ARTICLES

The Rigidity of Polyhedral Surfaces

1813 Cauchy: all convex polyhedral surfaces are rigid.
1974 Gluck: almost all triangulated surfaces are rigid.
1977 Connelly: not all triangulated surfaces are rigid.

ROBERT CONNELLY
Cornell University
Ithaca, NY 14853

Are triangulated polyhedral surfaces rigid? Euler apparently thought so for he said, "A closed spacial figure allows no changes, as long as it is not ripped apart." However, proving such "rigidity" statements is another matter. In 1813 the famous French mathematician A. L. Cauchy proved that a convex polyhedral surface is rigid if its flat polygonal faces are held rigid. In 1896 R. Bricard, a French engineer, showed that the only flexible octahedra had bad self-intersections, and so all embedded octahedra (i.e., those that can be represented in 3-dimensional space without self-intersections) were rigid. In the 1940's A. D. Alexandrov, a Russian geometer, showed that all triangulated convex polyhedral surfaces were rigid, if there were no vertices in the interior of the flat natural faces. Then in 1974 H. Gluck showed that "almost all" triangulated spherical surfaces were rigid.

Despite all this "evidence" for the rigidity conjecture, that all embedded polyhedral surfaces are rigid, in June 1977 I found a counterexample—an embedded polyhedral surface that flexes. After a brief introduction to what is meant by rigidity, we will see how to build some models and discuss why they work. We will conclude with some related results, a few conjectures, and several open questions.

Rigidity

In order to get a feeling for what is meant by rigidity, we present first a few definitions and examples in Euclidean n-space E^n. A **framework** F in E^3 or E^2 is a collection of **vertices**, points in E^3 or E^2, together with a collection of **rods** joining certain pairs of the vertices. We think of a framework as a collection of dowel rods joined with flexible rubber connectors at their endpoints. A framework F is called **rigid** if any continuous motion of the vertices that keeps the length of every rod fixed also keeps fixed the distance between every pair of vertices in the framework. For example, the frameworks of FIGURE 1 are rigid in E^3. The last of these frameworks, the convex octahedron, is one of the "surfaces" that Cauchy's theorem shows is

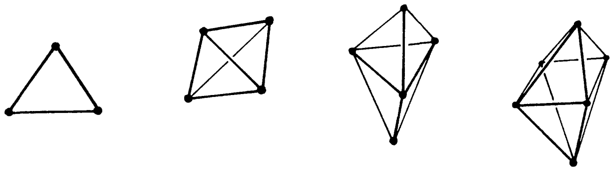


FIGURE 1

rigid. FIGURE 2 contains some examples of frameworks that are not rigid in E^3: each of these permit a continuous motion, or flex, of the framework that changes its shape.

Suppose we have a finite collection of triangles in E^3, where any two intersect in a common edge, a common vertex, or not at all, and at any vertex the triangles intersecting that vertex form a polygonal disk with the vertex in its interior. Then the union of the 2-dimensional triangles is called a **polyhedral surface**, and the collection of triangles is called a **triangulation** of that surface. For instance, the last three examples in FIGURE 1 are triangulated surfaces. The vertices and edges of these triangles form the framework associated with this triangulation of the surface. For our purposes when we talk about the rigidity or flexibility of a polyhedral surface, we will mean the rigidity or flexibility of the framework associated with some triangulation. So the rigidity conjecture says that any triangulation of a polyhedral surface in E^3 is rigid. This is false, and we can now start with a description of the counterexample.

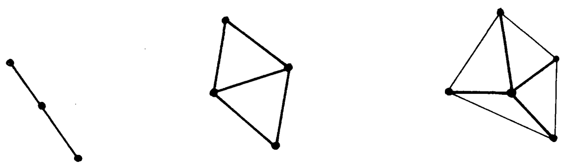


FIGURE 2

The construction

To understand why the examples flex it is helpful to describe one of the types of flexible octahedra of Bricard. The construction depends on a little lemma from Euclidean geometry, illustrated in FIGURE 3.

