Covering Curves by Translates of a Convex Set

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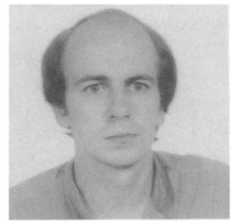
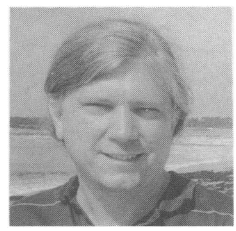
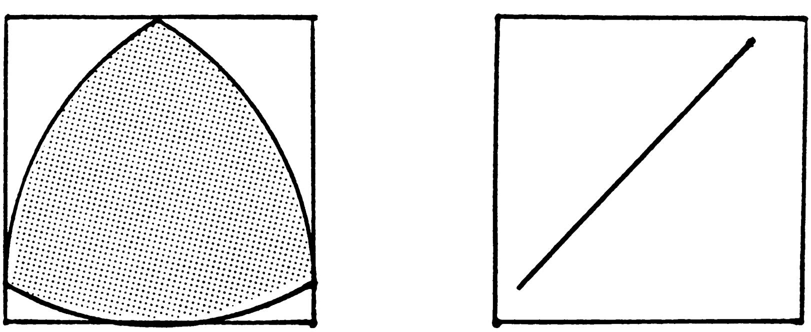
KÁROLY BEZDEK: I received my degrees in Budapest (Hungary), the M.S. Degree in 1978 and the University Doctorate Degree in 1980 at Eötvös Loránd University, the Kandidátus Degree (Ph.D.) in 1985 from the Hungarian Academy of Sciences. I have been teaching at the Department of Geometry of Eötvös Loránd University since 1978. I was on leave in the academic years 1985–86 and 1988–89 visiting Cornell University in Ithaca. My research focuses on different problems of discrete and convex geometry.



ROBERT CONNELLY: I received my Ph.D. in topology at the University of Michigan in 1969. Since then I have been at Cornell University on and off with excursions to I.H.E.S. in France, Syracuse University, the Max Planck-Institut für Math. in Bonn, West Germany, the University of Dijon and Savoie University in France, Eötvös University in Budapest, Hungary, and the Center for Research at the University of Montreal, Canada. My interests were compulsively attracted to geometry in the mid-1970's and since then I have been interested in the rigidity of frameworks and surfaces, reconstructing asteroid shapes, simplexwise-linear homeomorphisms of a two-disk, and the rigidity of packings.



I. Introduction. Turning a key in a lock is a familiar experience. Imagine a "key" that is a cylinder with convex planar cross-section. Imagine a "lock" which is a cylindrical hole with another convex cross-section. When can such a key fit inside the lock and turn 360°? Clearly for this problem we need only consider the cross-sections of the lock and key. For instance, the shaded set of Figure 1a can turn completely 360° inside the square, but the line segment of Figure 1b is longer than the length of a side of the square so it can never be rotated inside the square so as to be parallel to one of the sides of the square.



(a) (b)

FIG. 1

*Partially supported by NSF grant number MCS-790251.

Being mathematically curious, one might ask the following related basic question: What geometric conditions will insure that a convex set can be translated into another? For instance Wetzel [8] shows that for a given acute triangle, if a closed curve has length equal to or less than the length of the perimeter of the pedal triangle, then the closed curve can be translated into the given acute triangle. The pedal triangle is the triangle formed by the three feet of the altitudes of the triangle. See Figure 2.

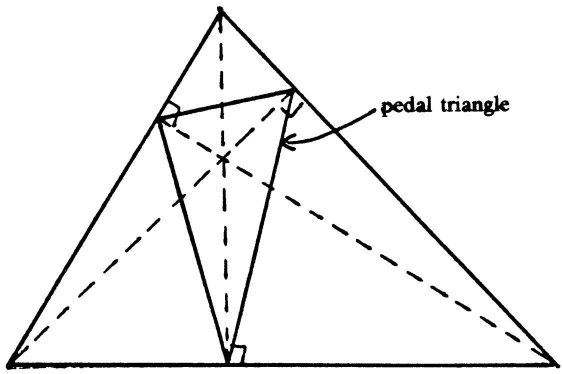


FIG. 2

Since rotating a curve does not change its length, any of the curves in Wetzel's Theorem can be rotated inside the given triangle. (It turns out for compact planar sets C and X, X convex, if for every $0 \leqslant \theta \leqslant 360°$, C rotated by θ can be translated into X, then C can be translated and then continuously rotated 360° inside X.)

Another example is when the covering set is a circular disk of diameter $1/2$. Then any closed curve of length one or less can be translated into the disk. This is an old result that can be found in the standard reference in this area, Bonnesen and Fenchel [2, p. 82]. (See also Nitche [14].) Note that since the covering set is a circular disk, the problem reduces simply to finding some congruent copy of the set to be covered in the disk. Any key can be turned in a round lock.

In section II we collect some principles and techniques that can possibly reduce the problem (of covering every element of a collection of sets by a translate of a fixed convex set) to a routine exercise.

Wetzel [21] looked at the problem of finding a plane convex set of minimum area that can cover any closed curve of length one or less with a translate. We do not know the answer to this problem either, but in section VI we find another set, with smaller area than the ones that Wetzel found, that can cover any closed curve of length one or less with a translate.

Another natural problem is to find a plane convex set of minimum perimeter that can cover any closed curve of length one or less with a translate. Our Theorem below answers this problem.

A line is called a *support line* for a compact set if the line has a non-empty intersection with the set, and the set is contained in one of the closed half planes with the line as boundary. A compact convex set has *constant breadth* b if the distance between any pair of distinct parallel support lines is the constant b.

We will prove the following:

THEOREM. *Let X be any compact convex set of constant breadth $1/2$ in the plane, and let C be a closed curve of length one or less in the plane. Then C can be covered by a translate of X. Furthermore, if Y is any compact convex set such that every closed curve of length one or less can be covered by a translate of Y, then the length of the perimeter of Y is equal to or larger than $\pi/2$ with equality if and only if Y has constant breadth $1/2$.*

Remark 1. An easy compactness argument shows that there must be a compact convex set X of minimum perimeter such that any closed curve in the plane with length one or less can be covered by a translate of X. What is surprising is that any compact convex set of constant breadth will serve as such a minimal X. A famous theorem of Barbier [**1**] says that all compact convex sets of constant breadth, $1/2$ say, have perimeters of the same length $\pi/2$, so they all are at least candidates.

Figure 3 shows some examples of the Theorem. Each of the sets on the top is a convex set of the same constant breadth and each curve on the bottom has length less than twice that breadth. Thus each curve on the bottom can be translated into each set on the top (and then turned 360° inside it).

