An Attack on Rigidity, I

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Abstract: The problem of the rigidity of polyhedral surfaces in three space is considered. In particular, for a suspension it is shown that, if the winding number of the equator about the axis through the north and south poles (the suspension points) is non-zero, then the suspension is rigid. This is also extended to the case of a suspension of a piecewise smooth curve. The formulae developed here are also used in Part II where, among other things, it is shown that any embedded suspension is rigid.

The method used is to derive a formula that must be satisfied if the polyhedron flexes and then extend the domain of the variables used and evaluate at a convenient point to gain the information desired.

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§1. Introduction

Occasionally in mathematics a simple, elegant problem, once stated, goes relatively unscathed for a long period of time. Such is the case for the problem of the rigidity of polyhedral surfaces in three space. After an initial success by Cauchy in 1813 there have been only limited extensions and new ideas; yet it seems to us that there is a great potential for important applications with new techniques. We believe the problem's anonimity is quite undeserved.

We propose to present some new ideas and techniques that are independent of Cauchy's and apply them to the special case of suspensions. The techniques described here are very elementary and in fact there is little that could not have been done 150 years ago. Practically none of the machinery usually associated with differential geometry is used, and possibly this might be an opportunity to develop a parallel theory or somehow extend the old (modern) theory. At this stage we believe the time is right for such developments.

Our philosophy for proving rigidity theorems differs somewhat from the usual philosophy in the smooth situation (see Stoker p. 360 [7]), where a certain "potential" function is needed to vanish. Instead we write down a formula which describes the motion of the polyhedron as it flexes, but we make no assumption about whether the polyhedron is immersed or embedded (this general situation seems inherently difficult to describe in the smooth category). It turns out that the variables used in the formula can be extended outside of their normal domains, with the formula still valid. Then by evaluating the formula at a convenient point we can gain both intrinsic and extrinsic

information about the original polyhedron. In the special cases we consider this is enough to show that the original polyhedron is not immersed. Thus, if the polyhedron is immersed, it is rigid.

Out attack is in two parts. In Part I we define the basic formulae and the variables and theory that go into them. The primary purpose is to prove (by two methods) that if a suspension flexes with the distance between the north and south poles changing, then the winding number of the equator about the line through the north and south poles is zero.

In Part II we apply the formulae and techniques of Part I. We apply the basic formula to obtain certain extrinsic (and intrinsic) information. We then use this information to calculate a certain generalized volume. It turns out that if the polyhedron flexes in a non-trivial way, then this volume is zero, showing that all embedded suspensions are rigid.

Next we go on to give a systematic classification of all flexible octahedra with explicit techniques allowing one to construct them (with a compass and straightedge say). The classification, useful for any suspension, involves the classical non-singular cubic of algebraic geometry and a flow graph constructed using the well known group operation defined on the non-singular cubic. In particular, this allows one to parametrize such flexible suspensions by means of the elliptic Wierstrass $\wp$ function. We hope such explicit construction will ultimately prove very useful.

The following is a table of contents for Part I.

§2. Basic Definitions and Previous Results. Here we state two versions of the rigidity conjecture.

§3. The Fundamental Relation - The variables and basic equations are defined.

§4. Suspensions of Polygons - This specializes to suspensions and proves the winding number theorem.

§5. Suspensions of Differentiable Curves - This is independent of all the rest and the reader may skip it if he wishes. This provides another proof (of a generalization to piecewise C^1 curves) of the winding number theorem. If one suspends C^1 curves and does not subdivide, rigidity comes very easily.

§6. Remarks and Comments - This contains a few deadends and hopeful directions for future attack.

Basic Definitions and Previous Results

Let K be a simplical complex. We shall slightly abuse some standard notation (following Gluck [5]) and regard a polyhedron in three-space as a map from K to R^3, linear on each simplex of K. If the vertices of K are $v_1,\ldots,v_V$, and if $P:K \to R^3$ is the polyhedron, then P is determined by the V points $p_1,p_2,\ldots,p_V$, where $P(v_j) = p_j$. We also regard P as the collection of V-tuples $p_1,\ldots,p_V$. Throughout, K will be fixed.

If P and Q are two polyhedra then we say P and Q are congruent iff there is a rigid motion of R^3 which takes the V-tuple corresponding to P into the V-tuple corresponding to Q. We say P and Q are isometric iff each edge of P has the same length as the corresponding edge of Q (i.e. if $\langle v_j,v_k\rangle$ is a 1-simplex of K then $|p_j-p_k| = |q_j-q_k|$, where we always use the standard euclidean norm, and $q_1,\ldots$ are the vertices of Q).

Following Gluck we have the following two equivalent definitions of a rigid polyhedron:

Definition 1: P is rigid if there is an $\epsilon > 0$ such that if another polyhedron, Q, is within ϵ of P (as maps or simply at each vertex) and isometric to P, then P and Q are congruent.

