Thus, by the second-order condition,

$$|\mathbf{p}'_i - \mathbf{p}'_j|^2 + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) \begin{cases} \leq 0 & \{i,j\} = \text{ cable} \\ = 0 & \{i,j\} = \text{ bar} \\ \geq 0 & \{i,j\} = \text{ strut} \end{cases}$$

for all members $\{i,j\}$ of G. In any case, since ω is proper for all members $\{i,j\}$ of G,

$$\omega_{ij}|\mathbf{p}'_i - \mathbf{p}'_j|^2 + \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \leq 0.$$

But since $\mathbf{p}'$ is a nontrivial first-order flex of $G(\mathbf{p})$ and since ω is a stabilizing self stress for $G(\mathbf{p})$,

$$\omega R(\mathbf{p}')\mathbf{p}' = \sum_{ij} \omega_{ij}|\mathbf{p}'_i - \mathbf{p}'_j|^2 = (\mathbf{p}')^T \Omega \mathbf{p}' > 0$$

by Proposition 3.4.2. Recalling that $\omega R(\mathbf{p}) = 0$,

$$\begin{aligned} 0 < \omega R(\mathbf{p}')\mathbf{p}' &= \omega R(\mathbf{p}')\mathbf{p}' + \omega R(\mathbf{p})\mathbf{p}'' \\ &= \sum_{ij} \omega_{ij}(\mathbf{p}'_i - \mathbf{p}'_j)^2 + \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) \leq 0, \end{aligned}$$

a contradiction. Thus $G(\mathbf{p})$ is second-order rigid. □

Remark 4.4.1. It turns out that there are tensegrity frameworks that are not prestress stable for any prestress yet are still second-order rigid. For instance, the example of Figure 9b has this property. The two "tetrahedral" blocks are prestress stable. Therefore, all second-order flexes are trivial on these blocks. Any such second-order flex must extend to a rotation about the common point 0. However, this violates either the strut or the cable condition on the unstressed connecting members at first-order.

However, in §5.3 we will see that if the space of first-order flexes or the space of proper self stresses is one-dimensional, then second-order rigidity and prestress stability are the same. This will also help us to find examples of bar frameworks which are second-order rigid but not prestress stable in §5.3.

5. The stress test.

5.1. Duality from linear algebra. We now formulate some well-known principles of duality in linear algebra which we will later interpret as a "stress" test for second-order rigidity. These duality principles are a special case of the duality principles used in linear programming.

In the following, let A be a d-by-e real matrix, where we write A in block form

$$A = \begin{bmatrix} A_0 \\ A_+ \end{bmatrix},$$

where A_0 and A_+ are some designated subsets of the rows of A. In our applications A will correspond to the rigidity matrix, A_0 will correspond to the rows indexed by the bars of G, and A_+ will correspond to the rows indexed by the struts and cables of G, with the strut rows multiplied by -1. However, for the general statements in this section we will not need any special properties of A.

We can now restate the first-order stress test in this somewhat more general context. We use the notation $[x_1, x_2, \ldots] < 0$ for vectors to mean $x_i < 0$ for all $i = 1, 2, \ldots$.

PROPOSITION 5.1.1. *There is a column vector* $\mathbf{x} \in \mathbb{R}^d$ *such that*

$$A_0\mathbf{x} = 0,$$
$$A_+\mathbf{x} < 0,$$

if and only if for all row vectors $\mathbf{y} \in \mathbb{R}^e$, $\mathbf{y} = [\mathbf{y}_0, \mathbf{y}_+]$, *such that*

$$\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0,$$

$$\mathbf{y}_+ \geq 0;$$

then $\mathbf{y}_+ = 0$.

This is a special case of the duality principle for homogeneous linear equalities, for instance, as found in [32, Thm. 6]. Note that the "only if" implication is easy, since if $A_0\mathbf{x} = 0$, $A_+\mathbf{x} < 0$, $\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0$, $\mathbf{y}_+ \geq 0$, then $0 = \mathbf{y}_0 A_0 \mathbf{x} + \mathbf{y}_+ A_+ \mathbf{x} \leq 0$. If $\mathbf{y}_+ \neq 0$ this gives a strict inequality and thus a contradiction. The other implication implicitly or explicitly uses the principle of "hyperplane separation." See, for example, [17, p. 10].

An important point is the strictness of the inequalities. However, in the following we instead concentrate on the duality principle itself, putting aside the strictness properties for the moment.

Let

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_+ \end{bmatrix} \in \mathbb{R}^e$$

be a column vector.

PROPOSITION 5.1.2. *There is a column vector* $\mathbf{x} \in \mathbb{R}^d$ *such that*

$$A_0\mathbf{x} = \mathbf{b}_0,$$
$$A_+\mathbf{x} \leq \mathbf{b}_+,$$

if and only if for all row vectors $\mathbf{y} \in \mathbb{R}^e$, $\mathbf{y} = [\mathbf{y}_0, \mathbf{y}_+]$, *such that*

$$\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0,$$
$$\mathbf{y}_+ \geq 0;$$

then $\mathbf{y}_0 \mathbf{b}_0 + \mathbf{y}_+ \mathbf{b}_+ \geq 0$.

Note that again the "only if" implication is easy since if $A_0\mathbf{x} = \mathbf{b}_0$, $A_+\mathbf{x} \leq \mathbf{b}_+$, $\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0$, and $\mathbf{y}_+ \geq 0$ then $0 = \mathbf{y}_0 A_0 \mathbf{x} + \mathbf{y}_+ A_+ \mathbf{x} \leq \mathbf{y}_0 \mathbf{b}_0 + \mathbf{y}_+ \mathbf{b}_+$, and again the "if" implication follows from the hyperplane separation principle.

In the terminology of linear programming this proposition is an asymmetric form of duality in the special case when the primal problem has the constant 0 objective function. See [15] for instance for a discussion of various such forms of the Farkas alternative as well as a proof.

We now sharpen this proposition to obtain an equivalent dual reformulation to determine when we get strict inequality. We fix A and $\mathbf{b}$.

PROPOSITION 5.1.3. *There is a column vector* $\mathbf{x} \in \mathbb{R}^d$ *such that*

$$A_0\mathbf{x} = \mathbf{b}_0,$$
$$A_+\mathbf{x} < \mathbf{b}_+,$$

if and only if for all row vectors $\mathbf{y} \in \mathbb{R}^e$, $\mathbf{y} = [\mathbf{y}_0, \mathbf{y}_+]$, *such that*

$$\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0,$$
$$\mathbf{y}_+ \geq 0;$$

then $\mathbf{y}_0 \mathbf{b}_0 + \mathbf{y}_+ \mathbf{b}_+ \geq 0$ *with equality if and only if* $\mathbf{y}_+ = 0$.

Proof. Again, the only if implication follows easily. $A_0\mathbf{x} = \mathbf{b}_0$, $A_+\mathbf{x} < \mathbf{b}_+$, $\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0$, $\mathbf{y}_+ \geq 0$ imply that $0 = \mathbf{y}_0 A_0 \mathbf{x} + \mathbf{y}_+ A_+ \mathbf{x} \leq \mathbf{y}_0 \mathbf{b}_0 + \mathbf{y}_+ \mathbf{b}_+$ and we have strict inequality, and a contradiction, if and only if $\mathbf{y}_+ \neq 0$.