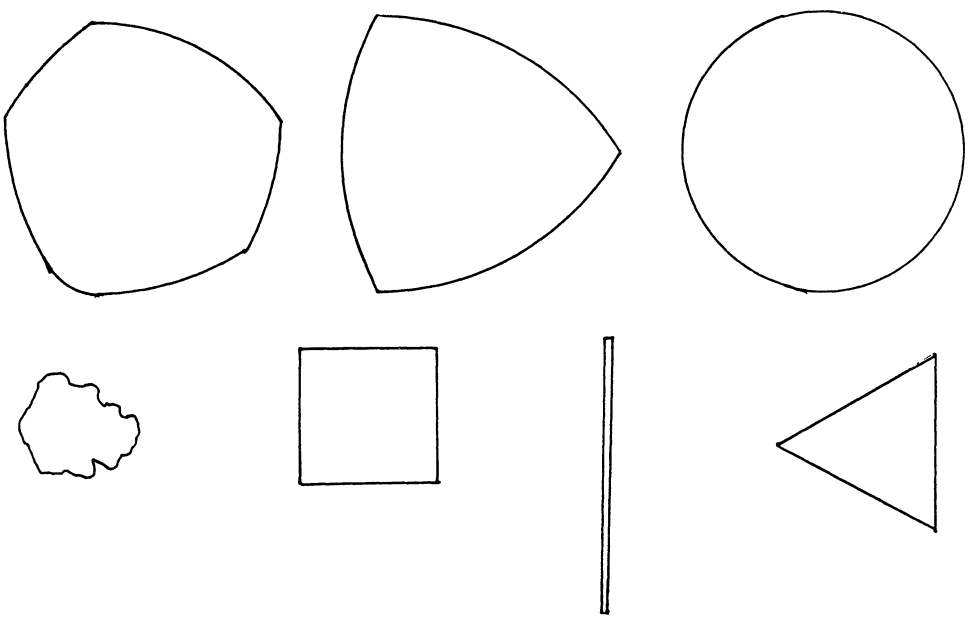


FIG. 3

Let $\mathscr{C}$ be any collection of sets in the plane. We say a set X is a *translation cover* for $\mathscr{C}$ if every set in $\mathscr{C}$ can be covered by a translate of X. Let $\mathscr{L}_1$ be the collection of closed curves in the plane of length one or less. Our theorem then can be restated as saying that the compact convex sets of constant breadth $1/2$ are all the translation covers for $\mathscr{L}_1$ with minimum perimeter.

The terminology "translation cover" comes from Wetzel [**21**, p. 368]. If a compact convex set X can cover any element of $\mathscr{C}$ by some congruent copy, then Wetzel calls X a *displacement cover* for $\mathscr{C}$. Eggleston [**6**, p. 131], uses the word "universal cover" instead of displacement cover. Chakerian [**3**, p. 759], and Lay [**12**, p. 85] use the word "strong universal cover" instead of translation cover. A displacement cover is just a lock that allows the key to enter, but the key might not be able to turn.

We prefer Wetzel's terminology. However, Chakerian and Lay have an interesting discussion of such covers, whatever they are called, as well as other related subjects. The reader is also referred to [**4**], [**5**], and [**17**]–[**21**] for further information about more related covering problems.

In the following proof of the main Theorem (which occupies sections II-V of this paper) the exposition will be self-contained, except for three results: Helly's Theorem (Lemma 1, below), Sperner's Lemma (Lemma 5, below), and Fagnano's Theorem about pedal triangles (Lemma 7, below). These are all basic results that have been amply discussed elsewhere, and we have nothing to add to their proofs. However, we can apply a result of Chakerian to shorten the last step of the proof of our main Theorem. Nevertheless, we will also provide another proof, without using Chakerian's result, following our ideas of using billiard triangles.

Section II contains a series of general lemmas that, roughly speaking, reduce the covering problem to covering triangles, and the "local" problem to the case of covering a triangle by a triangle. In our case this turns out to be the billiard condition as discussed in section III. This in turn gives us a way of calculating a number for a convex set X that is the upper bound for the length of a closed curve to insure that it can be covered by a translate of X. This is applied in section IV to the Reuleaux triangle, in section V to prove the main Theorem, and in section VI to investigate Wetzel's problem.

In section VII we look at other related problems and generalizations regarding translation covers, and mention several more results, without proof, that we can obtain with our techniques.

We thank the referees for many helpful comments, especially for informing us of Chakerian's result, and suggestions regarding an earlier version of this paper. We also thank John Wetzel for referring us to his papers [**18**], [**20**], and [**21**], and Maria Terrell for helpful advice.

II. General Helly-type facts. There are many reductions that are very helpful. We start with Helly's Theorem.

LEMMA 1 (Helly). *Let $\mathscr{C}$ be any collection of compact convex sets in the plane. If every* 3 *sets of $\mathscr{C}$ have a nonempty intersection, then the intersection of all the sets of $\mathscr{C}$ have non-empty intersection.*

A good reference for the many proofs and generalizations of this result is Danzer, Grünbaum, Klee [**5**].

For any set X in the plane, and a point p in the plane, define

$$p + X = \{ p + x | x \in X \}.$$

the *translate* of X by p. All points in the plane are regarded as vectors in some coordinate system.

The following observation changes our covering problem to an intersection problem.

LEMMA 2. *Let X and Y be subsets of the plane. A translate of X contains Y if and only if*

$$\bigcap_{y \in Y} (-y + X) \neq \varnothing .$$

Proof. $p \in \bigcap_{y \in Y}(-y + X)$ if and only if $p \in (-y + X)$ for all $y \in Y$ if and only if $p + y \in X$ for all $y \in Y$ if and only if $p + Y \subset X$. □

We can now reduce the covering problem to the case of 3 points. This can be found in Chakerian **[3]** and is due to Klee **[11]**. The next Lemma follows immediately from Lemma 1 and Lemma 2.

LEMMA 3. *Let Y be any set in the plane with at least three points. A compact convex set X has a translate which covers Y if and only if X has a translate which covers every three-point subset of Y.*

Proof. By Lemma 2 a translate of X covers Y if and only if $\bigcap_{y \in Y}(-y + X) \neq \varnothing$. Since every three-point subset of Y has a translate of X which covers it, every three of the sets in the intersection have a nonempty intersection. By Lemma 1 the whole intersection is nonempty. Thus Lemma 3 follows. □

Remark 1. Since any three points on a closed curve of length one or less form a triangle of perimeter one or less, by Lemma 3 we can assume, without loss of generality, that the curve we want to cover is a triangle with the understanding that we allow the triangle to possibly degenerate to a line segment.

To ease the development we introduce a bit of notation. A *triangle* in the plane is written as $\Delta = \langle p_1, p_2, p_3 \rangle$, where p_1, p_2, p_3 are the vertices of Δ. Note that Δ can degenerate into a line segment when some pair of the vertices coincide. We define the length of the perimeter of Δ as

$$l(\Delta) = |p_1 - p_2| + |p_2 - p_3| + |p_3 - p_1| .$$

For a set X in the plane and $L \geqslant 0$, let

$$\mathscr{T}_X = \{ \Delta \text{ a triangle} | \text{there is a point } p \text{ such that } p + \Delta \subset X \},$$
$$\mathscr{L}_L = \{ \Delta \text{ a triangle} | l(\Delta) \leqslant L \}.$$

int X denotes the topological interior of the set X, and similar to what was done with translates we let

$$\alpha X = \{ \alpha x | x \in X \}.$$

For sets A, B we denote the difference set

$$A \setminus B = \{ a | a \in A \text{ and } a \notin B \}.$$

$\mathbb{R}^2$ denotes the Euclidean plane.

