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Problems: 10306-10313

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# PROBLEMS AND SOLUTIONS

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Edited by:  
**Richard T. Bumby, Fred Kochman and Douglas B. West**

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*An asterisk ( \* ) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

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## PROBLEMS

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**10306.** *Proposed by Seung-Jin Bang, Seoul, Korea.*

Find all positive integers  $n$  such that the polynomial

$$a^n(b-c) + b^n(c-a) + c^n(a-b)$$

has  $a^2 + b^2 + c^2 + ab + bc + ca$  as a factor.

**10307.** *Proposed by John Calvin Williams, student, and I. Martin Isaacs, University of Wisconsin, Madison, WI.*

Can one construct a set  $\mathcal{X}$  of finite groups satisfying the two conditions:

- i.  $\mathcal{X}$  contains precisely one representative from each isomorphism class.
- ii. If  $A \in \mathcal{X}$  is isomorphic to a subgroup of  $B \in \mathcal{X}$ , then  $A$  is a subgroup of  $B$ .

**10308.** *Proposed by Robert Connelly and John H. Hubbard, Cornell University, Ithaca, NY, and Walter Whiteley, York University, North York, Ontario, Canada.*

Suppose that  $p_1, p_2, p_3, q_1, q_2, q_3$  are six points in the plane and that the distance between  $p_i$  and  $q_j$  ( $i, j = 1, 2, 3$ ) is  $i + j$ . Show that the six points are collinear.

**10309.** Proposed by Walter Rudin, University of Wisconsin, Madison, WI.

Compute

$$\exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(A + B \cos \theta) d\theta\right)$$

when  $A > B > 0$ . The answer should be given as an algebraic function of  $A$  and  $B$ .

**10310.** Proposed by E. Rodney Canfield, University of Georgia, Athens, GA.

Fix an integer  $r \geq 2$ . Using Stirling's formula we may find constants  $c_1$  and  $c_2$  such that

$$\binom{rm}{m} \sim \frac{c_1(c_2)^m}{m^{1/2}}$$

as  $m \rightarrow \infty$ . Prove that the ratio  $\binom{rm}{m} m^{1/2} / c_2^m$  is an increasing function of  $m$  for  $m \geq 1$ .

**10311.** Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, CA.

It is well-known that if  $g$  is a primitive root modulo  $p$ , where  $p > 2$  is prime, either  $g$  or  $g + p$  (or both) is a primitive root modulo  $p^2$  (indeed modulo  $p^k$  for all  $k \geq 1$ ).

(a) Find an example of a prime  $p > 2$ , and a primitive root  $g$  modulo  $p$  with  $1 < g < p$  such that  $g$  is *not* a primitive root modulo  $p^2$ .

(b) Show that, among all  $\phi(p - 1)$  primitive roots  $g$  modulo  $p$  with  $1 < g < p$ , at least half of them are also primitive roots modulo  $p^2$ .

**10312.** Proposed by Hongyuan Zha, IMA—University of Minnesota, Minneapolis, MN.

Let  $c$  and  $s$  be non-negative real numbers satisfying  $c^2 + s^2 = 1$ . Prove that, for  $n > 1$ ,

$$s^{n-2} \sqrt{1+c}$$

is the *second* smallest singular value of the  $n$  by  $n$  upper triangular matrix

$$T_n(c) = \text{diag}(1, s, \dots, s^{n-1}) \begin{pmatrix} 1 & -c & -c & \cdots & -c \\ & 1 & -c & \cdots & -c \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -c \\ & & & & 1 \end{pmatrix}.$$

**10313.** Proposed by O. Krafft and M. Schaefer, Rheinisch-Westfälische Technische Hochschule, Aachen, Germany.

Let  $a \in [-1/5, 1)$  and let  $\mathcal{X}_a$  denote the set of random variables  $X$  satisfying  $a \leq X \leq 1$ . Show that

$$\max\{EX^2EX^4 - (EX^3)^2 : X \in \mathcal{X}_a\} = 2^{-6}$$

if and only if  $a \in [-1/5, 1/2]$ .