LEMMA. *Let $aba'b'$ be a 4-gon in E^3 with equal opposite sides $ab=a'b'$, $a'b=ab'$. (Assume a,b,a',b' are not all on one line.) Then there is a unique line L meeting the diagonals aa', bb' in their centers such that rotation by $180°$ about L leaves the 4-gon invariant by interchanging a with a' and b with b'.*

The proof of the lemma is not hard, and is the only part of the whole construction that requires any concentration. If, on the one hand, the midpoints of the diagonals aa' and bb' coincide, then $aba'b'$ is planar (and in fact is a parallelogram). Then L must be the line through

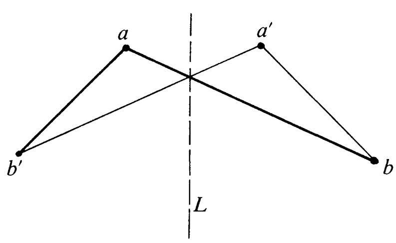


FIGURE 3

this common midpoint perpendicular to the plane of the 4-gon $aba'b'$. So suppose, on the other hand, that the midpoints x,y of aa' and bb', respectively, are distinct. Then all we have to show is that the line $L=xy$ is perpendicular to aa' and bb', for clearly a 180° rotation about such a line will interchange a with a', b with b'. But abb' is congruent to $a'b'b$ by the conditions on the lengths of the sides. So the median lengths $a'y$ and ay are equal (see FIGURE 4). So yaa' is an isosceles triangle and xy is the median to the base. Thus, $L=xy$ is perpendicular to aa'. Similarly L is perpendicular to bb'. This completes the proof.

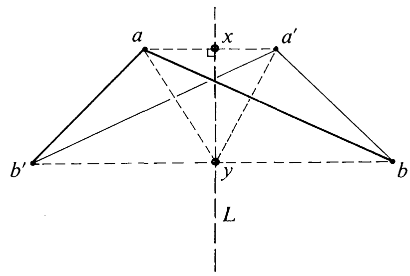


FIGURE 4

With this lemma in mind we can construct some of the octahedra discovered by Bricard and show why they are flexible. We regard an octahedron as a framework with a 4-gon $aba'b'$ and 2 other vertices c,c'. In addition to the rods on the 4-gon $aba'b'$, c and c' each have four rods connecting them to each vertex on the 4-gon.

Start with a 4-gon $aba'b'$ (as in the lemma) with opposite sides equal. Choose c anywhere not on L, its line of symmetry. With c joined to the 4-gon $aba'b'$ we get a framework something like the last framework in FIGURE 2. If $aba'b'$ is not coplanar or if $aba'b'$ is a parallelogram and c is inside the parallelogram, this framework with 5 vertices $c(aba'b')$ will flex in E^3. So flex it and join it at each instant to the congruent framework $c'(a'b'ab)$ obtained from the first by rotation by 180° about L. The union is one of the flexible octahedra of Bricard and is easy to build with straws and strings. It is illustrated in FIGURE 5, where L is vertical and the 4-gon $aba'b'$ is in the plane of the paper as a rectangle. Points c and thus c' are chosen inside the rectangle as shown. As this framework is flexed the vertices do not remain coplanar, but this is a very convenient position in which to start.

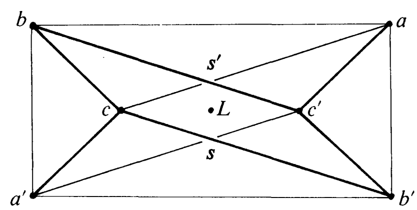


FIGURE 5

Another slight variation on this framework is to start with the points a and a' in a horizontal plane H as in FIGURE 5. Then choose points b,b' at a height $\varepsilon>0$ above H and c,c' at a height $\delta>\varepsilon$ above H so that all the points project orthogonally onto the picture of FIGURE 5. Line L is again perpendicular to H, the octahedral framework is still flexible, and the boundaries of the triangles $ab'c$ and $a'bc'$ link, i.e., they cannot be pulled apart without intersecting.

