PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before October 31, 1993 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgment is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10306. *Proposed by Seung-Jin Bang, Seoul, Korea.*

Find all positive integers n such that the polynomial

$$a^n(b-c)+b^n(c-a)+c^n(a-b)$$

has $a^2+b^2+c^2+ab+bc+ca$ as a factor.

10307. *Proposed by John Calvin Williams, student, and I. Martin Isaacs, University of Wisconsin, Madison, WI.*

Can one construct a set $\mathscr{X}$ of finite groups satisfying the two conditions:

i. $\mathscr{X}$ contains precisely one representative from each isomorphism class.

ii. If $A \in \mathscr{X}$ is isomorphic to a subgroup of $B \in \mathscr{X}$, then A *is* a subgroup of B.

10308. *Proposed by Robert Connelly and John H. Hubbard, Cornell University, Ithaca, NY, and Walter Whiteley, York University, North York, Ontario, Canada.*

Suppose that $p_1, p_2, p_3, q_1, q_2, q_3$ are six points in the plane and that the distance between p_i and q_j $(i, j = 1, 2, 3)$ is $i + j$. Show that the six points are collinear.

10309. *Proposed by Walter Rudin, University of Wisconsin, Madison, WI.*

Compute

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(A + B\cos\theta)\,d\theta\right)$$

when $A > B > 0$. The answer should be given as an algebraic function of A and B.

10310. *Proposed by E. Rodney Canfield, University of Georgia, Athens, GA.*

Fix an integer $r \geq 2$. Using Stirling's formula we may find constants c_1 and c_2 such that

$$\binom{rm}{m} \sim \frac{c_1(c_2)^m}{m^{1/2}}$$

as $m \to \infty$. Prove that the ratio $\binom{rm}{m} m^{1/2}/c_2^m$ is an increasing function of m for $m \geq 1$.

10311. *Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, CA.*

It is well-known that if g is a primitive root modulo p, where $p > 2$ is prime, either g or $g + p$ (or both) is a primitive root modulo p^2 (indeed modulo p^k for all $k \geq 1$).

(a) Find an example of a prime $p > 2$, and a primitive root g modulo p with $1 < g < p$ such that g is *not* a primitive root modulo p^2.

(b) Show that, among all $\phi(p-1)$ primitive roots g modulo p with $1 < g < p$, at least half of them are also primitive roots modulo p^2.

10312. *Proposed by Hongyuan Zha, IMA—University of Minnesota, Minneapolis, MN.*

Let c and s be non-negative real numbers satisfying $c^2 + s^2 = 1$. Prove that, for $n > 1$,

$$s^{n-2}\sqrt{1+c}$$

is the *second* smallest singular value of the n by n upper triangular matrix

$$T_n(c) = \operatorname{diag}(1, s, \cdots, s^{n-1})\begin{pmatrix} 1 & -c & -c & \cdots & -c \\ & 1 & -c & \cdots & -c \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -c \\ & & & & 1 \end{pmatrix}.$$

10313. *Proposed by O. Krafft and M. Schaefer, Rheinisch-Westfälische Technische Hochschule, Aachen, Germany.*

Let $a \in [-1/5, 1)$ and let $\mathscr{X}_a$ denote the set of random variables X satisfying $a \leq X \leq 1$. Show that

$$\max\left\{EX^2EX^4 - (EX^3)^2 : X \in \mathscr{X}_a\right\} = 2^{-6}$$

if and only if $a \in [-1/5, 1/2]$.