To show the converse we use the two previous propositions. Assume that the condition on the $\mathbf{y}$ vector holds. By Proposition 5.1.2 we know that there is an $\mathbf{x} \in \mathbb{R}^d$ such that $A_0\mathbf{x} = \mathbf{b}_0$, $A_+\mathbf{x} \leq \mathbf{b}_+$. If all of the $\mathbf{b}_+$ inequalities are strict we are done. If not, throw out those strict inequalities from A to get the condition $A_0\mathbf{x} = \mathbf{b}_0$, $A_+\mathbf{x} = \mathbf{b}_+$. Similarly, we can throw out the corresponding set of variables in the $\mathbf{y}$ vector. Now it is still true that if $\mathbf{y}_0 A_0 + \mathbf{y}_+ A_+ = 0$, then $\mathbf{y}_0 A_0 \mathbf{x} + \mathbf{y}_+ A_+ \mathbf{x} = \mathbf{y}_0 \mathbf{b}_0 + \mathbf{y}_+ \mathbf{b}_+ = 0$. Thus Proposition 5.1.1 applies, and there is a (small) $\mathbf{x}_\varepsilon \in \mathbb{R}^e$ such that $A_0\mathbf{x}_\varepsilon = 0$, $A_+\mathbf{x}_\varepsilon < 0$. Then $A_0(\mathbf{x} + \mathbf{x}_\varepsilon) = \mathbf{b}_0$, $A_+(\mathbf{x} + \mathbf{x}_\varepsilon) < \mathbf{b}_+$. If $\mathbf{x}_\varepsilon$ is small enough then $\mathbf{x} + \mathbf{x}_\varepsilon$ still satisfies those inequalities that were thrown out as well. ☐

5.2. Interpretation as the stress test. We now specialize the results of §5.1 to the case of the rigidity matrix. Let $G(\mathbf{p})$ be a tensegrity framework in $\mathbb{R}^d$, and let $R(\mathbf{p})$ be its d-by-e rigidity matrix. Let $R_0(\mathbf{p})$ denote those rows (regarded as a smaller matrix) corresponding to the bars of G. Let $R_+(\mathbf{p})$ denote the matrix obtained from those rows of $R(\mathbf{p})$ corresponding to cables and struts of G, with the rows corresponding to struts multiplied by -1.

Suppose ω is a proper stress for $G(\mathbf{p})$, so $\omega R(\mathbf{p}) = 0$. We then have a corresponding $[\omega_0, \omega_+]$, where ω_0 corresponds to the stresses on the bars and ω_+ to the stresses on the cables and struts, but with the opposite sign for struts only. Thus ω being proper translates into $\omega_+ \geq 0$, and being a self stress means $\omega_0 R(\mathbf{p}) + \omega_+ R_+(\mathbf{p}) = 0$.

In this terminology $\mathbf{p}'$ is a first-order flex if

$$R_0(\mathbf{p})\mathbf{p}' = 0,$$
$$R_+(\mathbf{p})\mathbf{p}' \leq 0,$$

and $\mathbf{p}'$, $\mathbf{p}''$ is a second-order flex if in addition

$$R_0(\mathbf{p}')\mathbf{p}' + R_0(\mathbf{p})\mathbf{p}'' = 0,$$
$$R_+(\mathbf{p}')\mathbf{p}' + R_+(\mathbf{p})\mathbf{p}'' \leq 0,$$

where an inequality need only hold when the corresponding inequality, in the first-order system, is an equality. Recall that $\mathbf{p}''$ (or $\mathbf{p}'$) is strict for $\{i,j\}$ a cable or strut if the corresponding inequality is strict.

We now have our strict second-order duality result.

COROLLARY 5.2.1 (the second-order stress test). *A first-order flex $\mathbf{p}'$ of $G(\mathbf{p})$ extends to a second-order flex $(\mathbf{p}', \mathbf{p}'')$ if and only if for all proper self stresses ω for $G(\mathbf{p})$, with stress matrix Ω,*

$$(\mathbf{p}')^T \Omega \mathbf{p}' \leq 0.$$

Furthermore, $\mathbf{p}''$ can be chosen to be strict on each cable and strut $\{i,j\}$ where $\mathbf{p}'$ is not strict, if and only if for all proper self stresses $\omega, \mathbf{p}'\Omega\mathbf{p}' = 0$ implies $\omega_{ij} = 0$, for each such $\{i,j\}$.

Proof. We apply Proposition 5.1.3, where

$$\mathbf{p} = \mathbf{x},$$

$$R_0(\mathbf{p}) = A_0, \qquad -R_0(\mathbf{p}')\mathbf{p}' = \mathbf{b}_0,$$
$$R_+(\mathbf{p}) = A_+, \qquad -R_+(\mathbf{p}')\mathbf{p}' = \mathbf{b}_+,$$
$$\omega_0 = \mathbf{y}_0, \quad \omega_+ = \mathbf{y}_+.$$

We may assume, without loss of generality, that $R(\mathbf{p})\mathbf{p}' = 0$, since any cable or strut where $\mathbf{p}'$ is strict can be disregarded as a cable or strut for the second-order conditions.

The second-order conditions translate into the hypothesis of Proposition 5.1.2, and the conclusion translates into the condition that ω is a proper self stress. Then

$$0 \leq \mathbf{y}_0\mathbf{b}_0 + \mathbf{y}_+\mathbf{b}_+ = -\omega_0 R_0(\mathbf{p}')\mathbf{p}' - \omega_+ R_+(\mathbf{p}')\mathbf{p}' = -\omega R(\mathbf{p}')\mathbf{p}' = -(\mathbf{p}')^T\Omega\mathbf{p}'$$

is the condition desired. The strictness follows from Proposition 5.1.3. □

We can simplify matters even further when G consists only of bars. This is our second-order duality result for bar frameworks.

COROLLARY 5.2.2 (second-order stress test for bars). *A first-order flex $\mathbf{p}'$ of a bar framework $G(\mathbf{p})$ extends to a second-order flex if and only if for all self stresses ω for $G(\mathbf{p})$, with stress matrix Ω,*

$$(\mathbf{p}')^T \Omega \mathbf{p}' = 0.$$

Remark 5.2.1. In the appendix, we show that we can always replace a framework (with a strict proper self stress) by an equivalent bar framework and use this to check second-order rigidity. However, it seems simpler to use Corollary 5.2.1 directly, rather than introduce so many extraneous members.

5.3. Second-order rigidity and prestress stability. When does second-order rigidity imply prestress stability? We begin with cases when the set of self stresses or the set of equilibrium first-order flexes is one-dimensional, the natural first cases to consider.

Note that for a fixed tensegrity framework $G(\mathbf{p})$, the proper self stress and first-order flexes each form a cone with the origin as the cone point.