To construct the embedded flexible surface we start with the surface that is used for FIGURE 5. Instead of filling in all the triangles with flat planar pieces we change the surface somewhat, but still keep the rods of the old surface. We regard the octahedral surface as being made of two pieces, a bottom and a top. Let us say the bottom is as in FIGURE 6. We push down on each of

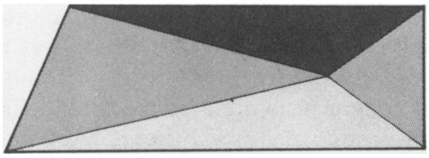


FIGURE 6

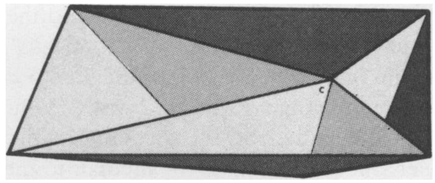


FIGURE 7

the triangular faces to get a new surface that looks like FIGURE 7, in which each triangle is replaced with an upsidedown bottomless tetrahedron, a pit.

Similarly we replace the "top" surface, FIGURE 8, with the surface in FIGURE 9, where each triangle is replaced by a bottomless pyramid, a mountain. Just as the surfaces of FIGURES 6 and 8 flex, so do the surfaces of FIGURES 7 and 9, with the extra vertices, the apexes of the pyramids, moving rigidly with respect to their bases.

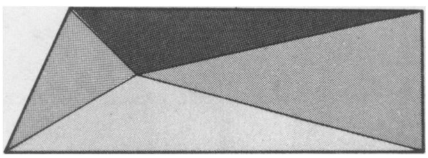


FIGURE 8

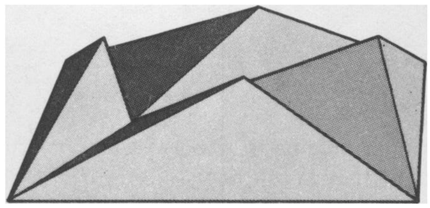


FIGURE 9

We next glue the surfaces of FIGURES 7 and 9 together along their common boundary to get the surface of FIGURE 10. This surface is flexible, just like the surface of FIGURE 5, but unfortunately it has a couple of self-interactions s and s'. FIGURE 11 shows the parts of the surface of FIGURE 10 that intersect: points s and s' correspond to the crossing points of FIGURE 5.

In order to get rid of s and s', we build what I call a crinkle, which is again based on the Bricard flexible octahedra. Choose a planar 4-gon $defg$ with opposite sides equal, $de=fg, ef=gd$ as in FIGURE 12 with the segment de intersecting fg. Choose a point h directly over the center of the circle through $defg$ and h' the same distance under the center. Thus $hd=he=hf=hg=h'd=h'e=h'f=h'g$. Then the frameworks $h(defg)$ and $h'(defg)$ flex in conjunction. (The 4-gon $defg$ actually remains coplanar.) The union of the triangular faces $hef, hfg, hgd, h'ef, h'fg, h'gd$ is the