Definition 2: P is rigid if for any continuous one parameter family of polyhedra, P_t, with $P_0 = P$, $0 \leq t \leq 1$ (a homotopy) such that each P_t is isometric to P_0, then each P_t is congruent to P. Gluck shows that in fact these two definitions are equivalent. We are now in a position to state 2 forms of the rigidity conjecture. (Originally stated by Euler [4], more or less).

STRONG RIGIDITY CONJECTURE: Let the underlying space of K be homeomorphic to a closed 2-manifold. Let P, a polyhedron, be an immersion of K. Then P is rigid.

WEAK RIGIDITY CONJECTURE: With K as above, let P be an embedding. Then P is rigid.

Clearly, we have a conjecture for each homeomorphism class of 2-manifolds.

The best results that we know are the well known and beautiful theorems of Cauchy, Dehn, [1] and [3], which state that if K is homeomorphic to a sphere and the polyhedron P is (strictly) convex (no flat vertices), then P is rigid and in fact infinitesimally rigid (see Gluck [5] for definitions and a beautiful proof of this).

Also, for K a sphere, Gluck has shown that almost all polyhedra are rigid (irregardless of whether they are even immersed).

Both of these results are encouraging, but Cauchy's methods do not seem to generalize (see Stoker [8] for an exception) and Gluck's methods (which use only elementary algebraic geometry) do not seem to be able to distinguish embedded or immersed polyhedra from arbitrary maps, and there are copious and subtle examples of non-immersed polyhedra which are not rigid, even for spheres.

We have shown in an earlier paper [2] that if P is an immersed orthogonal suspension, then P must be rigid. The methods used there inspired some of the techniques in this paper, but, a priori, they are independent.

Until now, as far as we know, there were no known examples, other than trivial ones of tetrahedra stuck together, where a given complex had all its embedded polyhedra rigid. In Part II we show it for the simplest non-trivial example, the octahedron.

§3. The Fundamental Relation

THE VECTOR EQUATION

Here we derive a relation which is extremely useful throughout.

Let A_1, A_2, A_3 be three vectors in three-space, where A_1 and A_2 are not scalar multiples of $A_3 \neq 0$. Let $A_3^{\perp}$ denote the space perpendicular to A_3, and let π denote the orthogonal projection onto $A_3^{\perp}$. Let θ denote the angle from πA_1 to πA_2. We wish to compute θ in terms of the six quantities $A_j \cdot A_k$, $1 \leq j \leq k \leq 3$. In fact, it is most convenient to compute $e^{i\theta}$. If we regard θ as the dihedral angle between the plane determined by A_1 and A_3 and the plane determined by A_2 and A_3, then it is easy to see that:

$$\cos\theta = \frac{(A_1 \times A_3)\cdot(A_2 \times A_3)}{|A_1 \times A_3|\,|A_2 \times A_3|} = \frac{(A_1\cdot A_2)(A_3\cdot A_3)-(A_1\cdot A_3)(A_2\cdot A_3)}{|A_1 \times A_3|\,|A_2 \times A_3|}$$

$$(\sin\theta)\frac{A_3}{|A_3|} = \frac{(A_1 \times A_3)\times(A_2 \times A_3)}{|A_1 \times A_3|\,|A_2 \times A_3|} = \frac{((A_1 \times A_3)\cdot A_3)A_2-((A_1 \times A_3)\cdot A_2)A_3}{|A_1 \times A_3|\,|A_2 \times A_3|}$$

$$= \frac{[A_1, A_2, A_3]A_3}{|A_1 \times A_3|\,|A_2 \times A_3|},$$

where $[A_1, A_2, A_3]$ denotes the scalar triple product.

Thus,

$$(3.1) \qquad e^{i\theta} = \frac{(A_1\cdot A_2)(A_3\cdot A_3)-(A_1\cdot A_3)(A_2\cdot A_3) + |A_3|[A_1, A_2, A_3]i}{|A_1 \times A_3|\,|A_2 \times A_3|},$$

where

$$|A_j \times A_3|^2 = (A_j\cdot A_j)(A_3\cdot A_3)-(A_j\cdot A_3)^2 \qquad j = 1,2,$$

and

$$[A_1,A_2,A_3]^2 = [\det\ (A_1,A_2,A_3)]^2 = \det(A_1,A_2,A_3)\det(A_1,A_2,A_3)^t$$

$$= \det(A_j\cdot A_k). \quad (^t \text{ denotes transpose}).$$

This provides the desired explicit expression for $e^{i\theta}$ in terms of the $A_j\cdot A_k$.

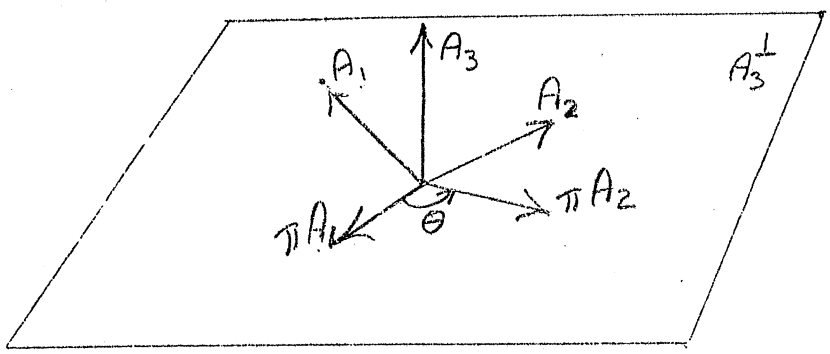


Figure 1

DEFINITION OF A CHAIN OF EDGES

With this diversion behind us, let us now derive some formulae that must be satisfied if a polyhedron, P, is to flex (not be rigid).

