group of order n, respectively. If S_4 acts as a permutation group on four letters, it is easy to see that S_3 must be the stabilizer of some letter and that C_4 is generated by a 4-cycle. Thus $S_3 \cap C_4$ is trivial. This is a contradiction, since both contain C_2.

Editorial comment. Vladimir Božin and Robin J. Chapman each noted that the answer would be *Yes* if $\mathcal{X}$ were restricted to finite abelian groups.

Solved also by D. Alvis, R. Barbara (Lebanon), V. Božin (student, Yugoslavia), R. J. Chapman (U. K.), S. M. Gagola Jr., R. Holzsager, O. P. Lossers (The Netherlands), the MMRS group of Oklahoma State University, and the proposers.

A Collinear Configuration

10308 [1993, 498]. *Proposed by Robert Connelly and John H. Hubbard, Cornell University, Ithaca, NY, and Walter Whiteley, York University, North York, Ontario, Canada.*

Suppose that $p_1, p_2, p_3, q_1, q_2, q_3$ are six points in the plane and that the distance between p_i and q_j $(i, j = 1, 2, 3)$ is $i + j$. Show that the six points are collinear.

Solution I by Ilias Kastanas, California State University, Los Angeles, CA. Let the coordinates of p_i, q_j be (x_i, y_i), (a_j, b_j) respectively. Without loss of generality, let $(x_2, y_2) = (2, 0)$ and $(a_2, b_2) = (-2, 0)$. Then we have the equations

$$(x_i - a_j)^2 + (y_i - b_j)^2 = (i + j)^2. \qquad (E_{ij})$$

By taking $E_{11} - (E_{12} + E_{21})$, $E_{33} - (E_{32} + E_{23})$, $E_{13} - (E_{12} + E_{23})$, and $E_{31} - (E_{32} + E_{21})$ we get

$$(x_1 - 2)(a_1 + 2) + y_1 b_1 + 1 = 0,$$
$$(x_3 - 2)(a_3 + 2) + y_3 b_3 + 1 = 0,$$
$$(x_1 - 2)(a_3 + 2) + y_1 b_3 - 1 = 0,$$
$$(x_3 - 2)(a_1 + 2) + y_3 b_1 - 1 = 0.$$

Therefore, $(1 + y_1 b_1)(1 + y_3 b_3) = (x_1 - 2)(a_1 + 2)(x_3 - 2)(a_3 + 2) = (1 - y_1 b_3)(1 - y_3 b_1)$, from which $(y_1 + y_3)(b_1 + b_3) = 0$. Suppose that $y_1 + y_3 = 0$ (the case $b_1 + b_3 = 0$ is similar). Then, adding the first and last of the four equation displayed above, we get $(a_1 + 2)(x_1 + x_3 - 4) = 0$. If $a_1 + 2 = 0$, E_{21} would give the contradiction $b_1^2 = -7$, so $x_1 + x_3 = 4$. By $E_{32} - E_{12}$, it follows that $8(x_3 - x_1) = 16$, so $x_1 = 1$, $x_3 = 3$, and E_{12}, E_{32} then imply $y_1 = y_3 = 0$. It follows that $a_1 = -1$, $a_3 = -3$, $b_1 = b_3 = 0$, and all six points are on the x-axis.

Solution II by MMRS, Oklahoma State University, Stillwater, OK. We shall view the points as complex numbers. Consider four points p_i, p_j, q_k, and q_l, with $i \neq j$ and $k \neq l$. Suppose the line segments $p_i q_k$ and $p_j q_l$ have a point x in common (which might be an endpoint). Then, by the triangle inequality, we have

$$|x - p_i| + |x - q_l| \geq |p_i - q_l|$$

and

$$|x - p_j| + |x - q_k| \geq |p_j - q_k|;$$

where equality holds in both inequalities if and only if p_i, p_j, q_k, q_l are all collinear and x is on the line segments $p_i q_l$ and $p_j q_k$. Adding the inequalities, we obtain