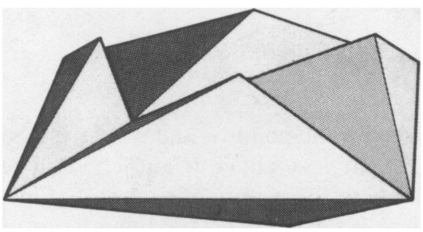


FIGURE 10

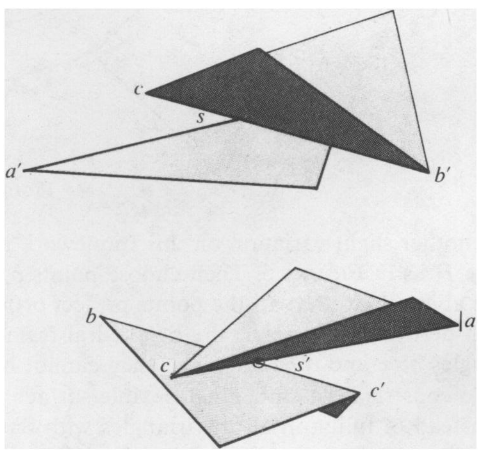


FIGURE 11

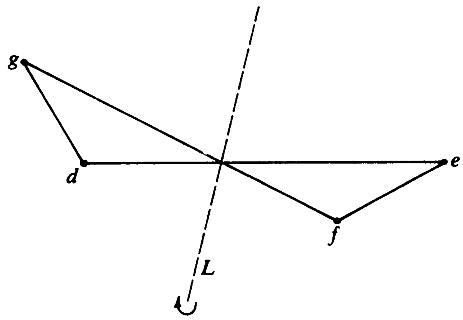


FIGURE 12

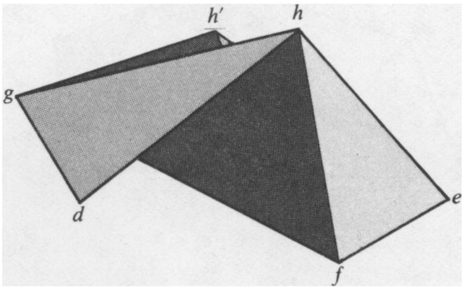


FIGURE 13

crinkle, an octahedron with two triangular faces removed (see FIGURE 13), with boundary $hdh'e$. The distance from d to e remains fixed during the flex, since it would still flex if the de rod were there.

To construct the final embedded flexible surface, take the surface of FIGURE 10 and cut out, as in FIGURE 14, one small quadrilateral hole around each of the self-intersection points. Then insert a crinkle of the appropriate size into each of the holes. If the crinkle is positioned properly, there will be no self-intersections in the resulting surface (FIGURE 15). Since de remains at a fixed distance in the crinkle, the surface with the two holes removed and the crinkle flex in conjunction. Thus the whole crinkled surface flexes. It looks something like FIGURE 16.