Let $e_{\pm 1}, e_{\pm 2}, \ldots, e_{\pm E}$ denote the edges of P, i.e. $e_j = p_k - p_\ell$ where $\langle v_k, v_\ell \rangle$ is a 1-simplex of K. Note that edges are directed so that $-e_j = e_k$ for some k. (We assume here $j = -k$ for simplicity.) We define e_j and e_k to be <u>adjacent</u> iff the vertices of K corresponding to e_j and e_k form a 2-simplex of K. So in particular e_j and e_k have one common vertex and $e_j \pm e_k = e_\ell$ for some ℓ.

Let $\mathcal{C} = \{(j,k)\}$ be a finite collection of ordered pairs of non-zero integers $0 < |j|, |k| \leq E$. We say $\mathcal{C}$ is a <u>chain</u> of <u>edges</u> iff

(i) $\forall j$, j appears in the first position of the ordered pair the same number of times that it appears in the second position in $\mathcal{C}$.

(ii) $\forall (j,k) \in \mathcal{C}$, e_j, e_k are adjacent.

EXAMPLE

The most important example of a chain of edges is the following: Let the underlying space of K be homeomorphic to a two manifold, and let v be a vertex of K. Then each edge with v as a vertex is adjacent to two others with v' as a vertex (neglect the sign here). Orient the neighborhood of v. Then define the chain of edges $\mathcal{C}_v$ to be all the pairs (j,k) where e_k is obtained from e_j be proceeding in a clockwise fashion around v, where both e_j and e_k have v as a vertex.

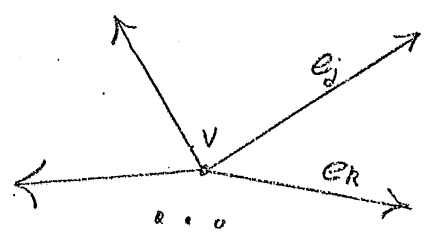


Figure 2

THE VARIABLES AND THE FUNDAMENTAL EQUATION

To set up the basic equation corresponding to a polyhedron we first choose an arbitrary vector $R \neq 0$. Later we define R in terms of the polyhedron itself, but for the purposes of setting up the equations it is just a reference vector.

Variables are defined as follows:

$$x \equiv R \cdot R$$

$$(3.2) \qquad z_j \equiv R \cdot e_j$$

$$y_{jk} \equiv |R|\,[e_j, e_k, R]\,i \qquad (e_j, e_k \text{ usually adjacent here}).$$

In view of (3.1)

$$(3.3) \qquad y_{jk}^2 = -x \det \begin{pmatrix} e_j \cdot e_j & e_j \cdot e_k & z_j \\ e_k \cdot e_j & e_k \cdot e_k & z_k \\ z_j & z_k & x \end{pmatrix}.$$

Suppose $\mathcal{C}$ is a chain of edges. Let π be the orthogonal projection onto the plane perpendicular to R. Then, if $\pi e_j \neq 0$ for all j, define θ_{jk} to be the angle between πe_j and πe_k. So $e^{i\theta_{jk}} = \frac{\pi e_j}{|\pi e_j|} \cdot \frac{|\pi e_k|}{\pi e_k}$, where the plane is regarded as the complex numbers. So, by (3.1)

$$(3.4) \qquad 1 = \prod_{(j,k)\in\mathcal{C}} e^{i\theta_{jk}} = \prod_{(j,k)\in\mathcal{C}} \frac{((R\cdot R)e_j\cdot e_k - (R\cdot e_j)(R\cdot e_k) + |R|\,[e_j, e_k, R]i}{|R\times e_j|\;|R\times e_k|}$$

and using (3.2),

$$(3.5) \qquad \prod_{(j,k)\in\mathcal{C}} (x(e_j\cdot e_k) - z_j z_k + y_{jk}) = \prod_{(j,k)\in\mathcal{C}} (x(e_j\cdot e_j) - z_j^2).$$

Notice that there are also linear relations among the z_j's generated by the linear relations among the e_j's (eg. every triangle, $e_j + e_k = e_\ell$, generates a relation $z_j + z_k = z_\ell$). Thus we can regard isometric polyhedra as generating points on the variety defined by (3.3), (3.5) and the linear relations. This is in contrast to the variety defined by the quadratic equations arising from the distances from adjacent vertices (being constant).

(3.5) is the crutical equation vital to the analysis to follow. It seems to capture the local information much more succinctly than the quadratic equations.