We now wish to look at the covering problem from another point of view. We investigate the triangles that have no translate in the interior of X and relate the minimum of their lengths to the maximum length that insures that a triangle (or any closed curve) with that length can be translated into X.

LEMMA 4. *Let X be a compact convex set with non-empty interior in the plane. Then there is a triangle (possibly degenerating into a line segment) $\triangle_0 \in \mathscr{T}_X \setminus \mathscr{T}_{\operatorname{int} X}$ such that*

$$l(\triangle_0) = \inf\{l(\triangle) | \triangle \text{ is a triangle with } \triangle \notin \mathscr{T}_{\operatorname{int} X}\}$$
$$= \sup\{L | \mathscr{L}_L \subset \mathscr{T}_X\}.$$

Proof. Since triangles with sufficiently small perimeter clearly can be translated into X, for some $\varepsilon > 0$, $\mathscr{L}_\varepsilon \subset \mathscr{T}_X$. Thus $\sup\{L | \mathscr{L}_L \subset X\} = L_0$ exists. Let $\triangle \in \mathscr{L}_{L_0}$ with $l(\triangle) = L_0$. Then for every $0 < \alpha < 1$, $\alpha\triangle$ has a translate in X, since $l(\alpha\triangle) = \alpha L_0 < L_0$. Thus $\alpha\triangle \in \mathscr{L}_{\alpha L_0} \subset \mathscr{T}_X$. As $\alpha \to 1$, a limit of translates of $\alpha\triangle$ will be contained in X, and since X is compact, $\triangle \in \mathscr{T}_X$. Thus $\mathscr{L}_{L_0} \subset \mathscr{T}_X$.

For every $\delta > 0$, let $\triangle_\delta$ be a triangle with $\triangle_\delta \notin \mathscr{T}_X$ with $L_0 < l(\triangle_\delta) \leqslant L_0 + \delta$. By fixing a vertex of $\triangle_\delta$ and bounding all the $\triangle_\delta$, we may choose $\triangle_\delta$ such that $\lim_{\delta \to 0} \triangle_\delta = \triangle_0$, a triangle in the plane. Then by the continuity of l, $l(\triangle_0) = L_0 = \sup\{L | \mathscr{L}_L \subset \mathscr{T}_X\}$, and $\triangle_0 \in \mathscr{T}_X$.

If a triangle $\triangle$ has $l(\triangle) < L_0$, then for some $\alpha > 1$, $l(\alpha\triangle) = \alpha l(\triangle) < L_0$, $\alpha\triangle \in \mathscr{T}_X$, and $\triangle \in \mathscr{T}_{\operatorname{int} X}$. Thus

$$L_0 \leqslant \inf\{l(\triangle) | \triangle \notin \mathscr{T}_{\operatorname{int} X}\}.$$

Similarly for every $\delta > 0$, for $\triangle_\delta$ chosen as above, $\triangle_\delta \notin \mathscr{T}_{\operatorname{int} X}$ implies that $L_0 = l(\triangle_0) \geqslant \inf\{l(\triangle) | \triangle \notin \mathscr{T}_{\operatorname{int} X}\}$. Thus $\sup\{L | \mathscr{L}_L \subset \mathscr{T}_X\} = L_0 = \inf\{l(\triangle) | \triangle \notin \mathscr{T}_{\operatorname{int} X}\}$.

Since $\triangle_0 = \lim_{\delta \to 0} \triangle_\delta$ and $\triangle_\delta \notin \mathscr{T}_{\operatorname{int} X}$, then $\triangle_0 \notin \mathscr{T}_{\operatorname{int} X}$. Thus $\triangle_0 \in \mathscr{T}_X \setminus \mathscr{T}_{\operatorname{int} X}$. $\square$

The following Lemma is a version of Sperner's Lemma for open (convex) sets. We show how to deduce this from the statement in Lyusternik [**13**, p. 162], where it is stated for closed sets. The following Lemma is the contrapositive stated for the complementary open sets.

LEMMA 5 (Sperner). *Let $\triangle = \langle p_1, p_2, p_3 \rangle$ be a triangle in the plane with non-empty interior. Let U_1, U_2, U_3 be open sets in the plane with the line segment $\langle p_{i-1}, p_{i+1} \rangle \subset U_i$, $i = 1, 2, 3$, (indices taken modulo 3). If $\triangle \subset U_1 \cup U_2 \cup U_3$ then $\triangle \cap U_1 \cap U_2 \cap U_3 \neq \varnothing$.*

Proof. Suppose for some $i = 1, 2, 3$, $U_i \cap U_{i+1} \cap \langle p_i, p_{i+1} \rangle \neq \varnothing$. Then $U_1 \cap U_2 \cap U_3 \neq \varnothing$. If $p_i \in U_i$, for some $i = 1, 2, 3$, then $p_i \in U_1 \cap U_2 \cap U_3$.

Let $\triangle \subset U_1 \cup U_2 \cup U_3$, i.e., $\bigcap_{i=1}^3 (\triangle \setminus U_i) = \varnothing$, and from the above we may assume that for all $i = 1, 2, 3$, $p_i \in \triangle \setminus U_i$ and $\langle p_i, p_{i+1} \rangle \subset (\triangle \setminus U_i) \cup (\triangle \setminus U_{i+1})$. Thus by Sperner's Lemma for closed sets $\cup_{i=1}^3 (\triangle \setminus U_i)$ does not contain $\triangle$, i.e., $\triangle \cap U_1 \cap U_2 \cap U_3 \neq \varnothing$, finishing the Lemma. $\square$

Let $H = \langle H_1, H_2, H_3 \rangle$ be three closed half planes. We say that H is *nearly bounded* if $H_1 \cap H_2 \cap H_3 = \bar{\triangle}$ is contained between two parallel lines. In the case when two of the H_i's are the same we can write $H = \langle H_1, H_2 \rangle$. If $\bar{\triangle}$ is bounded, then H is clearly nearly bounded. It is easy to show that the following are equivalent:

1. H is nearly bounded.
2. $\langle p_1 + H_1, p_2 + H_2, p_3 + H_3 \rangle$ is nearly bounded, for all p_1, p_2, p_3 in the plane.

3. $(p_1 + H_1) \cap (p_2 + H_2) \cap (p_3 + H_3) = \varnothing$, for some p_1, p_2, p_3 in the plane.
4. $\langle(\mathbb{R}^2 \setminus \operatorname{int} H_1), (\mathbb{R}^2 \setminus \operatorname{int} H_2), (\mathbb{R}^2 \setminus \operatorname{int} H_3)\rangle$ is nearly bounded.
5. $\overline{\triangle}$ has no translate contained in $\operatorname{int} \overline{\triangle}$.

If we look at the inward pointing vector n_i perpendicular to the boundary of H_i, $i = 1, 2, 3$, we see that H is nearly bounded if and only if the 0 vector is in the convex hull of the n_i.