PROPOSITION 5.3.1. *If a tensegrity framework $G(\mathbf{p})$ is second-order rigid with either a one-dimensional cone of equilibrium first-order flexes or a one-dimensional cone of proper self stresses, then $G(\mathbf{p})$ is prestress stable.*

Proof. Suppose $\mathbf{p}'$ is any nontrivial equilibrium first-order flex of $G(\mathbf{p})$ generating the one-dimensional cone of all equilibrium first-order flexes. If for all proper self stresses ω with stress matrix Ω we have

$$t^2(\mathbf{p}')^T\Omega\mathbf{p}' = (t\mathbf{p}')^T\Omega t\mathbf{p}' \leq 0$$

for all first-order flexes $t\mathbf{p}'$ of $G(\mathbf{p})$ (t a real scalar), then by Corollary 5.2.1 $\mathbf{p}'$ extends to a second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $G(\mathbf{p})$, which contradicts $G(\mathbf{p})$ being second-order rigid. Thus for some proper self stress ω, $(\mathbf{p}')^T\Omega\mathbf{p}' > 0$, ω stabilizes $G(\mathbf{p})$ (by Proposition 3.4.2) and $G(\mathbf{p})$ is prestress stable.

Suppose ω is a proper nonzero self stress in the one-dimensional cone of proper self stresses. Suppose there is a nontrivial first-order flex $\mathbf{p}'$ such that $(\mathbf{p}')^T\Omega\mathbf{p}' \leq 0$. If $-\omega$ is not a proper stress, then $t\omega$, $t \geq 0$, are the only proper self stresses for $G(\mathbf{p})$. Then by Corollary 5.2.1 again $G(\mathbf{p})$ would not be second-order rigid, contradicting the hypothesis. Therefore either $(\mathbf{p}')^T\Omega\mathbf{p}' > 0$ for all nontrivial first-order flexes $\mathbf{p}'$ *or* $-\omega$ is a proper self stress. If $-\omega$ is a proper self stress, then $(\mathbf{p}')^T\Omega\mathbf{p}' = 0$ and again $\mathbf{p}'$ would extend to a second-order flex. Thus $(\mathbf{p}')^T(\pm\Omega)\mathbf{p}' > 0$ and $\pm\omega$ stabilizes $G(\mathbf{p})$. □

We have already seen an example of tensegrity framework in the plane, Figure 9b, which is easily seen to be second-order rigid, directly from the definition, but is not prestress stable for any proper self stress. Here we present another example, but one which is a bar framework in three-space. It also serves as an example of how to calculate using the stress test.

If we have any bar framework $G(\mathbf{p})$, let

$$\mathbf{p}'(1), \ldots, \mathbf{p}'(n)$$

denote a basis for a space of nontrivial first-order flexes of $G(\mathbf{p})$. Let

$$\Omega(1), \ldots, \Omega(m)$$

denote a basis for the space of self stresses of $G(\mathbf{p})$. If $G(\mathbf{p})$ is prestress stable, some linear combination of the stress matrices must be positive definite on the space generated by the first-order flexes $\mathbf{p}'(1), \ldots, \mathbf{p}'(n)$. From Corollary 5.2.2, the second-order stress test for bar frameworks, $G(\mathbf{p})$ is not second-order rigid if and only if all of the stress matrices have a common nonzero vector on which they evaluate to be 0 in this same space generated by $\mathbf{p}'(1), \ldots, \mathbf{p}'(n)$. When $n = 2$, both of these criteria can be checked with certain easily calculated expressions.

Example 5.3.1. We define a specific example in three-space. Let $G(\mathbf{p})$ be the following bar framework in three-space with the following seven vertices:

$$\begin{array}{lll} \mathbf{p}_1 = (0,0,0), & \mathbf{p}_4 = (1,0,0), & \mathbf{p}_7 = (1,0,0), \\ \mathbf{p}_2 = (0,1,0), & \mathbf{p}_5 = (1,1,0), & \\ \mathbf{p}_3 = (0,0,1), & \mathbf{p}_6 = (0,1,0), & \end{array}$$

and the following bars:

$$\{1,2\},\{1,3\},\{1,4\},\{1,6\},\{1,7\},$$
$$\{2,3\},\{2,5\},\{2,7\},$$
$$\{3,4\},\{3,5\},$$
$$\{4,5\},\{4,6\},$$
$$\{5,6\},\{5,7\},$$
$$\{6,7\}.$$

See Figure 17a, where although $\mathbf{p}_2 = \mathbf{p}_6$ and $\mathbf{p}_4 = \mathbf{p}_7$ we have separated them slightly so that the framework can be more easily understood.

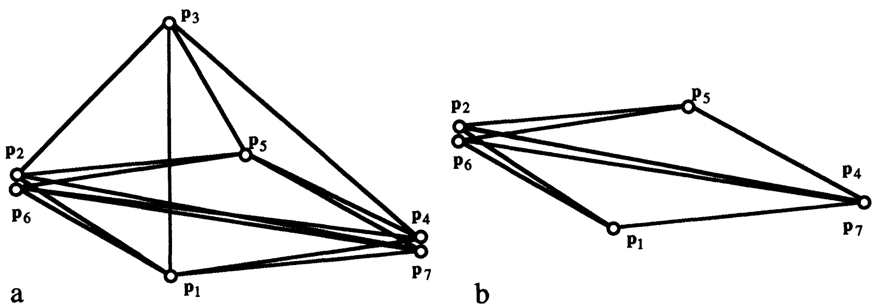

FIG. 17. *A bar framework* (a) *in* 3*-space that is second-order rigid but not prestress stable. A portion of the planar base is shown in* (b).

Note that this framework is made of the framework of Figure 17b and its symmetric copy, with appropriate identifications. Then $\mathbf{p}_3$ is added along the z-axis.

We consider those first-order flexes $\mathbf{p}'(k)$, where $\mathbf{p}_1'(k) = \mathbf{p}_2'(k) = \mathbf{p}_3'(k) = 0$, which clearly determines a complement of the space of trivial flexes, since $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ determines a bar triangle. Since $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_7)$ and $(\mathbf{p}_2, \mathbf{p}_5, \mathbf{p}_7)$ are bar triangles in the same plane, sharing a common bar $\mathbf{p}_1$, $\mathbf{p}_2$, any first-order flex $\mathbf{p}'$ must have

$$(\mathbf{p}_1 - \mathbf{p}_5) \cdot (\mathbf{p}_1' - \mathbf{p}_5') = 0.$$

In other words, $\{1,5\}$ is an "implied bar." Thus the first-order rigid tetrahedron $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_5)$ is implied and $\mathbf{p}_5' = 0$. Similarly $\mathbf{p}_4' = 0$, from the implied tetrahedron $(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_5, \mathbf{p}_4)$. See [9]. So the following first-order flexes are a basis for a complementary space:

$$\mathbf{p}_i'(1) = \left\{ \begin{array}{ll} (0,0,1) & \text{if} \quad i = 6 \\ (0,0,0) & \text{otherwise} \end{array} \right\},$$

$$\mathbf{p}_i'(2) = \left\{ \begin{array}{ll} (0,0,1) & \text{if} \quad i = 7 \\ (0,0,0) & \text{otherwise} \end{array} \right\}.$$

We can also find two independent stresses for $G(\mathbf{p})$:

$$\omega_{ij}(1) = \left\{ \begin{array}{lll} 1 & \text{if} \quad \{i,j\} = \{2,7\}, \{5,6\}, & \text{or} \ \{1,6\} \\ -1 & \text{if} \quad \{i,j\} = \{2,5\}, \{6,7\}, & \text{or} \ \{2,1\} \\ 0 & \text{otherwise} & \end{array} \right\},$$

$$\omega_{ij}(2) = \left\{ \begin{array}{lll} 1 & \text{if} \quad \{i,j\} = \{1,7\}, \{5,7\}, & \text{or} \quad \{4,6\} \\ -1 & \text{if} \quad \{i,j\} = \{1,4\}, \{6,7\}, & \text{or} \quad \{4,5\} \\ 0 & \text{otherwise} & \end{array} \right\}.$$

These are easy to see by looking at Figure 17b. Notice that $e = 15 = 3v - 6$ and that the space of first-order nontrivial (equilibrium) flexes must be of dimension 2, so the dimension of the space of self stresses is 2 as well. Thus $\omega(1)$ and $\omega(2)$ generate all the self stresses.