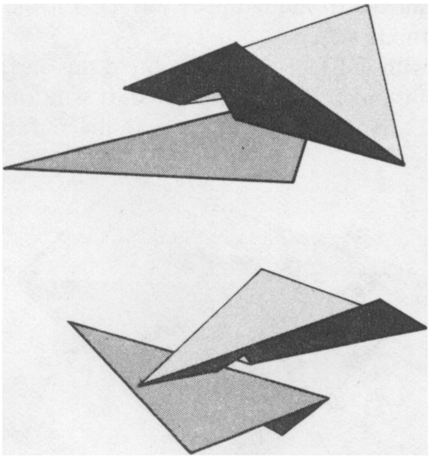


FIGURE 14

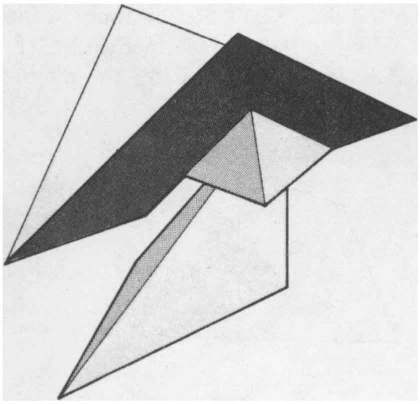


FIGURE 15

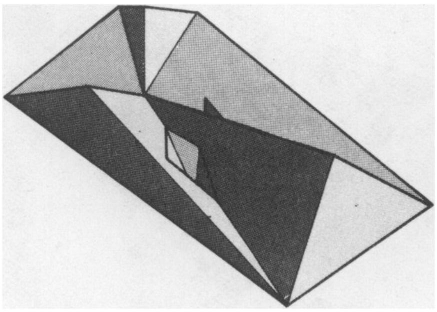


FIGURE 16

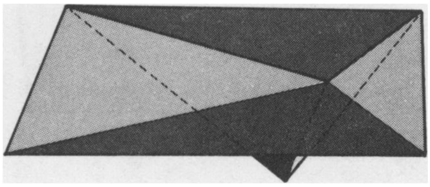


FIGURE 17

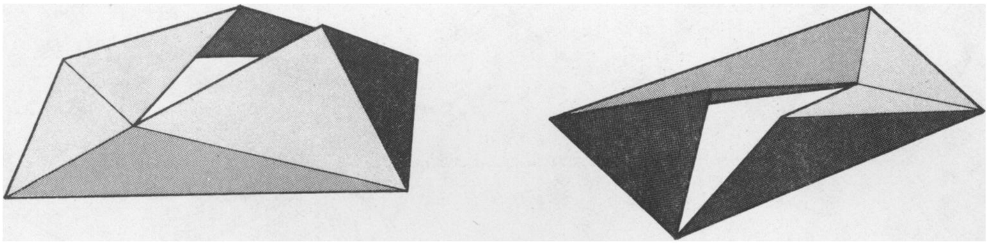


FIGURE 18

This surface was one of the first I found, and I paid no attention to the simplicity of construction or how few vertices would be needed. Subsequently N. H. Kuiper and Pierre Deligne modified my construction to get a surface with 11 vertices and 18 faces. They start with the framework described just after FIGURE 5. Instead of adding four pits to the bottom surface, they add only one as in FIGURE 17. The other three triangles are kept flat. For the upper surface they only add two mountains as in FIGURE 18 (shown with two views with two sides removed).

When these new upper and lower surfaces are glued together along their common boundary, the line segment $c'b$ intersects the sides of the two mountains above ca. Due to the slight raising of c, c', b, b' this is the only place where the surface intersects itself. They then remove ca and the inside of the two triangles with ca as an edge and place a carefully proportioned crinkle, as in FIGURE 13, in the hole that was created. Points d and e in FIGURE 13 fit into c and a respectively, and the apexes of the two mountains are h and h'. FIGURE 19 shows two views of the upper and lower surfaces glued together with ca removed from the upper surface. FIGURE 20 shows two views of the final flexible surface with the crinkle added.

To top this, Klaus Steffen found a flexible surface with only 9 vertices. To build this surface, start with the flat surface in FIGURE 21. If it is cut out and folded as indicated, it will form a closed surface something like the picture in FIGURE 22. Note that in FIGURE 21 the surface is

