PROBLEMS


Find all positive integers $n$ such that the polynomial

$$a^n(b - c) + b^n(c - a) + c^n(a - b)$$

has $a^2 + b^2 + c^2 + ab + bc + ca$ as a factor.

10307. Proposed by John Calvin Williams, student, and I. Martin Isaacs, University of Wisconsin, Madison, WI.

Can one construct a set $\mathcal{X}$ of finite groups satisfying the two conditions:

i. $\mathcal{X}$ contains precisely one representative from each isomorphism class.

ii. If $A \in \mathcal{X}$ is isomorphic to a subgroup of $B \in \mathcal{X}$, then $A$ is a subgroup of $B$.

10308. Proposed by Robert Connelly and John H. Hubbard, Cornell University, Ithaca, NY, and Walter Whiteley, York University, North York, Ontario, Canada.

Suppose that $p_1, p_2, p_3, q_1, q_2, q_3$ are six points in the plane and that the distance between $p_i$ and $q_j$ $(i, j = 1, 2, 3)$ is $i + j$. Show that the six points are collinear.
10309. Proposed by Walter Rudin, University of Wisconsin, Madison, WI.

Compute
\[ \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(A + B \cos \theta) \, d\theta \right) \]
when \( A > B > 0 \). The answer should be given as an algebraic function of \( A \) and \( B \).

10310. Proposed by E. Rodney Canfield, University of Georgia, Athens, GA.

Fix an integer \( r \geq 2 \). Using Stirling’s formula we may find constants \( c_1 \) and \( c_2 \) such that
\[ \left( \frac{rm}{m} \right) \sim \frac{c_1(c_2)^m}{m^{1/2}} \]
as \( m \to \infty \). Prove that the ratio \( \left( \frac{rm}{m} \right) m^{1/2} / c_2^m \) is an increasing function of \( m \) for \( m \geq 1 \).

10311. Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, CA.

It is well-known that if \( g \) is a primitive root modulo \( p \), where \( p > 2 \) is prime, either \( g \) or \( g + p \) (or both) is a primitive root modulo \( p^2 \) (indeed modulo \( p^k \) for all \( k \geq 1 \)).

(a) Find an example of a prime \( p > 2 \), and a primitive root \( g \) modulo \( p \) with \( 1 < g < p \) such that \( g \) is not a primitive root modulo \( p^2 \).

(b) Show that, among all \( \phi(p - 1) \) primitive roots \( g \) modulo \( p \) with \( 1 < g < p \), at least half of them are also primitive roots modulo \( p^2 \).

10312. Proposed by Hongyuan Zha, IMA—University of Minnesota, Minneapolis, MN.

Let \( c \) and \( s \) be non-negative real numbers satisfying \( c^2 + s^2 = 1 \). Prove that, for \( n > 1 \),
\[ s^{n-2} \sqrt{1+c} \]
is the second smallest singular value of the \( n \) by \( n \) upper triangular matrix
\[ T_n(c) = \text{diag}(1, s, \cdots, s^{n-1}) \begin{pmatrix} 1 & -c & \cdots & -c \\ 1 & -c & \cdots & -c \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -c \\ 1 \end{pmatrix} \]


Let \( a \in [-1/5, 1) \) and let \( \mathcal{B}_a^* \) denote the set of random variables \( X \) satisfying \( a \leq X \leq 1 \). Show that
\[ \max \left\{ EX^2EX^4 - (EX^3)^2 : X \in \mathcal{B}_a^* \right\} = 2^{-6} \]
if and only if \( a \in [-1/5, 1/2] \).