MORE VARIABLES

For future reference let us rewrite the factors in (3.4) as

$$(3.6)\quad e^{i\theta_{jk}} = F_{jk} = \frac{(R\cdot R)(e_j\cdot e_k)-(R\cdot e_j)(R\cdot e_k) + |R|\,[e_j,e_k,R]i}{|R\times e_j|\;|R\times e_k|}$$

$$= \frac{x(e_j\cdot e_k)-z_jz_k+y_{jk}}{\sqrt{xe_j\cdot e_j-z_j^2}\;\sqrt{xe_k\cdot e_k-z_k^2}}\,,$$

where (j,k) represent adjacent edges.

So (3.4) becomes:

$$(3.7)\qquad \prod_{(j,k)\in\mathcal{C}} F_{jk} = 1.$$

Also we define

$$G_{jk} = (R\cdot R)(e_j\cdot e_k)(R\cdot e_j)(R\cdot e_k) + |R|\,[e_j,e_k,R]i$$

$$= x(e_j\cdot e_k) - z_jz_k + y_{jk}.$$

$$(3.8)\qquad H_j = |R\times e_j| = \sqrt{x(e_j\cdot e_j)-z_j^2}\,.$$

So (3.5) becomes

$$\prod_{(j,k)\in\mathcal{C}} G_{jk} = \prod_{(j,k)\in\mathcal{C}} H_j^2 \quad \text{since} \quad F_{jk} = \frac{G_{jk}}{H_jH_k}\,.$$

§4. Suspensions of Polygons

DEFINITION OF SUSPENSIONS

Here we take up the case of the simplest nontrivial examples of polyhedra which may flex, suspensions. Let K be defined as follows: K has vertices $v_0, v_1, \ldots v_n, v_{n+1}$, where $v_1, v_2, \ldots, v_n$ form a circle (v_j adjacent to v_{j+1} (mod n), $j = 1, \ldots, n$) and v_0 and v_{n+1} are each adjacent to all of $v_1, \ldots, v_n$. Call $P(v_0) = N$ the north pole, and $P(v_{n+1}) = S$ the south pole, and $P(v_j) = p_j$, $j = 1, \ldots, n$, vertices of the equator. Such a K is called a suspension.

We assume P, a given suspension, flexes. If $N-S$ is constant during the flex, we may regard $N-S$ as an extra edge and P becomes n tetrahedra stuck together cyclicly around the edge $N-S$. It is easy to see that then either P is not an immersion of a 2-sphere, or P must have been rigid in the first place. Thus we assume $N-S$ is variable during a flex.

DEFINITION OF R AND CALCULATION OF z_j.

We define $R = N-S$ (which may be assumed to be nonzero). Let $\mathcal{C} = \mathcal{C}_N$ be the chain of adjacent edges $\mathcal{C}_N = \{(j, j+1)\}$, where $e_j = N-p_j$, $j, j+1 = 1, 2, \ldots, n$ (mod n). Thus (3.4), (3.5) and (3.6) apply. However, there are some special relations which we must calculate. Define $e_j' = p_j - S$. So

$$R = e_j + e_j' \qquad j = 1, 2, \ldots, n,$$

$$z_j = R \cdot e_j = e_j \cdot e_j + e_j \cdot e_j', \quad R \cdot R = e_j \cdot e_j + 2e_j \cdot e_j' + e_j' \cdot e_j' = x.$$

Eliminating the variable $e_j e_j'$,

(4.1) $$z_j = \frac{1}{2}(x + e_j \cdot e_j - e_j' \cdot e_j').$$

So each z_j is a linear function of x. Thus y_{jk}, F_{jk}, G_{jk}, H_j can be regarded as functions of x and since x varies as P flexes, (3.4), (3.5), (3.6) can be regarded as identities in x, and can be defined over an appropriate Riemann surface for complex x. To gain understanding of these formulae and for future reference we make some observations about the nature of the F_{jk}'s.

THE ROOTS

First we characterize geometrically the roots of F_{jk}, G_{jk}, and H_j. Note that F_{kj} (and G_{jk}) is the same as F_{jk} (G_{jk}) except that

$$y_{jk} = -y_{kj}.$$

$\{(j,k),(k,j)\}$ is a chain so,

$$G_{jk}G_{kj} = H_j^2 H_k^2$$

so the roots of G_{jk} are included in the roots of H_j and H_k. H_j^2 is a quadratic polynomial in x and thus has at most two roots. Let $r_j = (|e_j| + |e_j'|)^2$, $r_j' = (|e_j| - |e_j'|)^2$. When $|R| = \sqrt{r_j}$ or $|R| = \sqrt{r_j'}$ it is easy to see that $R \times e_j = 0$ as well as being defined. Thus $r_j \neq r_j'$ are the two roots of H_j^2. (In particular they are real.) This can be verified also by direct substitution into (3.7). Thus,

(4.2) $$H_j^2 = -\frac{1}{4}(x - r_j')(x - r_j).$$

THE BRANCH POINTS

Next we find the branch points of y_{jk} and G_{jk}. By (3.3) y_{jk}^2 is a cubic function of x (recall z_j, z_k are linear in x) with $x = 0$ as a root. Consider the two triangles determined by e_j, e_k

and e'_j, e'_k. Note that the two triangles share a common edge, say $e_{jk} = e_j - e_k$. As above, when discussing the roots, we can flex just this part of the polyhedron to gain information about G_{jk} and y_{jk}. In particular let b_{jk}, b'_{jk} be the maximum and minimum value of $x = |R|^2$ for just these two triangles. Namely when $x = b_{jk}$ the two triangles are planar with N,S in the plane on opposite sides of the line determined by e_{jk}. Similarly, when $x = b'_{jk}$, N,S are on the same side of the line.