We next calculate the stress matrices $\Omega(1)$, $\Omega(2)$ corresponding to $\omega(1)$, $\omega(2)$, relative to the vectors $\mathbf{p}'(1)$, $\mathbf{p}'(2)$. Note that for $a = 1, 2$, $b = 1, 2$, $k = 1, 2$,

$$\mathbf{p}'(a)^T \Omega(k) \mathbf{p}'(b) = \Omega_{ab}(k) = \sum_{ij} \omega_{ij}(k)(\mathbf{p}'_i(a) - \mathbf{p}'_j(a)) \cdot (\mathbf{p}'_i(b) - \mathbf{p}'_j(b)).$$

Then

$$\Omega(1) = \begin{bmatrix} \sum_i \omega_{6i}(1) & -\omega_{67}(1) \\ -\omega_{76}(1) & \sum_i \omega_{7i}(1) \end{bmatrix} = \begin{bmatrix} 1 & +1 \\ +1 & 0 \end{bmatrix},$$

$$\Omega(2) = \begin{bmatrix} \sum_i \omega_{6i}(2) & -\omega_{67}(2) \\ -\omega_{76}(3) & \sum_i \omega_{7i}(2) \end{bmatrix} = \begin{bmatrix} 0 & +1 \\ +1 & 1 \end{bmatrix}.$$

To see if any linear combination of these is positive definite, we calculate, for any real λ_1, λ_2,

$$\det[\lambda_1 \Omega(1) + \lambda_2 \Omega(2)] = \det \begin{bmatrix} \lambda_1 & \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 & \lambda_2 \end{bmatrix} = -\lambda_1^2 - \lambda_1 \lambda_2 - \lambda_2^2,$$

which is a negative definite quadratic form itself, since $(-1)^2 - 4(-1)(-1) = -3 < 0$. Thus for each choice of $(\lambda_1, \lambda_2) \neq (0,0)$, $\det[\lambda_1 \Omega(1) + \lambda_2 \Omega(2)] < 0$, which implies that none of the forms $\lambda_1 \Omega(1) + \lambda_2 \Omega(2)$ are positive definite, and thus no stress $\lambda_1 \omega(1) + \lambda_2 \omega(2)$ can serve as a stable prestress.

On the other hand recall that the second-order stress test for bar frameworks, Corollary 5.2.2, says that a first-order flex $\mathbf{p}'$ will extend to a second-order flex if and only if $\mathbf{p}'$ is in the zero set of all the proper self stresses (regarded as quadratic forms) of $G(\mathbf{p})$. If some $\mathbf{p}'$ does extend, then so does $\mathbf{p}'$ plus any trivial first-order flex, and so we can assume that $\mathbf{p}'$ is in the space spanned by $\mathbf{p}'(1)$ and $\mathbf{p}'(2)$. Thus $G(\mathbf{p})$ will be second-order rigid if and only if $\Omega(1)$ and $\Omega(2)$ (and thus all $\lambda_1 \Omega(1) + \lambda_2 \Omega(2)$) have a common zero. The zeros of $\Omega(1)$ (as a quadratic form) are scalar multiples of

$$(0, 1) \quad \text{or} \quad (2, -1).$$

For $\Omega(2)$ we get

$$(1, 0) \quad \text{or} \quad (-1, 2).$$

None of the above four vectors are scalar multiples of any of the others, so $G(\mathbf{p})$ is second-order rigid but not prestress stable. □

Remark 5.3.1. If the dimension of the space of nontrivial (equilibrium) first-order flexes I is two, then it is easy to determine when the framework is prestress stable. Calculate a basis of stress matrices restricted to I. Choosing an orthonormal basis for

I, we see that there are at most three such distinct stress matrices (even if the space of self stresses is higher-dimensional), since they correspond to symmetric 2×2 matrices. If there are just two such matrices, one can perform an analysis similar to Example 5.3.1. If there are three such stress matrices, independent over I, there always is a stabilizing self stress.

Also for Example 5.3.1 it is possible to vary the points by a small amount, keeping all the points except $\mathbf{p}_3$ in a plane, and obtain many other examples of second-order rigid but not prestress stable bar frameworks in three-space. □

Note that the underlying graph of the example above is a triangulated sphere. Figure 18a shows a realization of this graph as a triangulated convex surface. By [6], this realization is also second-order rigid. In fact it is prestress stable as well. All members adjacent to $\mathbf{p}_6$ and $\mathbf{p}_7$ have a positive stress in the stabilizing self stress. This brings up the question: are all triangulations of a convex polyhedron in three-space, with edges as bars, prestress stable? In [6] it is only shown that such frameworks are second-order rigid. The answer is yes and will be shown elsewhere.

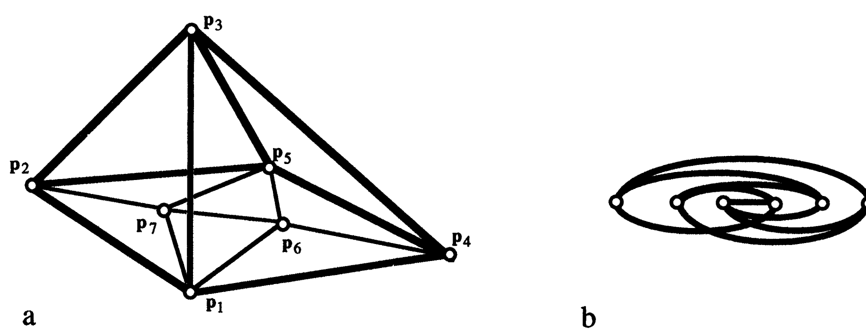

FIG. 18. *Example* (a), *with the same graph as Figure* 17a, *is prestress stable in three-space. Example* (b) *is second-order rigid in the plane but not prestress stable.*

In the plane, it turns out that if we take the bipartite graph $K_{3,3}$ with its six points on the line and $\{1, 2, 3\}$, $\{4, 5, 6\}$ as the partition (Figure 18b), then this framework $K_{3,3}(\mathbf{p})$ is second-order rigid but not prestress stable. We omit this nontrivial calculation. It also turns out that $K_{3,3}(\mathbf{p})$ is a mechanism in R^3.

In [26], as well as in [27, p. 50], there is another example of a second-order rigid but not prestress stable bar framework in the plane. The calculation for that example turns out to be quite simple.