Let $\langle p_1, p_2, p_3 \rangle = \triangle$ be a triangle in the plane, and let $H = \langle H_1, H_2, H_3 \rangle$ be closed half planes with $\overline{\triangle} = H_1 \cap H_2 \cap H_3$. If H is nearly bounded and $p_i \notin \operatorname{int} H_i$, for $i = 1, 2, 3$, then we say that H (or $\overline{\triangle}$) *separates* $\triangle$ (from any set contained in $\overline{\triangle}$). See Figure 4.

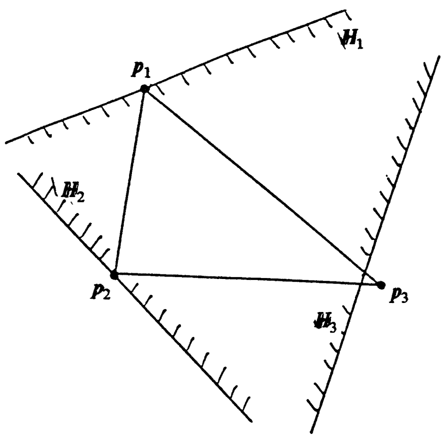


Fig. 4

We can now provide a criterion for when a triangle cannot be translated into a convex set.

LEMMA 6. *Let X be a compact convex set with non-empty interior. A triangle (possibly degenerating into a line segment) $\triangle \notin \mathscr{T}_{int\,X}$ if and only if there are closed (nearly bounded) half planes $H = \langle H_1, H_2, H_3 \rangle$ that separate a translate of $\triangle$ from X, or $H = \langle H_1, H_2 \rangle$ (with parallel boundaries) separates a translate of an edge of $\triangle$ from X.*

Proof. Suppose $H = \langle H_1, H_2, H_3 \rangle$, which is nearly bounded, separates $\triangle = \langle p_1, p_2, p_3 \rangle$ with $H_1 \cap H_2 \cap H_3 = \overline{\triangle} \supset X$. If a translate of $\triangle$, $p + \triangle \subset \operatorname{int} X$, $p + H_i \subset \operatorname{int} H_i$, $i = 1, 2, 3$, and $p + \overline{\triangle} \subset \operatorname{int} \overline{\triangle}$, which is impossible by property 5. Thus $\triangle \notin \mathscr{T}_{\operatorname{int} X}$. If $H = \langle H_1, H_2 \rangle$ separates $\langle p_1, p_2 \rangle$, the proof is similar.

Suppose $\triangle \notin \mathscr{T}_{\operatorname{int} X}$. By Lemma 2, $\bigcap_{i=1}^{3}(-p_i + \operatorname{int} X) = \varnothing$.

If for some i, $(-p_{i-1} + \operatorname{int} X) \cap (-p_{i+1} + \operatorname{int} X) = \varnothing$ (indices taken mod 3), then a line B separates $(-p_{i-1} + \operatorname{int} X)$ and $(-p_{i+1} + \operatorname{int} X)$ in the usual sense. Thus there is a point p on B and closed half planes H_{i-1} and H_{i+1} such that $p \in bdy\, H_{i-1} \cap bdy\, H_{i+1} = B$, $-p_{i-1} + \operatorname{int} X \subset H_{i-1}$, and $-p_{i+1} + \operatorname{int} X \subset H_{i+1}$, where *bdy* denotes the topological boundary. Thus $p_{i-1} + p \in bdy(p_{i-1} + H_{i-1})$, $p_{i+1} + p \in bdy(p_{i+1} + H_{i+1})$, $p_{i-1} + H_{i-1} \supset X$, $p_{i+1} + H_{i+1} \supset X$, and thus $\langle p_{i-1} + H_{i-1}, p_{i+1} + H_{i+1} \rangle$ is nearly bounded and separates $\langle p + p_{i-1}, p + p_{i+1} \rangle$, an edge of $p + \triangle$, from X.

So we are left with the case when $(-p_{i-1} + \text{int } X) \cap (-p_{i+1} + \text{int } X) \neq \varnothing$, for all $i = 1, 2, 3$. Let $q_i \in (-p_{i-1} + \text{int } X) \cap (-p_{i+1} + \text{int } X)$, for $i = 1, 2, 3$. We choose $\Delta' = \langle q_1, q_2, q_3 \rangle$ with non-empty interior as a triangle. Then $\langle q_{i-1}, q_{i+1} \rangle \subset -p_i + \text{int } X$, for all $i = 1, 2, 3$. Thus Lemma 5 and Lemma 2 imply that $\bigcup_{i=1}^3(-p_i + \text{int } X)$ does not contain Δ'. Let $q \in \Delta' \setminus \bigcup_{i=1}^3(-p_i + \text{int } X)$. Then for $i = 1, 2, 3$, let H_i be a closed half plane with $q \in bdy\, H_i$ and $(-p_i + \text{int } X) \subset \text{int } H_i$. Thus $q \in \bigcap_{i=1}^3(\mathbb{R}^2 \setminus \text{int } H_i) = \Delta''$, and $\Delta'' \cap \langle q_{i-1}, q_{i+1} \rangle = \varnothing$, for $i = 1, 2, 3$, because $H_i \supset \langle q_{i-1}, q_{i+1} \rangle$. Thus $\Delta'' \cap bdy \Delta' \neq \varnothing$ and $\Delta'' \subset \Delta'$. Thus Δ'' is bounded. (It turns out that $\Delta'' = \{q\}$.) Thus $H = \langle H_1, H_2, H_3 \rangle$ is nearly bounded by property 4. $p_i + q \in bdy(p_i + H_i)$ and so $\langle p_1 + H_1, p_2 + H_2, p_3 + H_3 \rangle$ is (nearly) bounded by property 2 and separates $q + \Delta = \langle q + p_1, q + p_2, q + p_3 \rangle$ from X. See Figure 5. $\square$

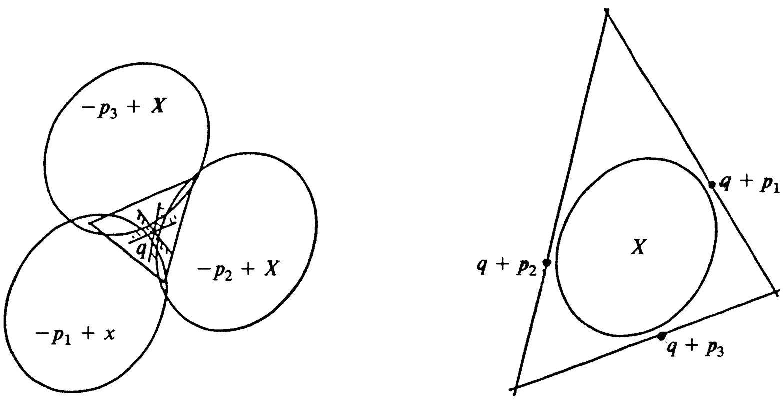


FIG. 5

Remark 2. All of the above Lemmas and their proofs generalize to Euclidean space of any (positive) dimension. Both Helly's Theorem and Sperner's Lemma are true and well known in this generality.