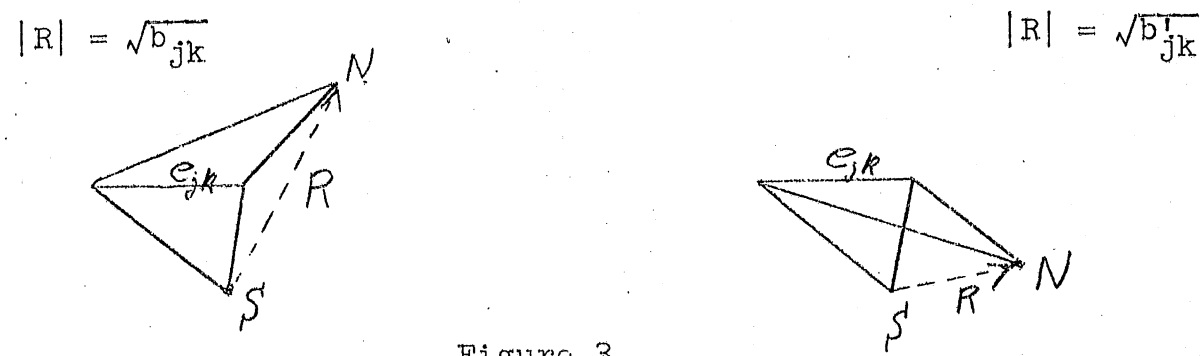


Figure 3.

Thus, when $x = b_{jk}$ or b'_{jk}, $y_{jk} = |R|[e_j, e_k, R]i = 0$, and 0, b'_{jk}, b_{jk} are the roots of y^2_{jk}. (It is easy to see that 0 is a double root if $b'_{jk} = 0$). Hence

$$(4.3) \qquad y^2_{jk} = \frac{1}{4}\, e_{jk} \cdot e_{jk}\; x(x-b'_{jk})(x-b_{jk})$$

by (3.3), (4.1) and the definition of $e_{jk} = e_j - e_k$. In particular b'_{jk}, b_{jk} are nonnegative, real and

$$0 \leq r'_j \leq b'_{jk} < b_{jk} \leq r_j,$$

since the r'_j, r_j are obtained as the value of $|R|^2$ when we restrict to a smaller portion of P. We see now that G_{jk} and G_{kj}

have the branch points, $0, b_{jk}, b'_{jk}$ and each has four roots (perhaps multiple roots), r'_j, r_j, r'_k, r_k, where these points are regarded as being on the appropriate 2 sheeted Riemann surface of G_{jk} or G_{kj}. If a point is a root of G_{jk}, the corresponding point on the "other" sheet is a root of G_{kj} and not for G_{jk} (unless the root is a branch point of course).

THE WINDING NUMBER

Let E be the equator of P, and let πE be the projection into $R^{\perp}$ (regard this as a map of the circle into $R^{\perp}$). We still assume (which is justified if P is flexible) that $\pi(E) \cap 0 = \emptyset$. We define ω to be the winding number of this map. Then with $-\pi < \theta_{jk} < \pi$ defined as in §3, we see

$$2\pi\omega = \sum_{(j,k)\in\mathcal{C}_N} \theta_{jk}.$$

But by (3.6)

$$\theta_{jk} = \frac{1}{i} \log F_{jk}, \tag{4.4}$$

where that branch of log is taken so that $-\pi < \theta_{jk} < \pi$.

THE WINDING NUMBER THEOREM

Recall that F_{jk} is regarded as a function of x, and so $\theta_{jk} = \theta_{jk}(x)$ is an analytic function of x with branch points or singularities at $r_j, r'_j, r_k, r'_k, b_{jk}, b'_{jk}$, all real. If P is flexible then

$$\sum_{(j,k)\in\mathcal{C}_N} \theta_{jk} = \frac{1}{i} \sum_{(j,k)\in\mathcal{C}_N} \log F_{jk} = \frac{1}{i} \log \prod_{(j,k)\in\mathcal{C}_N} F_{jk}$$

is constant and equal to $2\pi\omega$ for all complex x, except at the singularities (which are finite in number). Now it is easy to show the following remarkable fact:

<u>Theorem 1</u>: If P flexes (with a non-constant x) then $\omega = 0$.