5.4. Applications in *b-c* polygons. Here we apply the second-order stress test to a special class of frameworks $G(\mathbf{p})$: the *b-c* polygons of §3.5. That is, $G(\mathbf{p})$ is a convex polygon in the plane with bars as edges and only cables on the inside.

PROPOSITION 5.4.1. *Let* $\mathbf{p}'$ *be any nontrivial first-order flex of* $G(\mathbf{p})$, *a b-c polygon in the plane. Then* $\mathbf{p}'$ *extends to a strict second-order flex* $(\mathbf{p}', \mathbf{p}'')$ *of* $G(\mathbf{p})$.

Proof. Let Ω be any stress matrix coming from a proper self stress ω of $G(\mathbf{p})$. Since $\mathbf{p}'$ is nontrivial it is easy to check that $\mathbf{p}'$ is not an affine image of $\mathbf{p}$. By [7]

$$(\mathbf{p}')^T \Omega \mathbf{p}' < 0,$$

since for the reversed polygon (with struts on the inside) the corresponding matrix $(-\Omega)$ is positive semidefinite. Thus by Corollary 5.2.1, $\mathbf{p}'$ extends to a strict second-order flex $(\mathbf{p}', \mathbf{p}'')$. □

Remark 5.4.1. We will use this result in the next section to prove Roth's conjecture about *b-c* polygons. It is interesting (although painful) to calculate $\mathbf{p}''$ directly for the $\mathbf{p}'$ as indicated in Figure 14.

Note, however, with this result alone we can prove a weak form of Roth's conjecture. Namely, if $\mathbf{p}'$ is a nontrivial first-order flex of a strut-cable polygon (the outside edges are struts), then an argument similar to the one above can be used to find a strict second-order flex $(\mathbf{p}', \mathbf{p}'')$ as well. Then $\mathbf{p}(t) = \mathbf{p} + t\mathbf{p}' + \frac{1}{2}t^2\mathbf{p}''$ is a flex at $G(\mathbf{p})$ as required.

The only problem left in the stronger form of Roth's conjecture is to find a way of handling the bars. We will treat this in §6.

5.5. Interpretation for triangulated spheres. For any triangulated sphere $G(\mathbf{p})$ in $\mathbb{R}^3$, there is a natural correspondence between first-order flexes $\mathbf{p}'$ of $G(\mathbf{p})$ (modulo trivial first-order flexes) and self stresses ω of $G(\mathbf{p})$. In fact for each edge $\{i, j\}$ there is a dihedral angle θ_{ij}, which itself "varies" and thus there is a θ'_{ij} defined as the derivative of θ_{ij}. Then

$$\omega_{ij} = \frac{\theta'_{ij}}{|\mathbf{p}_i - \mathbf{p}_j|}, \quad \{i, j\} \quad \text{and edge at} \quad G$$

serves as a self stress for $G(\mathbf{p})$. Conversely, given a self stress it is possible to define a first-order flex $\mathbf{p}'$ with θ'_{ij} as above. See [16] or [14] for a discussion of this.

Thus using Corollary 5.2.1 we can state the dual condition for a second-order flex.

COROLLARY 5.5.1. *A first-order flex* $\mathbf{p}'$ *of a triangulated sphere* $G(\mathbf{p})$ *in* $\mathbb{R}^3$ *extends to a second-order flex if and only if for every* θ'_{ij} *(coming from a possibly different first-order flex) we have*

$$\sum_{ij} \frac{\theta'_{ij}}{|\mathbf{p}_i - \mathbf{p}_j|} |\mathbf{p}'_i - \mathbf{p}'_j|^2 = 0.$$

5.6. Interpretation in terms of packings. For the rigidity of packings as in [11] or [12] we see that the associated framework has certain vertices pinned and all the members are struts. For any proper self stress ω, $\omega_{ij} \leq 0$ for all $\{i, j\}$ struts, and thus such a $G(\mathbf{p})$ has for all $\mathbf{p}'$, a first-order flex,

$$(\mathbf{p}')^T \Omega \mathbf{p}' = \sum_{ij} \omega_{ij} (\mathbf{p}'_i - \mathbf{p}'_j)^2 \leq 0,$$

since we can take $\mathbf{p}'$ to be 0 on the pinned vertices. In fact, we get strict inequality assuming G is connected and $\mathbf{p}' \neq 0$. Thus there is a strict second-order flex $(\mathbf{p}', \mathbf{p}'')$, and it is easy to see that such a $G(\mathbf{p})$ is rigid if and only if it is first-order rigid. This was observed directly in [11].

6. Extending second-order flexes.

6.1. The general result. Some second-order flexes extend to continuous flexes of the framework. For example, if a second-order flex shortens all cables and lengthens all struts and there are no bars, then it is clear that we can complete these first two derivatives to a real analytic path. We describe a less restrictive situation where we can still extend the second-order flex. This result is an extension of and motivated by some of the results in [1, 2]. A bar framework $G(\mathbf{p})$ is called *independent* if the only self stress for $G(\mathbf{p})$ is the zero self stress.

PROPOSITION 6.1.1. *Let $G(\mathbf{p})$ be any independent bar framework with a second-order flex $(\mathbf{p}', \mathbf{p}'')$ in $\mathbb{R}^d$. Then there is an analytic flex $\mathbf{p}(t)$ of $G(\mathbf{p})$ with*

$$\mathbf{p}(0) = \mathbf{p},$$
$$D_t[\mathbf{p}(t)]\Big|_{t=0} = \mathbf{p}',$$
$$D_t^2[\mathbf{p}(t)]\Big|_{t=0} = \mathbf{p}''.$$

Proof. Let

$$M_{G(\mathbf{p})} = \Big\{\mathbf{q} \in \mathbb{R}^{dv} \;\Big|\; |\mathbf{q}_i - \mathbf{q}_j| = |\mathbf{p}_i - \mathbf{p}_j|, \quad \{i,j\} \quad \text{a member of} \quad G\Big\}$$

be the set of all configurations equivalent to $\mathbf{p}$. By [1] or [29], since $G(\mathbf{p})$ is independent, $M_{G(\mathbf{p})}$ is a smooth analytic manifold of dimension at least $d(d+1)/2$ in a neighborhood of $\mathbf{p}$ (when the dimension of the affine span of $\mathbf{p}$ is at least $d-1$), and we may naturally identify the tangent space $T_{\mathbf{p}}$ of $M_{G(\mathbf{p})}$ at $\mathbf{p}$ with the first-order flexes of $G(\mathbf{p})$.

Let $h : T_{\mathbf{p}} \to M_{G(\mathbf{p})}$ be a smooth analytic map such that the following hold:

(a) On a neighborhood of $\mathbf{p}$ in $T_{\mathbf{p}}$, h is a real analytic diffeomorphism onto a neighborhood of $\mathbf{p}$ in $M_{G(\mathbf{p})}$.

(b) Identifying the tangent space of $T_{\mathbf{p}}$ with itself, $h(\mathbf{p}) = \mathbf{p}$ and $dh_{\mathbf{p}}(\mathbf{p}') = \mathbf{p}'$ for all $\mathbf{p}' \in T_{\mathbf{p}}$, where $dh_{\mathbf{p}}$ is the differential of h at $\mathbf{p}$.

For instance, the exponential map has these properties.