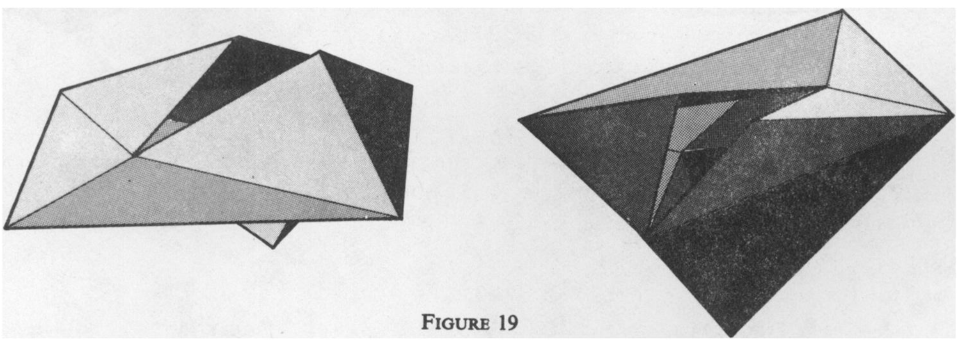


FIGURE 19

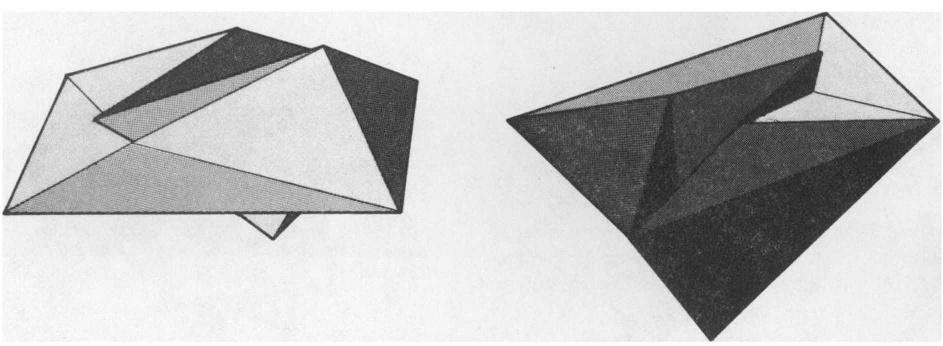


FIGURE 20

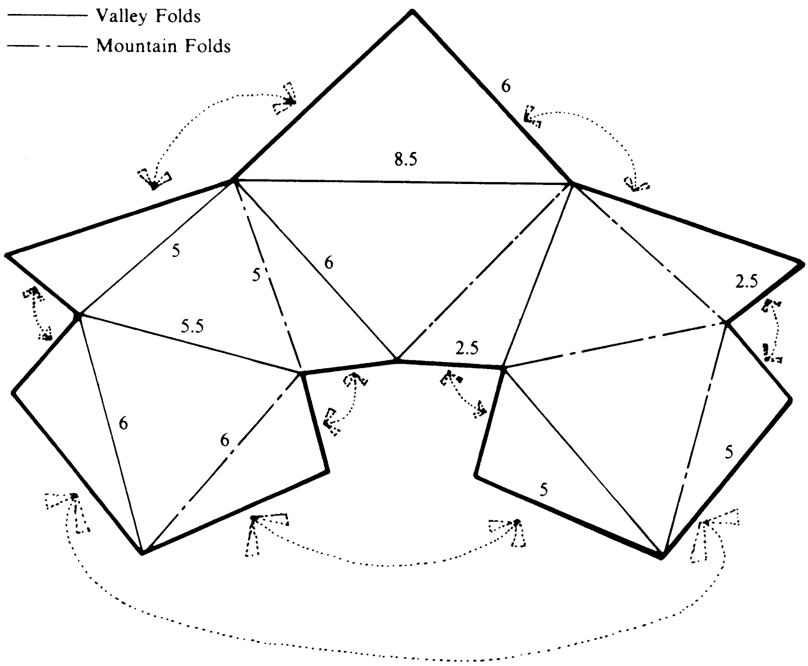


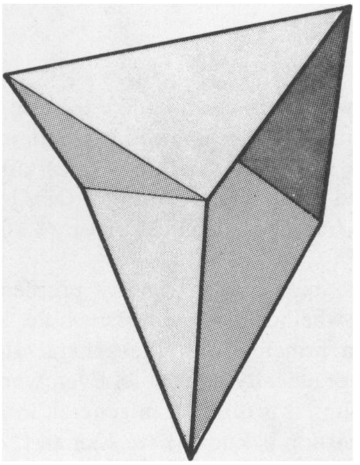
FIGURE 21

symmetric about a horizontal line and the numbers indicate appropriate lengths. If a line is not labeled, then its length can be found by looking at its symmetric length above or below, which is labeled. The two large central triangles separate the surface into two crinkles. To flex the surface hold the top two triangles in one hand and move the bottom vertex left and right.

The first model is what I describe in [**9**]. The second model is described in [**10**] and [**21**]. A nice description of the first model and related results can be found in [**18**].

Some conjectures

An interesting property of the previous examples is that as they flex, the volume enclosed by these surfaces remains constant. However, I do not know how to prove that the volume is constant for *every* possible flexible surface.



Cut and fold Figure 21 to obtain this flexible polyhedron with only nine vertices. This surface, found by Klaus Steffen, represents the simplest known flexible polyhedron in 3-space. Is it the best possible?

FIGURE 22

CONJECTURE 1: *If a triangulated polyhedral surface flexes, its volume remains constant during the flex.*