<u>Proof</u>: Let x_0 be some fixed value of x in the interior of some real interval, where $|R|^2 = x$. Let A be a path starting at x_0 and going to infinity in the upper half plane with x_0 as its only real point. We show that

$$\lim_{x\to\infty} F_{jk} = 1,$$

where the limit is taken along A. Clearly

$$G_{jk}/x^2 = \frac{e_j \cdot e_k}{x} - \frac{z_j}{x} \cdot \frac{z_k}{x} + \frac{y_{jk}}{x^2} \to -\frac{1}{4}$$

regardless of the path.

$$H_j^2/x^2 = \frac{e_j \cdot e_j}{x} - \left(\frac{z_j}{x}\right)^2 \to -\frac{1}{4}.$$

So $H_j/x \to i\frac{1}{2}$ and $H_k/x \to i\frac{1}{2}$, since the same path is used both cases. Putting this together we obtain $F_{jk} \to 1$. Thus

$$\lim_{x\to\infty} \log F_{jk} = 2\pi i\, \omega_{jk},$$

where ω_{jk} is some integer. (A kind of local winding number.) Clearly $\sum_{(j,k)\in\mathcal{C}_N} \omega_{jk} = \omega$. Thus we are done if we can show each $\omega_{jk} = 0$. Notice that the definition of ω_{jk} depends only on P_{jk}, that part of P which is the suspension of e_{jk} (namely, the lengths of $e_j, e_j', e_k, e_k', e_{jk}$), and certainly ω_{jk} does not depend on whether P flexes or not. (Flexing $\Rightarrow \sum_{(j,k)\in\mathcal{C}_N} \omega_{jk} = \omega$.)

This is the key observation. At this stage one could choose a convenient path, A, and carefully compute that $\omega_{jk} = 0$. However, instead we observe that ω_{jk} itself is a continuous function of the lengths of $e_j, e_k, e'_j, e'_k, e_{jk}$ $(e_j \cdot e_k = \frac{1}{2}(e_{jk} \cdot e_{jk} - e_j \cdot e_j - e_k \cdot e_k))$, assuming a continuous choice of the sign of y_{jk}. Thus, if we deform the suspensions of e_{jk} continuously through polyhedra, ω_{jk} remains constant. Here, let us assume that x_0 remains fixed through the deformation so that the limit along A is defined. Let us subdivide P_{jk} by choosing a point in the interior of the arc corresponding to e_{jk}. Say the new edge from the north pole is e_ℓ (So $P_{jk} = P_{j\ell} \cup P_{\ell k}$).

From the remarks above, since $P_{j\ell}$ and $P_{\ell k}$ deform to P_{jk}, and $\omega_{j\ell} + \omega_{\ell k} = \omega_{jk} = \omega_{j\ell} = \omega_{\ell k}$ we see $\omega_{jk} = 0$. (Alternately we could deform e_j to e_k and since $\theta_{kk} \equiv 0$, we obtain $\omega_{jk} = \omega_{kk} = 0$.)

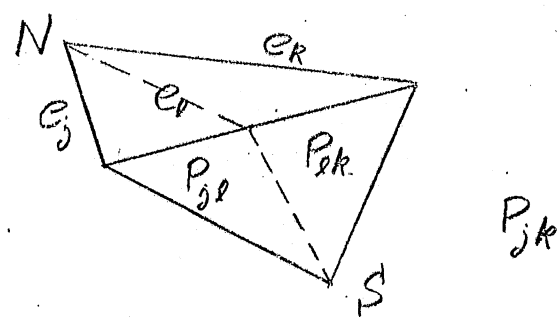


Figure 4

<u>Corollary 1</u>: If P is embedded with the line segment from N to S inside of P, then P is rigid.

THE OCTAHEDRON

Observe that when $n = 4$, we may regard P, the octahedron, as a suspension from any one of the three pairs of non-adjacent vertices

So, it is natural to hope that if the octahedron is embedded, then at least one of the diagonals is on the inside. (This is the same as saying that triangulation of the surface can be extended to the inside without adding any new vertices.) Unfortunately this hope is false as one can see from the following figure.

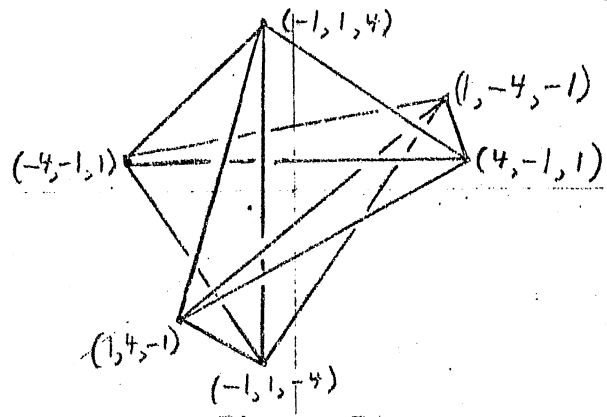


Figure 5

Later in Part II we shall make a more detailed study of the octahedron, and then we shall be able to show that all the embedded ones are rigid.

§5. Suspensions of Differentiable Curves.