Let $\mathbf{q}(t) = \mathbf{p} + t\mathbf{p}' + \frac{1}{2}t^2\mathbf{q}$ be a smooth analytic path in $T_{\mathbf{p}}$ where we see that $\mathbf{q}(0) = \mathbf{p}$. $D_t[\mathbf{q}(t)]\big|_{t=0} = \mathbf{p}'$ and $D_t^2[\mathbf{q}(t)]\big|_{t=0} = \mathbf{q}''$, which will be determined later. Then define $\mathbf{p}(t) = h(\mathbf{q}(t))$, which is a smooth analytic flex of $G(\mathbf{p})$ with $\mathbf{p}(0) = \mathbf{p}$. Also

$$D_t[\mathbf{p}(t)] = D_t[h(\mathbf{q}(t))] = dh_{\mathbf{q}(t)}(D_t[\mathbf{q}(t)]) \tag{9}$$

and

$$D_t[\mathbf{p}(t)]\Big|_{t=0} = dh_{\mathbf{p}}(\mathbf{p}') = \mathbf{p}',$$

by condition (b). Since $\mathbf{p}(t) \in M_{G(\mathbf{p})}$, $\mathbf{p}(t)$ is an analytic flex of $G(\mathbf{p})$ and thus the second derivatives of the squares of the edge lengths are zero. Restating this in terms of the matrix $R(\mathbf{p})$ we get

$$R(D_t[\mathbf{p}(t)])D_t[\mathbf{p}(t)] + R(\mathbf{p}(t))D_t^2[\mathbf{p}(t)] = 0.$$

Evaluating when $t = 0$, we get

$$R(\mathbf{p}')\mathbf{p}' + R(\mathbf{p})\mathbf{r}'' = \mathbf{0},$$

where $\mathbf{r}'' = D_t^2[\mathbf{p}(t)]\big|_{t=0}$.

We must choose $\mathbf{q}''$ such that $\mathbf{r}'' = \mathbf{p}''$ the preassigned vector. Recalling that $(\mathbf{p}', \mathbf{p}'')$ is a second-order flex of $G(\mathbf{p})$ we see that for any $\mathbf{q}''$,

$$R(\mathbf{p}')\mathbf{p}' + R(\mathbf{p})\mathbf{p}'' = \mathbf{0} = R(\mathbf{p}')\mathbf{p}' + R(\mathbf{p})\mathbf{r}''.$$

So

$$R(\mathbf{p})(\mathbf{p}'' - \mathbf{r}'') = \mathbf{0}. \tag{10}$$

Differentiating (9) again we obtain

$$D_t^2[\mathbf{p}(t)] = D_t[dh_{\mathbf{q}(t)}]D_t[\mathbf{q}(t)] + dh_{\mathbf{q}(t)}D_t^2[\mathbf{q}(t)].$$

Applying the chain rule to each entry of the matrix $dh_{\mathbf{q}(t)}$, we see that $D_t[dh_{\mathbf{q}(t)}]\big|_{t=0}$ depends only on $\mathbf{p}'$ and not on $\mathbf{q}''$. Let $\mathbf{s}''$ be the value of the second derivative of $\mathbf{p}(t)$ (that is $\mathbf{r}''$) when $\mathbf{q}'' = \mathbf{0}$, evaluated when $t = 0$. In other words

$$\mathbf{s} = D_t[dh_{\mathbf{q}(t)}]D_t[\mathbf{q}(t)]\big|_{t=0}$$

is independent of the choice of $\mathbf{q}''$. We must then solve the linear equation

$$\mathbf{p}'' = \mathbf{s}'' + dh_{\mathbf{p}}(\mathbf{q}'') = \mathbf{s}'' + \mathbf{q}''$$

for $\mathbf{q}''$. Recall that $dh_{\mathbf{p}}$ is the identity map by condition (b). By (10), $(\mathbf{p}', \mathbf{s}'')$ is a second-order flex of $G(\mathbf{p})$ and $\mathbf{p}'' - \mathbf{s}''$ is a first-order flex of $R(\mathbf{p})$ and thus is in $T_{\mathbf{p}}$. Thus we can define $\mathbf{q}'' = \mathbf{p}'' - \mathbf{s}''$. Then $\mathbf{p}(t)$ as defined is the desired flex. □

Remark 6.1.1. One way of looking at the above proof is to think of adding some curvature via $\mathbf{q}''$ to the curve $\mathbf{q}(t)$ to cancel the curvature in $M_{G(\mathbf{p})}$ in order to achieve the given second derivative $\mathbf{p}''$.

For our purposes we do not need $\mathbf{p}(t)$ to be real analytic. It only needs to be twice differentiable. In the spirit of Definition 2.1.2 (c) we stated things in this more general form.

It seems natural that there also should be a generalization of this result involving any number of derivatives.

6.2. Roth's conjecture. We can now prove Roth's conjecture in its full generality.

PROPOSITION 6.2.1. *A convex b-c polygon in the plane is rigid if and only if it is first-order rigid.*

Proof. Let $\mathbf{p}'$ be any nontrivial first-order flex of a *b-c* polygon $G(\mathbf{p})$. If $G(\mathbf{p})$ has no nonzero proper self stress, then by the first-order stress test, we can choose $\mathbf{p}'$ such that $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) < 0$ for all cables $\{i, j\}$. If ω is any proper nonzero self stress for $G(\mathbf{p})$, then, by [7], the associated stress matrix Ω is negative semidefinite with only the affine motions in the kernel. It is easy to check, due to the convex nature of the polygon and the cabling, that $(\mathbf{p}')^T\Omega\mathbf{p}' \neq 0$. Thus $(\mathbf{p}')^T\Omega\mathbf{p}' < 0$. Thus by the second-order stress test, Corollary 5.2.1, $\mathbf{p}'$ extends to a second-order flex $(\mathbf{p}', \mathbf{p}'')$ that is strict on all cables. But the subframework $G_0(\mathbf{p})$ of $G(\mathbf{p})$ consisting of just the bars is just a convex polygon and so is independent. Thus Proposition 6.1.1 applies to show that there is a flex $\mathbf{p}(t)$ of $G(\mathbf{p})$, such that $D_t[\mathbf{p}(t)]\big|_{t=0} = \mathbf{p}'$ and $D_t^2[\mathbf{p}(t)]\big|_{t=0} = \mathbf{p}''$. But since $(\mathbf{p}', \mathbf{p}'')$ is strict on all cables, $\mathbf{p}(t)$ is a nontrivial continuous flex of $G(\mathbf{p})$ as well. Thus $G(\mathbf{p})$ is not rigid, and the result is proved. □

COROLLARY 6.2.2. *If a convex b-c polygon in the plane with v vertices has less than $v - 2$ cables, then it is not rigid.*

Proof. At least $v-2$ cables are needed to make the tensegrity framework infinitesimally rigid. □

Remark 6.2.1. When one is attempting to show directly that a particular convex *b-c* polygon is not rigid in the plane, one might be tempted to force some of the stressed cables to be bars in order to decrease the "degrees of freedom" and simplify the calculation. For example, the framework $G(\mathbf{p})$ of Figure 19 is not rigid. It is

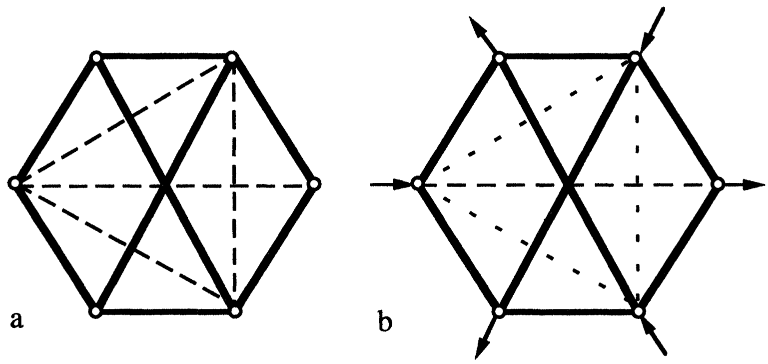

FIG. 19. *A flexible framework in the plane* (a), *with the corresponding first-order flex* (b).

obtained by forcing two of the cables of Figure 14a to be bars. (If the horizontal cable is changed to a strut (or bar), then the framework becomes rigid.)