Remark 3. We can use other functions for l besides the length of the perimeter. The properties of l that we need are that l is a real-valued function defined for at least convex sets in Euclidean space and the following hold.

1. l is continuous.
2. $l(\Delta) < l(\alpha\Delta)$, for $\alpha > 1$.
3. $l(p + \Delta) = l(\Delta)$ for all points p.
4. $\{\Delta \mid l(\Delta) \leqslant 1, 0 \in \Delta\}$ is bounded.

For instance we could take

$$l(X) = \text{diameter of } X = \sup\{|x - y| \mid x, y \in X\}.$$

III. Billiard triangles. We can now further reduce the set of triangles that need to be considered for our covering problem. Let X be a compact convex set with non-empty interior in the plane. For $i = 1, 2, 3$, let p_i be three distinct points on the boundary of X, and let H_i be a closed half plane containing X with $p_i \in bdy\, H_i$. If

$p_{i+1} - p_i$ and $p_{i-1} - p_i$ (regarded as vectors) make equal angles with the line *bdy* H_i at p_i, for all $i = 1, 2, 3$ (indices taken modulo 3), then we say $\triangle = \langle p_1, p_2, p_3 \rangle$ is a *billiard triangle* for X. Note that the H_i containing p_i as above, in general, may not be unique. However, we will only consider billiard triangles that are separated from X by a unique $\langle H_1, H_2, H_3 \rangle$, where $\overline{\triangle} = H_1 \cap H_2 \cap H_3$ is a triangle. See Figure 6 below.

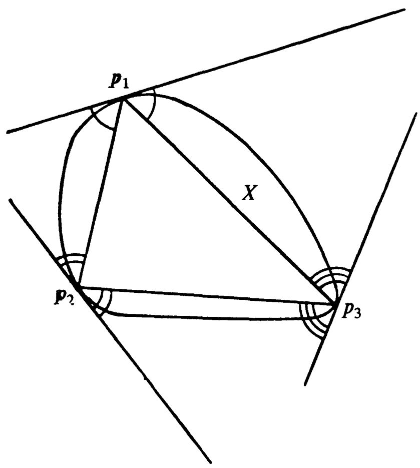


FIG. 6

We look at our covering problem for the special case when X is a triangle. This is a very old problem of Fagnano with some pleasant solutions involving a "reflection principle." Some very elementary proofs are described in Rademacher and Toeplitz [**16**, p. 27–34], as well as Kazarinoff [**10**, pp. 75–77].

LEMMA 7 (Fagnano). *Let $\overline{\triangle}$ be a triangle in the plane. If $\overline{\triangle}$ is acute then the triangle $\triangle$ with smallest perimeter inscribed in $\overline{\triangle}$ is the pedal triangle formed by the feet of the three altitudes of $\overline{\triangle}$. Furthermore $\triangle$ is the unique billiard triangle for $\overline{\triangle}$. If $\overline{\triangle}$ is not acute, then there is no billiard triangle in $\overline{\triangle}$, and any inscribed triangle has perimeter greater than twice the minimum altitude.*

We state the main criterion by which we can calculate which closed curves can be covered by a given compact convex set X.

Let $m_\triangle = \inf\{l(\triangle) | \triangle$ is a billiard triangle for $X\}$. If X has no billiard triangle $m_\triangle = \infty$. Let b be the minimum breadth of the set X, i.e., b is the minimum distance between two distinct parallel support lines of X.

LEMMA 8. *Let X be a compact convex set. Any closed curve of length one or less can be covered by a translate of X if and only if* $\min\{m_\triangle, 2b\} \geqslant 1$.

Proof. We will show that $\min\{m_\triangle, 2b\} = l(\triangle_0)$, where $\triangle_0 \in \mathscr{T}_X \setminus \mathscr{T}_{\operatorname{int} X}$ is as in Lemma 4. The result will then follow using Lemma 3 as well as Lemma 4.

Suppose that $\triangle_0$ does not degenerate to a line segment. If $\langle H_1, H_2 \rangle$ separates an edge of $\triangle_0$ from X, then $\triangle_0$ degenerates to a line segment. Thus by Lemma 6 there are half planes $H = \langle H_1, H_2, H_3 \rangle$ that separate a translate of $\triangle_0$ from X. We take $\triangle_0 \subset X$. By Lemma 7 (Fagnano) the pedal triangle of $\overline{\triangle} = H_1 \cap H_2 \cap H_3$ is the shortest triangle separated by (i.e. inscribed in) $\overline{\triangle}$, if there is such a shortest triangle. Thus $\triangle_0$ must be this pedal triangle for $\overline{\triangle}$. So $\triangle_0$ has the billiard property at each p_i, $i = 1, 2, 3$, $\overline{\triangle}$ is unique, and thus $l(\triangle_0) \geqslant m_\triangle$. By Lemma 4, Lemma 6, and the definition of a billiard triangle $l(\triangle_0) \leqslant m_\triangle$. Thus $l(\triangle_0) = m_\triangle \leqslant 2b$.

If $\triangle_0$ is a line segment, then clearly $l(\triangle_0) = 2b \leqslant m_\triangle$. □

Remark 4. If we replace l (the function that gives the length of the perimeter) by the diameter function, we can also provide a similar analysis as above. The critical triangle $\triangle_0$ of Lemma 4 can be taken to be either an equilateral triangle, an isosceles triangle with its base shorter than its two legs, or a line segment, since any such $\triangle_0$ is contained in one such triangle.

A corollary of the above observation is that if every circular wedge with a 60° vertex angle and leg length one can be translated into X, a compact convex set, then any set of diameter one or less can be translated into X. Of course the same is true for the Reuleaux triangle (see the next section) of diameter one, replacing the wedge. This is a curious dual to Chakerian's Theorem mentioned in section V.

We can continue with an analogue of Lemma 8. Namely $\triangle_0$ must be separated by $\overline{\triangle}$ in only a particular way.

If $\triangle_0$ is an equilateral triangle, then the lines perpendicular to the edges of $\overline{\triangle}$ at p_i, $i = 1, 2, 3$, must meet at a point in $\triangle_0$. See Figure 7. If the perpendiculars do not meet at a point as above, $\triangle$ can be "rotated" out of $\overline{\triangle}$ and then contracted to obtain another $\triangle'$, still separated by $\overline{\triangle}$ but with $l(\triangle') < l(\triangle)$ (l now being the diameter).

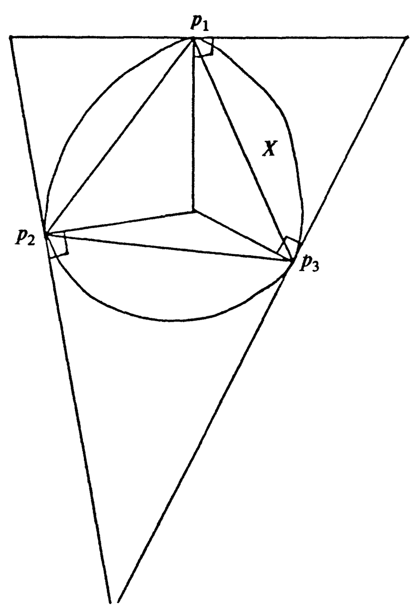


FIG. 7

If $\triangle_0$ is an isosceles triangle, then the equal sides of $\triangle_0$ must be perpendicular to the corresponding sides of $\bar{\triangle}$, as in Figure 8 below.