$$|x - p_i| + |x - q_l| + |x - p_j| + |x - q_k| \geq |p_i - q_l| + |p_j - q_k|,$$

and, since x is on both $p_i q_k$ and $p_j q_l$, this simplifies to

$$|p_i - q_k| + |p_j - q_l| \geq |p_i - q_l| + |p_j - q_k|.$$

However, by hypothesis both sides are equal to $i+j+k+l$. Therefore, equality does hold in all inequalities above, and so p_i, p_j, q_k, q_l are collinear. Furthermore, since x is on all four segments connecting p's to q's, these four points must be ordered so that the p's are on the opposite side of x from the q's.

Now, the line segments connecting all three p's to all three q's form a complete bipartite graph $K_{3,3}$. Since this graph is well known to be nonplanar, there must be two segments p_iq_k and p_jq_l with $i \neq j$ and $k \neq l$ that have a point x in common. By the above, the four points p_i, p_j, q_k, q_l are then collinear. We may also assume that they appear on their common line in this order. Consider the remaining two points p_m and q_n. Apply the above argument to p_i, p_m, q_l, and q_k with $x = q_k$, we see that p_m is also on the line containing p_i, p_j, q_k, and q_l. Similarly, q_n is on this line.

Editorial comment. Most solvers used coordinates of the points, sometimes encoded as vectors or complex numbers, leading to a system of equations resembling that considered in Solution I. Geometric considerations often guided the solution of these equations. Another approach was taken by Frank Schmidt. Nine of the fifteen distances between the six points are given, and he introduced six variables representing the remaining distances. Using the vanishing of the Cayley-Menger determinant (see M. Berger, *Geometry I*, Springer-Verlag, 1977, p. 239) as a criterion for four points to lie in a plane, he obtained equations relating them . These equations have a unique solution. These distances can then be used in general to locate the points in the plane; and in this case, to verify that they lie on a line.

Solved also by E. Aichinger (student, Austria), D. Alvis, J. Anglesio (France), R. Barbara (Lebanon), V. Božin (student, Yugoslavia), M. Brahm, R. J. Chapman (U. K.), H. S. Gunaratne (Brunei), J. G. Heuver (Canada), R. D. Hurwitz, O. P. Lossers (The Netherlands), A. D. Melas (Greece), I. Praton & E. P. Venugopal (student), P. Rennert, F. Schmidt, N. S. Thornber, PCC Math Problem Solvers Group, and the proposers.

Binomial Coefficient Growth

10310 [1993, 499]. *Proposed by E. Rodney Canfield, University of Georgia, Athens, GA.*

Fix an integer $r \geq 2$. Using Stirling's formula we may find constants c_1 and c_2 such that

$$\binom{rm}{m} \sim \frac{c_1(c_2)^m}{m^{1/2}}$$

as $m \to \infty$. Prove that the ratio $\binom{rm}{m} m^{1/2}/c_2^m$ is an increasing function of m for $m \geq 1$.

Solution by MMRS, Oklahoma State University, Stillwater, OK. Taking logarithms, we see that it suffices to prove that

$$f(m) = \log\Gamma\big(rm+1\big) - \log\Gamma\big(m+1\big) - \log\Gamma\big((r-1)m+1\big) + \frac{1}{2}\log m - m\log c_2$$

is increasing in m when $m \geq 1$ and $r \geq 2$. By Binet's formula for $\log\Gamma(z)$ (see E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth Edition, Cambridge, 1927, p. 249),

$$\log\Gamma(z+1) = \log z + \log\Gamma(z) = g(z) + h(z)$$

where

$$g(z) = \left(z + \frac{1}{2}\right)\log z - z + C,$$

with $C = (1/2)\log 2\pi$, and

$$h(z) = \int_0^\infty j(t)e^{-tz}\,\frac{dt}{t}$$

with