The subframework $G_0(\mathbf{p})$, consisting of just the bars, is independent: $G(\mathbf{p})$ has a first-order flex, and from first-order considerations, as in [1], $G_0(\mathbf{p})$ has a continuous flex. The length of the horizontal cable cannot increase under this flex (by [7]), and the other cable lengths decrease strictly in their first derivative. Thus we obtain a continuous flex of $G(\mathbf{p})$.

However, one must be careful in deciding which of the cables to force to be bars. For example, consider the framework of Figure 20a.

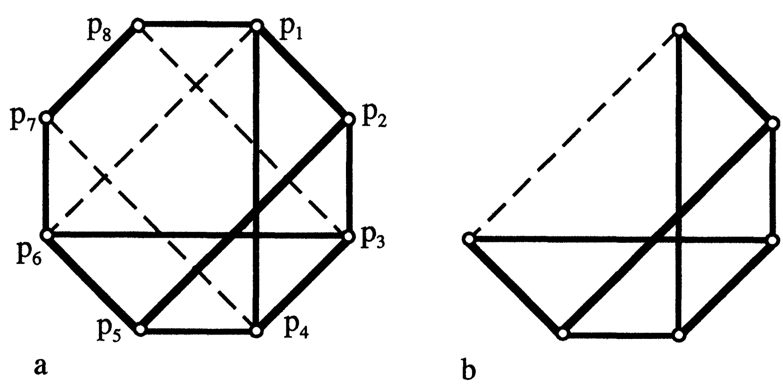

FIG. 20. *Framework* (a) *is rigid because subframework* (b) *is rigid.*

We have changed three of the stressed cables of Figure 3c to bars, and we consider only $\{1,6\}$, $\{4,7\}$, and $\{3,8\}$ from the rest of the cables of Figure 3c. There is a proper stress ω involving only members among the pairs of the first six vertices, since $\mathbf{p}_1, \ldots, \mathbf{p}_6$ lie on a circle. See Figure 20b for this *c-b* subframework, which is rigid by [7].

Considering $\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4$, and $\mathbf{p}_6$ as a pinned rigid subset, then the vertices $\mathbf{p}_1$,$\mathbf{p}_3$,$\mathbf{p}_4$,$\mathbf{p}_6$,$\mathbf{p}_7$,$\mathbf{p}_8$ determine an infinitesimally rigid framework. So the whole framework $G(\mathbf{p})$ in Figure 20a is rigid.

On the other hand, in $G(\mathbf{p})$, the subframework determined by just the bars is independent, as with our proof of Roth's conjecture. Also, all the proper self stresses of Figures 3c are proper self stresses of $G(\mathbf{p})$ as well, and there is a nontrivial first-order flex $\mathbf{p}'$ of Figure 3c that is a nontrivial first-order flex of $G(\mathbf{p})$. Why can't we then conclude, as we did with our proof of Roth's conjecture, that $G(\mathbf{p})$ is flexible by extending $\mathbf{p}'$ to s strict second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $G(\mathbf{p})$ (using the strict second-order stress test, Corollary 5.2.1)? The reason we cannot apply the second-order stress test is because we must consider *all* proper self stresses of $G(\mathbf{p})$, not just those coming from Figure 3c. In fact, $G(\mathbf{p})$ has more self stresses to consider due to the added bars. When the added stresses *are* considered, it turns out that $G(\mathbf{p})$ is even prestress stable.

The moral of the story is that if one wants to add interior bars, as in Figure 19, and keep the framework flexible, then one must not only be careful that the added bars keep the bar subframework independent and do not destroy the infinitesimal flexes, but also be sure that the new bars do not introduce any new proper self stresses to the whole framework.

A. Appendix on replacement principles.

A.1. From bar frameworks to cables and struts. Recall from §2.3 that if a bar framework is infinitesimally rigid, with a nonzero self stress, we can replace some of the bars with cables and struts, following the signs of the self stress. (See [29].)

Similarly, from §3.4, if a bar framework is prestress stable with a nontrivial self stress ω, then we are able to replace some of the bars with cables or struts, following the signs of this self stress ω.

For a second-order rigid bar framework, we have no such replacement principle. If the framework is not prestress stable, we must check the signs of all stresses used to block the cone of first-order flexes. If these all agree on a specific sign, then the corresponding bar can be replaced, while preserving second-order rigidity.

For a framework which is rigid, by some other test, we know of no general replacement principle. In the following, we show how to replace cables and struts with bars.

A.2. Equivalent bar frameworks for prestress stability. Given a tensegrity framework with cables and struts, we can always replace all members with bars. This replacement will, of course, preserve any rigidity in the framework. In fact it may increase the rigidity, turning a nonrigid framework into a rigid framework, a second-order rigid framework into a prestress stable framework, or a prestress stable framework into a first-order rigid framework. We would like a more delicate replacement principle which leaves the rigidity, prestress stability, or second-order rigidity unchanged.

We associate a special bar framework with a tensegrity framework, which does not depend on fixing a self stress. Suppose some framework $G(\mathbf{p})$ has a cable $\{i, j\}$ with $\mathbf{p}_i \neq \mathbf{p}_j$. We can then replace the cable by two bars $\{i, k\}$ and $\{k, j\}$ and place $\mathbf{p}_k$ on the open line segment between $\mathbf{p}_i$ and $\mathbf{p}_j$ to get a framework as in Figure 21a.

Similarly, a strut $\{i, j\}$ can be replaced by two bars $\{i, k\}$ and $\{k, j\}$, but now we insist that $\mathbf{p}_k$ be on the line through $\mathbf{p}_i$ and $\mathbf{p}_j$ but outside the closed line segment between $\mathbf{p}_i$ and $\mathbf{p}_j$ as in Figure 21b.

We call the above processes *splitting a cable* and *splitting a strut*, respectively. It is clear that such splittings do not change the rigidity of a framework in $\mathbb{R}^d$ for $d \geq 2$, but any such splitting creates a framework that is not first-order rigid. In fact we can split all the cables and struts of $G(\mathbf{p})$ to create what we shall call an *equivalent bar framework* $\hat{G}(\hat{\mathbf{p}})$, where $\hat{G}(\hat{\mathbf{p}})$ is rigid if and only if $G(\mathbf{p})$ is rigid.