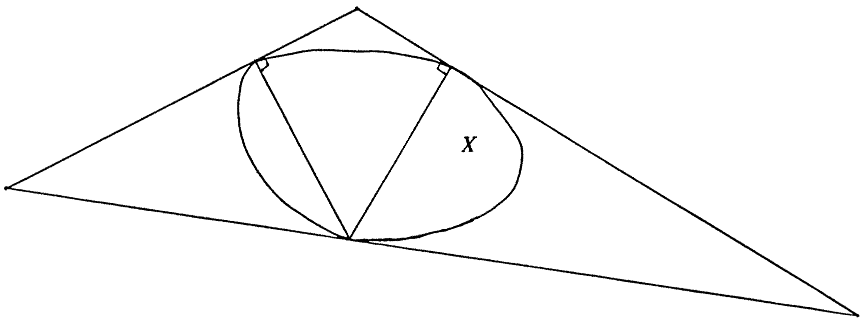


FIG. 8

If $\triangle_0$ is a line segment, then $\triangle_0$ is of minimal breadth, as before.

IV. Reuleaux triangles. We now apply Lemma 8 to prove a special case of the Theorem, namely when X is a Reuleaux triangle. A *Reuleaux triangle* is the convex set of constant breadth r obtained as the intersection three circular disks of the same radius r with the centers of the circles at the vertices of an equilateral triangle of side length r. See Figure 9.

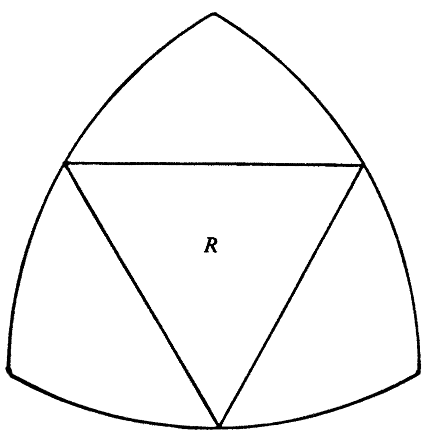


FIG. 9

LEMMA 9. *Let R be a Reuleaux triangle of constant breadth $1/2$. Then R has only the one symmetric billiard triangle joining the midpoints of the circular arcs as below. Thus $m_{\triangle} = 3(\sqrt{3} - 1)/2 \approx 1.098 > 1$ and every closed curve of length one or less is covered by a translate of R.*

Proof. From Lemma 7 the only billiard triangle in a triangle $\bar{\triangle}$ is the pedal triangle P formed by the feet of the three altitudes of $\bar{\triangle}$, and this billiard triangle exists only when $\bar{\triangle}$ is acute.

Suppose $\triangle = \{p_1, p_2, p_3\}$ forms a billiard triangle in R. Let L_i, $i = 1,2,3$, be the corresponding support lines. Clearly each p_i is on a separate circular arc of the Reuleaux triangle R. Let r_i be the vertex of R opposite p_i, and let C be the center of the inscribed circle of $\triangle$. Then r_i, C, p_i, are collinear because $r_i - p_i$ and $C - p_i$ are both perpendicular to L_i. See Figure 10.

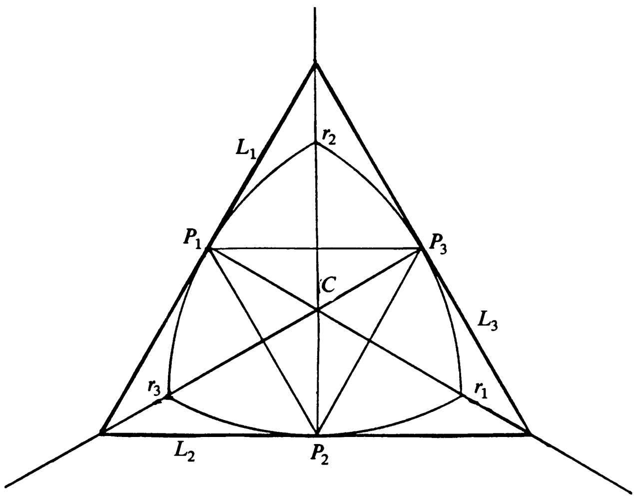


FIG. 10

Thus $\triangle$ is a billiard triangle for the triangle $\bar{\triangle}$ formed by L_1, L_2, L_3, as well as R. Thus $\triangle$ is the pedal triangle of $\bar{\triangle}$, and C is the altitude center of $\bar{\triangle}$. With all this information we wish to show that C is the center of R. This will show that $\triangle$ is the desired symmetric triangle.

Let C_R be the center of R. Let q_i be the midpoint of the circular arc from r_{i+1} to r_{i-1}, as in Figure 11 below.

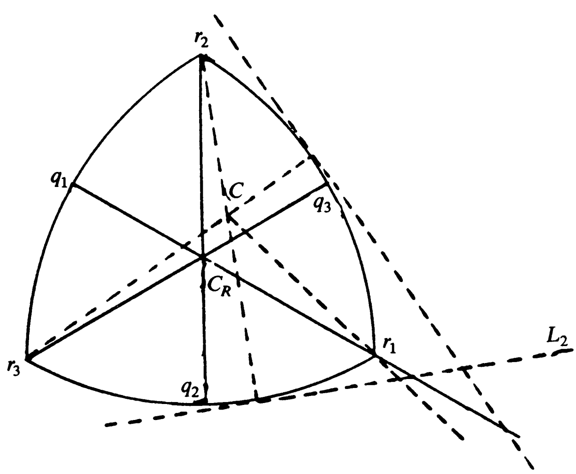


FIG. 11

Assume, without loss of generality, that C lies in the closed region of R defined by C_R, q_3, r_2. It is clear that the support lines L_3 and L_2 intersect at a point on the q_3 side of the line $C_R r_1$. On the other hand the ray from r_1 outside R on the line Cr_1 lies on the other side of $C_R r_1$. Thus the three lines L_2, L_3, Cr_1 intersect only if $C = C_R$ and $p_i = q_i$, $i = 1, 2, 3$. $\square$

V. Proof of the theorem. We thank one of the referees for pointing out that, at this point, we can complete the proof of the Theorem by using the following result of Chakerian [**3**, p. 760].

THEOREM (Chakerian). *Let R be a Reuleaux triangle of constant breadth b in the Euclidean plane, and let P be a compact subset of the plane. Suppose that each congruent copy of P can be covered by a translate of R. Then if K is any compact convex set of constant breadth b, each congruent copy of P can be covered by a translate of K.*

Remark 5. The proof of Chakerian's Theorem above uses yet another variation of Helly's Theorem and a simple property of sets of constant breadth, and it is not difficult. However, we include another proof using the ideas of finding billiard triangles using Lemma 8 above.