Even more startling things seem to be true. Let P and P' be two 3-dimensional polyhedra in 3-space. We say P is equivalent to P' by dissection, and write $P \sim P'$, if we can dissect P into a finite number of polyhedral pieces, $P_1, P_2, \ldots, P_k$ and then reassemble them to get P'. (So $P = P_1 \cup \cdots \cup P_k$ $P_i \cap P_j \subset$(boundary P_i)$\cup$(boundary P_j), for $i \neq j$, $P' = P'_1 \cup \cdots \cup P'_k, P'_i \cap P'_j \subset$ (boundary P'_i)$\cup$(boundary P'_j), for $i \neq j$, and P_i is congruent to P'_i for $i = 1, \ldots, k$.) It turns out that a result of M. Dehn [12] answering Hilbert's third problem [14] shows that the regular cube and tetrahedron of the same volume are *not* equivalent by dissection. However, suppose that P_t is the 3-dimensional solid enclosed by one of the above flexible surfaces at time t. A result of Sydler [27] implies that $P_0 \sim P_t$ for all t in the flexing interval. A good discussion of Hilbert's third problem and this (non-trivial) result of Sydler can be found in the book of Boltianskii [4]. Still the general question remains.

CONJECTURE 2: *If P_t is the polyhedral solid enclosed at time t by any flexing polyhedral surface, then $P_0 \sim P_t$ for all t.*

Even to see explicitly the dissections for the surfaces described above would be interesting.

Other results and open questions

The rigidity conjecture was an attempt to show that there was a more general setting for rigidity than convexity. However, since the conjecture is false, perhaps convex surfaces are natural objects to study after all. Yet despite Cauchy's result [6] and work of Alexandrov [1], it was not known until recently, Connelly [11], that an arbitrarily triangulated convex polyhedral surface (for example, the triangulated cube of FIGURE 23) was rigid.

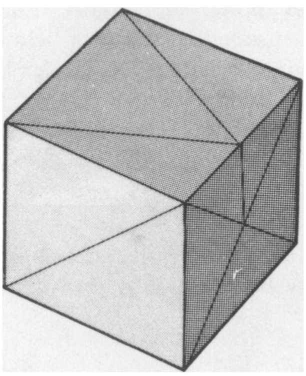


FIGURE 23

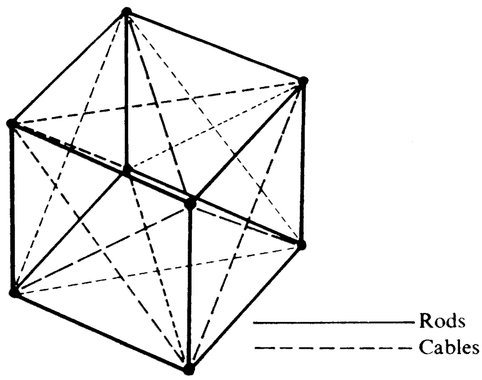


FIGURE 24

To show that frameworks like the one in FIGURE 23 are rigid, it is often helpful to show first that certain cabled frameworks are rigid (see B. Grünbaum [16]). A cabled framework is a framework with rods together with certain pairs of vertices, connected by tightened cables, that are allowed to get closer during a flex, but are not allowed to lengthen. FIGURE 24 shows one such rigid framework conjectured by Grünbaum to be rigid and proved to be rigid in [11] and [28].

Another approach to rigidity problems is to define some specific framework and try to determine whether it is rigid or flexible. Recent work of P. Kahn [17] shows that this is always possible in principle, but the general algorithm takes an enormously long number of steps, making it practically infeasible. Even when you are allowed to change the lengths of the rods a small amount, it is difficult in general, in E^3, to tell if it is possible to find a rigid framework. (In E^2 this question is known; see Laman [22] or Asimow and Roth [3].)

In the category of smooth surfaces the situation is similar but more difficult. The analogue of Cauchy's theorem, that a smooth convex surface is rigid, was first proved by Cohn-Vossen [**8**], and now there is a very short proof by Herglotz [**13**]. (See also Stoker [**26**, p. 365], Chern [**7**], or Pogorelov [**24**], for example.) The methods of my counterexample do not apply in the smooth category. However, if by smooth we mean that the map that defines the embedding of the smooth manifold is just C^1 (that is, continuously differentiable), then methods of Kuiper [**18**], [**19**] following Nash [**23**] provide counterexamples even for the standard, but strangely embedded, round 2-sphere. In the class of C^2 or C^∞ embeddings, the rigidity conjecture remains open.

This research was partially supported by a National Science Foundation grant. The art work was done by Ken Gruskin, a draftsman at Cornell.

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