We next look at the relation between splitting members and prestress stability. Suppose ω is a proper self stress for the tensegrity framework $G(\mathbf{p})$, and $\{i, j\}$ is a

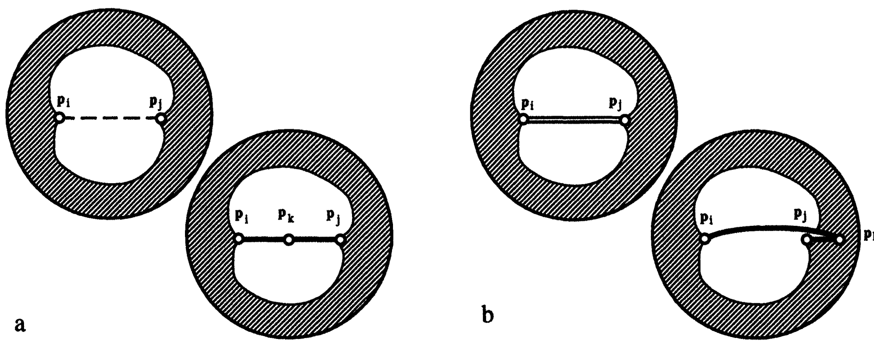

FIG. 21. *Replacing a cable* (a) *and a strut* (b) *with "equivalent" pairs of bars preserving prestress stability.*

cable or strut for G. Suppose $G(\mathbf{p})$ is split along $\{i,j\}$ at $\mathbf{p}_k$. Define

$$\hat{\omega}_{ik} = \omega_{ij}\frac{|\mathbf{p}_j - \mathbf{p}_i|}{|\mathbf{p}_i - \mathbf{p}_k|},$$

$$\hat{\omega}_{jk} = \left\{ \begin{array}{ll} \omega_{ij}\dfrac{|\mathbf{p}_j - \mathbf{p}_i|}{|\mathbf{p}_i - \mathbf{p}_k|} & \text{if} \quad \omega_{ij} > 0 \\ -\omega_{ij}\dfrac{|\mathbf{p}_j - \mathbf{p}_i|}{|\mathbf{p}_i - \mathbf{p}_k|} & \text{if} \quad \omega_{ij} < 0 \end{array} \right\},$$

and $\hat{\omega}_{\ell m} = \omega_{\ell m}$ for $\{\ell, m\} \neq \{i,k\}$ and $\{\ell, m\} \neq \{j,k\}$, where $\mathbf{p}_j$ is between $\mathbf{p}_i$ and $\mathbf{p}_k$ when $\omega_{ij} < 0$. It is easy to check that $\hat{\omega}$ defined above is a self stress for $\hat{G}(\hat{\mathbf{p}})$, the split framework.

PROPOSITION A.2.1. *Let $G(\mathbf{p})$ be any tensegrity framework with a proper strict self stress ω. Let $\hat{G}(\hat{\mathbf{p}})$ be the framework split along any cable or strut. Then ω stabilizes $G(\mathbf{p})$ if and only if $\hat{\omega}$ stabilizes $\hat{G}(\hat{\mathbf{p}})$.*

Proof. By Proposition 3.4.2 we need only consider a space of first-order flexes $\mathbf{p}'$ of $G(\mathbf{p})$ that are complementary to the trivial first-order flexes and then evaluate them on the form determined by Ω. A similar statement holds for $\hat{G}(\hat{\mathbf{p}})$. By the first-order stress test since $\omega_{ij} \neq 0$ we know that $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$. By adding a trivial first-order flex to $\mathbf{p}'$ we may consider some space of first-order flexes of $G(\mathbf{p})$ that has $\mathbf{p}'_i = \mathbf{p}'_j = 0$. Similarly for $\hat{G}(\hat{\mathbf{p}})$ we may consider only those first-order flexes $\hat{\mathbf{p}}'$ that are the direct sum of $\mathbf{p}'$ on the vertices of $G(\mathbf{p})$ and $\mathbf{p}'_k$ which is perpendicular to $\mathbf{p}_i - \mathbf{p}_j$, where k is the splitting vertex.

We evaluate $\hat{\Omega}$ at $\hat{\mathbf{p}}'$:

$$\begin{aligned} (\hat{\mathbf{p}}')^T\hat{\Omega}\hat{\mathbf{p}}' &= \sum_{\ell m} \hat{\omega}_{\ell,m}|\hat{\mathbf{p}}'_\ell - \hat{\mathbf{p}}'_m|^2, \\ &= \sum_{\ell m} |\mathbf{p}'_\ell - \mathbf{p}'_m|^2 + \omega_{ik}|\mathbf{p}'_k|^2 + \omega_{jk}|\mathbf{p}'_k|^2, \\ &= (\mathbf{p}')^T\Omega\mathbf{p}'. \end{aligned}$$

It is easy to check (even for struts) that $\omega_{ik} + \omega_{jk} > 0$. Thus $\hat{\Omega}$ is positive definite on its complementary space of nontrivial first-order flexes if and only if Ω is positive definite on its corresponding space. □

COROLLARY A.2.2. *Let $G(\mathbf{p})$ be any tensegrity framework and $\hat{G}(\hat{\mathbf{p}})$ the equivalent bar framework obtained by splitting all the cables and struts of nonzero length.*

Then $G(\mathbf{p})$ is prestress stable with a strict proper self stress if and only if $\hat{G}(\hat{\mathbf{p}})$ is prestress stable.

Thus, if we wish, we can "reduce" the problem of when a framework is prestress stable to the case when all the members are bars.

A.3. Equivalent bar frameworks for second-order rigidity. If we have a tensegrity framework with no strict proper self stress, such as in Figure 9b, then replacing this with the equivalent pair of bars can destroy second-order rigidity (but not rigidity). For example, a second-order flex is indicated on Figure 22.

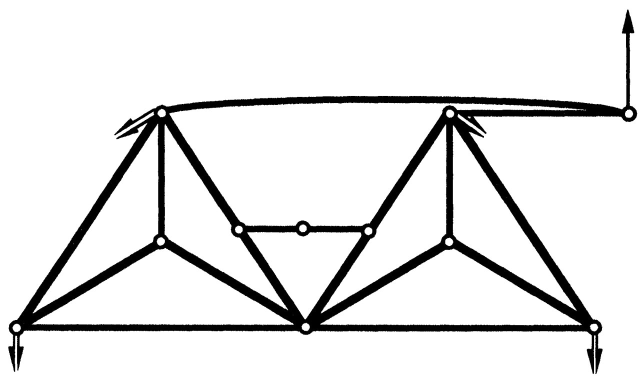

FIG. 22. *A second-order flexible (but rigid) bar framework equivalent to a second-order rigid tensegrity framework.*

If we restrict ourselves to tensegrity frameworks in which all cables and struts have nonzero coefficients in some self stress, then we can switch to the equivalent bar framework to check second-order rigidity. (Recall that for a tensegrity framework a proper self stress is strict if $\omega_{ij} \neq 0$ for every cable or strut.) We omit the proof, which is not difficult. See §5.2 and the second-order stress test.

PROPOSITION A.3.1. *A tensegrity framework with a strict proper self stress is second-order rigid if and only if the equivalent bar framework is second-order rigid.*

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