By Lemma 8 we need only prove that all billiard triangles $\triangle = \langle p_1, p_2, p_3 \rangle \subset X$ have $l(\triangle) \geqslant 1$, since we assume $b = 1/2$. Thus we suppose $\triangle$ is a billiard triangle with $l(\triangle) < 1$, and we look for a contradiction.

Let q_1, q_2, q_3 be the points on the boundary of X such that the segment $\langle p_i, q_i \rangle$ is perpendicular to the support line for X at p_i, $i = 1, 2, 3$. Note that the segments are coincident at the incenter of $\triangle$, each q_i is outside $\triangle$ since q_i is on the boundary of X, and $|p_i - q_i| = 1/2$ since X is of constant breadth $1/2$.

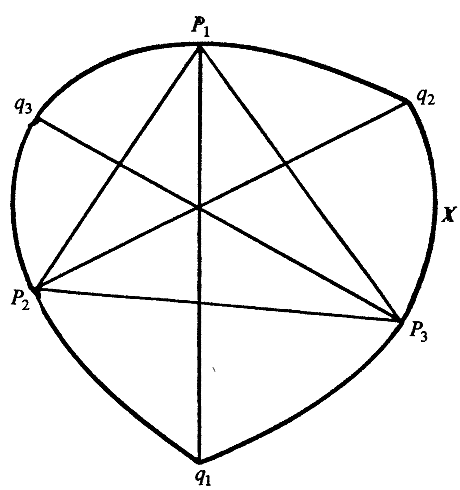


FIG. 12

Consider for $t \geqslant 1$, $r_i(t) = tq_i + (1-t)p_i$. Let

$$S_t = \{p_1, p_2, p_3, r_1(t), r_2(t), r_3(t)\},$$
$$t_0 = \sup\{t \mid \text{diameter of } S_t \leqslant t/2\}.$$

Note that $|r_i(t) - p_i| = t/2$. The supremum exists since the diameter of $S_1 = 1/2$, and for some i and j the angle between $q_i - p_i$ and $q_j - p_j$ is greater than 60° and thus

$$\lim_{t \to \infty} |r_i(t) - r_j(t)|/t > 1.$$

Let $r_i = r_i(t_0)$, $i = 1, 2, 3$. Then for some $i \neq j$ either $|p_i - r_j| = t_0/2 =$ diameter S_{t_0} or $|r_i - r_j| = t_0/2$. By renumbering, one of the following cases occurs.

Case 1. $|p_1 - r_2| = t_0/2$.
We then have the following diagram where •——• indicates that the distance is $t_0/2$.

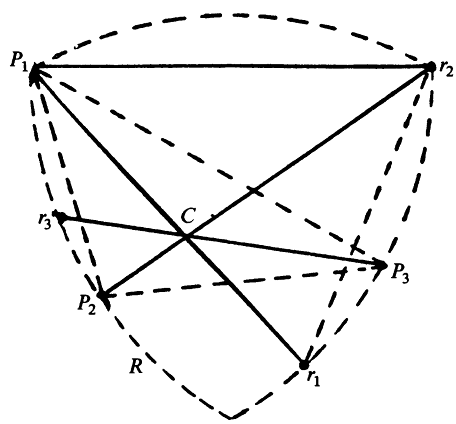


FIG. 13

Let R be the Reuleaux triangle with p_1 and r_2 as vertices with the third vertex on the same side of $[p_1, r_2]$ as p_2 and r_1. Thus the breadth of R is $t_0/2$. Then p_2, r_1 are on the boundary of R on the opposite sides of r_2, p_1 respectively, since $|p_1 - r_1| = |p_2 - r_2| = t_0/2$, $|p_1 - p_2| \leqslant t_0/2$, and $|r_1 - r_2| \leqslant t_0/2$. Thus $\langle p_2, r_2 \rangle \cap \langle p_1, r_1 \rangle = C$ is a point in the interior of R. Thus r_3 and p_3 lie in the open cones determined by Cp_2, Cp_1 and Cr_2, Cr_1 respectively. Thus $|r_3 - r_2| \leqslant t_0/2$, $|r_3 - p_1| \leqslant t_0/2$ implies r_3 is in R and $|p_3 - p_1| \leqslant t_0/2$, $|p_3 - r_2| \leqslant t_0/2$ implies p_3 is in R as well. The diameter of R is $t_0/2$. Thus r_3 and p_3 must be on the boundary with at least one of them equal to p_1 or r_2, which is impossible. Thus Case 1 cannot occur.

Case 2. $|r_1 - r_2| = t_0/2$.
We then have the following diagram, Figure 14.

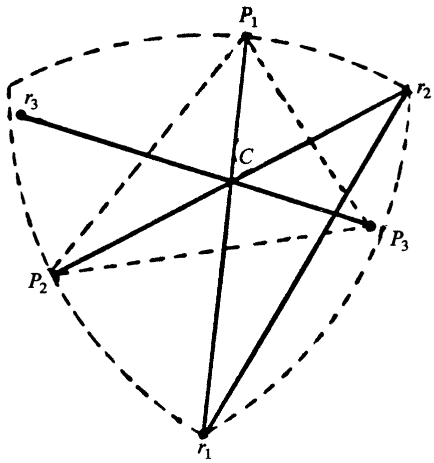


FIG. 14

Let R be the Reuleaux triangle with r_1, r_2 as vertices as before and thus with breadth $t_0/2$ again. As in Case 1 $\langle p_1, r_1\rangle \cap \langle p_2, r_2\rangle = C$, a point in the interior of R on the same side of $\langle r_1, r_2\rangle$ as p_1, p_2, r_3. Since $|r_3 - r_2| \leqslant t_0/2$ and $|r_3 - r_1| \leqslant t_0/2$, r_3 must be in R. Since $|r_3 - p_3| = t_0/2$, either p_3 is outside of R altogether, or p_3 is on the arc of R from r_1 to r_2 with r_3 as the opposite vertex of R. In either case let $\bar{p}_3$ be the point on the circular arc from r_1 to r_2 of R on $[r_3, p_3]$. This point exists since Cp_3 is in the cone determined by Cr_2 and Cr_1.

Let H_1, H_2, H_3 be the support half planes for R at $p_1, p_2, \bar{p}_3$ respectively. Clearly $H_1 \cap H_2 \cap H_3$ is bounded and thus by Lemma 6, $\triangle$ cannot be covered by int R. Thus by Lemma 9

$$1 > l(\triangle) \geqslant t_0 \geqslant 1,$$

the final contradiction.

To finish the Theorem suppose any closed curve of length one or less is covered by a translate of X. Then the minimum breadth of X is $b \geqslant 1/2$ by Lemma 8. Let $b(\theta)$ be the breadth in direction θ. Then it is a well-known formula of Cauchy, see Hurwitz [**8**] for example (or Chakerian and Klamkin [**4**] for a similar use), that the length of the perimeter of X is

$$l(X) = (1/2)\int_0^{2\pi} b(\theta)\, d\theta \geqslant (1/2)2\pi(1/2) = \pi/2,$$

with equality if and only if $b(\theta) = 1/2$, a constant. $\square$

VI. Wetzel's problem. We have found all the sets of shortest perimeter whose translates cover any closed curve of length one or less. As mentioned in the introduction, Wetzel considered the question: What is a set of smallest area whose translates cover any closed curve of length one or less? In other words, what is a translation cover of smallest area for $\mathscr{L}_1$ (the collection of closed curves of length one or less)?