Here we investigate a situation which is, in a sense, a generalization of the previous results, but more basically represents a slightly different (more classical) point of view. However, we still adhere to the philosophy that the way to prove rigidity is write down a formula that must hold if the surface flexes, and then extend the definition of the variables involved to gain specific (geometric) information about the parameters used to define that formula. In this case the formula differs (by a log) from the previous product formula, and the way of describing the intrinsic parameters is different.

We also run into a curious anomaly that forces us to be careful with our definition of rigidity. Namely, if we define a C^1 suspension in the natural way, it becomes much too easy for a suspension to be rigid. Since the surface is only "piecewise smooth" to start with, it seems natural to allow "subdivision" in the definition of rigidity in our category, and this in fact makes more sense (although some surfaces may still be rigid for C^1 reasons rather than the more appropriate "geometric" reasons). With this more appropriate definition we obtain a result generalizing Theorem 1; namely, the equator has winding number zero about the north-south axis, and in fact this supplies "another" proof of Theorem 1.

DEFINITION OF A C^1 SUSPENSION

Let $Y':S^1 \to \mathbb{R}^3$ be a C^1 map, and $N,S \in \mathbb{R}^3$ two points, the north and south poles. Let $Y:R^1 \to \mathbb{R}^3$ be a periodic map defining Y' parametrized by arc length. The (image of a) surface obtained by joining all the lines from N to $Y(s)$, and S to $Y(s)$ is

what we shall call a C^1 suspension, Σ, associated to Y, N,S. A flex of Σ (on more accurately of Y,N,S) will be the continuous homotopies Y_t, $N(t)$, $S(t)$, such that $Y_0 = Y$, $N(0) = N$, $S(0) = S$, each Y_t is C^1 for a fixed t $(0 \leq t < \epsilon)$, and $Y_t(s)-N(t)$ and $Y_t(s)-S(t)$ are constant for each fixed s. Also we require that the length of the arc from $Y_t(s_0)$ to $Y_t(s_1)$, along the curve defined by Y, be constant for all fixed s_0, s_1. (We regard Y as parameterized by arc length so $\dot{Y}\cdot\dot{Y} = \frac{dY}{ds}\cdot\frac{dY}{ds} = 1$, and s is the parameter refered to above for $Y_t(s) - S(t)$, etc.)

We also remark that this definition does not take into account a certain kind of flexing that one might want to consider. Namely if Y is a circle lying in a plane through N,S, then we can bend Σ along generators parallel to the line through N,S. However, this flex does not preserve the suspension property of Σ, and presumably, if Y does not lie in such a plane, (which must happen if Σ is immersed) any "reasonable" definition of a flex will preserve the suspension nature of Σ and fit into our definition, if Σ is immersed.

DEFINITION OF A PIECEWISE C^1 SUSPENSION AND RIGIDITY

As mentioned before, the requirement that $Y(s)$ remain C^1 everywhere turns out to be too restrictive, so with the same apparatus as discussed above we define a suspension, Σ, (associated with Y,N,S) to be piecewise C^1 if for a finite number of points $s_0 < s_1 < \ldots < s_n = s_0 + L$, $Y|[s_j, s_{j+1}]$ is C^1. (The derivative from the left at s_j may not be the derivative from the right at s_j)(L = length of $Y'(S^1)$.)

As before we define a <u>flex</u> of Σ as a homotopy of Y_t, $N(t)$, $S(t)$ with the generators constant, but only require the Y_t to be C^1 on $[s_j, s_{j+1}]$. Note that this takes into account the polyhedral suspension as a special case, and in fact it will turn out to be an equivalent notion of flexing in that case.

We can regard Y, N, S as defining the image of a surface homeomorphic to the two-sphere, S^2, in $\mathbb{R}^3$. We are interested in the question of when Y is rigid. I.e. when is every flex of Y obtained by restricting a rigid motion of $\mathbb{R}^3$ (a congruence). Oddly enough, in the C^1 rigid sense practically every map is rigid. Yet there are C^1 maps, Y, such that by subdividing and considering them as piecewise C^1 maps, they flex.

THE VARIABLES

We shall treat both cases simultaneously. As before define $R = N-S$. Let $X(s)$ (or $X_t(s)$) be the orthogonal projection of $Y(s)$ (or $Y_t(s)$) into the plane orthogonal to R. Let $x = |R|^2 = R\cdot R$, $\ell_N(s) = |N-Y(s)|^2$, $\ell_S(s) = |S-Y(s)|^2$, which are constant during a flex. To make the notation easier let us assume (without loss of generality) that $S = 0$ (for all t).