Among all compact convex sets of constant breadth $1/2$, the Reuleaux triangle is the set of smallest area $(\pi - \sqrt{3})/8 \approx 0.17619$, and of course is a translation cover for $\mathscr{L}_1$. Wetzel [**21**] found a set shown in Figure 15 (with some of the billiard triangles indicated), which is also a translation cover for $\mathscr{L}_1$ and has area ≈ 0.17141 (and thus is necessarily not of constant breadth). (Wetzel claimed incorrectly that the area is ≈ 0.15900.)

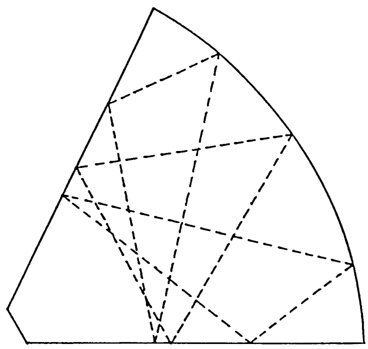


FIG. 15

Figure 16 shows a set, looking a bit like a church window, with area $1/6 \approx 0.16667$ which is also a translation cover for $\mathscr{L}_1$. The base has length $1/2$ and the height is also $1/2$. The curves are parabolas with their line of symmetry about the base. Since the common focus of the two parabolas is the midpoint of the base, it is easy to show that the indicated isosceles triangles (with horizontal base and vertex at the midpoint of the base of the church window) are indeed billiard triangles. It is a bit more difficult to show that there are no more shorter billiard triangles.

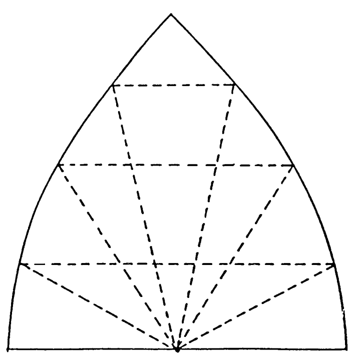


FIG. 16

We do not know what the minimum area is for translation covers for $\mathscr{L}_1$, but there are still other translation covers for $\mathscr{L}_1$ that are more complicated but have area smaller than $1/6$, the area of the church window. The best translation cover that we have found has area ≈ 0.16526. However, Wetzel [**21**], modifying an argument of Pal [**15**], derived a lower bound ≈ 0.15544 for the area of any translation cover for $\mathscr{L}_1$.

Note that the Reuleaux triangle, Figure 9, and the church window, Figure 16, are *minimal* with respect to being a translation cover for $\mathscr{L}_1$, i.e., they do not contain a proper closed convex subset that is also a translation cover for $\mathscr{L}_1$. A compact convex set X is a minimal translation cover for $\mathscr{L}_1$ if and only if it is the convex hull of all its billiard triangles of length 1, including the degenerate case of a doubly covered minimum breadth of length $1/2$. Wetzel's wedge, Figure 15, is not minimal since a small amount of the set, near the lower left-hand corner, can be shaved off, and it will still remain a translation cover for $\mathscr{L}_1$.

VII. Related problems. As mentioned in Remark 3 and Remark 4, it is possible to use a similar analysis as in section II to consider other classes of objects to be covered, as well as minimizing with respect to functions other than the length of the perimeter. We mention, without proof, some results for some of these combinations.

We consider three classes of objects, in the plane, to be covered:

1. $\mathscr{L}_1$, closed curves of length one or less.
2. Compact sets of diameter one or less.
3. Arcs of length one or less.

Then we consider translation covers for each of the classes above and minimize with respect to the following:

1. Length of the perimeter.
2. Diameter, the largest distance between pairs of points in the set.
3. Area.

The results are summarized in the following table.

	minimizing functions for the translation cover		
sets to be covered	length of perimeter	diameter	area
closed curves of length $\leqslant 1$	all sets of constant breadth $1/2$, length $= \pi/2$	all sets of constant breadth $1/2$, diameter $= 1/2$	? $.155 <$ area $< .167$
sets of diameter $\leqslant 1$	a circle of diameter $2\sqrt{3}/3$, length $= 2\pi\sqrt{3}/3$	a circle of diameter $2\sqrt{3}/3$, (Jung's Thm. [**9**])	?
arcs of length $\leqslant 1$	all sets of constant breadth 1, length $= \pi$	all sets of constant breadth 1, diameter $= 1$	the equilateral triangle of side length $2\sqrt{3}/3$ (Pal's Thm. [**15**]) area $= \sqrt{3}/3$

Each entry in the table represents the whole collection of minimizing sets, if known. A ? indicates that the minimizing object is unknown to us.

The line corresponding to closed curves of length $\leqslant 1$ has already been discussed, except for the entry under diameter. However, from our main Theorem, any set of constant breadth $1/2$ forms a translation cover for all sets of perimeter of length $\leqslant 1$. In order to cover any doubly covered line segment, any translation cover X for $\mathscr{L}_1$ must have minimum breadth $\geqslant 1/2$, and thus X must have diameter $\geqslant 1/2$. If X has diameter $= 1/2$ and minimum breadth $\geqslant 1/2$, then X must be a set of constant breadth $1/2$.

For the line corresponding to the sets of diameter $\leqslant 1$, Jung's Theorem **[9]** implies that the circle of diameter $2\sqrt{3}/3$ is a translation cover for such sets, but it does not imply, and it is nontrivial to prove, that, in the case of the perimeter and the diameter, this circle is the only such minimal convex set.

For the line corresponding to arcs of length $\leqslant 1$, it turns out that X is a translation cover for this collection if and only if X has minimum breadth 1. A theorem of Pal **[15]** states that the equilateral triangle has the least area among all sets of minimum breadth 1.

In dimensions greater than two we have very little information. For a given compact convex set X in $\mathbb{R}^d$, if one can calculate the length L of the shortest billiard path with $d+1$ or fewer bounces, X is a translation cover for curves of length L or less. For example, in *Martin Gardner's Sixth Book of Mathematical Diversions* **[7**, pp. 29–38], it is stated that the regular tetrahedron, with side length one, has three billiard paths, each of total length $4/\sqrt{10}$, with four bounces. It turns out that these billiard paths are the shortest, and thus the regular tetrahedron of side length $\sqrt{10}/4$ is a translation cover for closed curves of length one or less. Similarly the regular simplex of side length one in $\mathbb{R}^d$ has $d!/2$ billiard paths of minimal length $(d+1)\sqrt{6/d(d+1)(d+2)}$; and thus the regular simplex of side length $\sqrt{d(d+1)(d+2)/6}/(d+1)$ is a translation cover for closed curves of length one or less in $\mathbb{R}^d$.

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