THE WINDING ANGLE AND WINDING NUMBER

Let s', s'' be two points in the domain of Y. Let $\theta_{s',s''}$ denote the angle (if it makes sense) from $X(s')$ to $X(s'')$ in $R^{\perp}$. Then it is easy to see that, if $X \neq 0$ for $s \in [s',s'']$, then (cf Stoker [7]).

$$\theta_{s',s''} = \int_{s'}^{s''} \epsilon(s) \left|\frac{d}{ds}\left(\frac{X}{|X|}\right)\right| ds \tag{5.1}$$

where

$$\epsilon(s) = \begin{cases} +1 & \text{if } \det\begin{pmatrix} X \\ \frac{d}{ds}\left(\frac{X}{|X|}\right) \end{pmatrix} > 0 \\ -1 & \text{if } \det\begin{pmatrix} X \\ \frac{d}{ds}\left(\frac{X}{|X|}\right) \end{pmatrix} < 0 \\ 0 & \text{if } \det(\text{"}) = 0. \end{cases}$$

Note that $\theta_{s',s''}$ is a continuous function of s',s'' (even in the piecewise C^1 case) even though the integrand is only piecewise continuous.

A simple calculation shows:

$$\left|\frac{d}{ds}\left(\frac{X}{|X|}\right)\right|^2 = \frac{(X\cdot X)(\dot{X}\cdot\dot{X})-(X\cdot\dot{X})^2}{(X\cdot X)^2},$$

so

$$\theta_{s',s''} = \int_{s'}^{s''} \frac{\sqrt{(X\ X)(\dot{X}\ \dot{X}) - (X\ \dot{X})^2}}{X\cdot X}\, ds \tag{5.2}$$

where the $\sqrt{\ }$ takes the appropriate sign, and $(\dot{\ })$ indicates differentiation with respect to s.

Note that, if Σ is a polygonal suspension of the previous section, where the s_j's correspond to the vertices of the equator, then by labeling such that $k = j+1$, by (4.4)

$$\frac{1}{i}\log F_{jj+1} = \theta_{jj+1} = \theta_{s_j s_{j+1}} = \int_{s_j}^{s_{j+1}} \frac{\sqrt{(X\cdot X)(X\cdot X) - (X\cdot X)^2}}{X\cdot X}\, ds.$$

Thus, we have come upon the same quantity as before, but from a somewhat different direction.

We observe that $\theta_{s',s''}$ is simply the lift of the map $\frac{X}{|X|}$, so the winding number, ω, of X is (if $X \neq 0$)

$$(5.3) \qquad \omega = \frac{1}{2\pi}\int_0^L \frac{\sqrt{(X\cdot X)(\dot{X}\cdot\dot{X}) - (X\cdot\dot{X})^2}}{X\cdot X}\, ds.$$

Thus our task is set for us. We must compute $X\cdot X$, $X\cdot\dot{X}$, $\dot{X}\cdot\dot{X}$ in terms of $x = |R|^2$, ℓ_N, ℓ_S. When this is done we shall be able to show that $\omega = 0$, if Σ flexes in a non-trivial way.

THE COMPUTATION OF $X\cdot X$, $X\cdot\dot{X}$, $\dot{X}\cdot\dot{X}$.

Observe $X = Y - \frac{R\cdot Y}{x}R$, (recall $S = 0$), and $R = Y + Z$, $Z = N-Y$. So $Y\cdot Y = \ell_S$, $Z\cdot Z = \ell_N$, the squares of the lengths of the generators. By the same computation as in 4.1("polarization" identity) we see

$$R\cdot Y = \frac{1}{2}(x + \ell_S - \ell_N)$$

$$X = Y - \frac{1}{2x}(x + \ell_S - \ell_N)R$$

$$\dot{X} = \dot{Y} - \frac{1}{2x}(\dot{\ell}_S - \dot{\ell}_N)R$$

$$R\cdot\dot{Y} = \frac{1}{2}(\dot{\ell}_S - \dot{\ell}_N).$$

$$(5.4) \qquad X\cdot X = \ell_S - \frac{1}{2x}(x + \ell_S - \ell_N)^2 + \frac{1}{4x}(x + \ell_S - \ell_N)^2$$

$$= -\frac{1}{4}x + \frac{1}{2}(\ell_S + \ell_N) - \frac{1}{4x}(\ell_S - \ell_N)^2.$$

Differentiating,

$$(5.5) \qquad X\cdot\dot{X} = \frac{1}{4}(\dot{\ell}_S + \dot{\ell}_N) - \frac{1}{4x}(\ell_S - \ell_N)(\dot{\ell}_S - \dot{\ell}_N).$$

Similar to the above

$$(5.6) \qquad \dot{X}\cdot\dot{X} = 1 - \frac{1}{2x}(\dot{\ell}_S - \dot{\ell}_N)^2 + \frac{1}{4x}(\dot{\ell}_S - \dot{\ell}_N)^2$$

$$= 1 - \frac{1}{4x}(\dot{\ell}_S - \dot{\ell}_N)^2 \qquad \text{(recall } \dot{Y}\cdot\dot{Y} = 1\text{).}$$

Note if we regard the ℓ's as constants, all the scalar quantities above are rational functions of x.

RIGIDITY WHEN x IS CONSTANT

Suppose that Σ flexes (in the piecewise C^1 sense), where $|N-S|$ is constant. Let $[s',s'']$ be an interval where $X \neq 0$. Then, if we fix s', say, and let s'' vary, $\theta_{s's''}$ is determined by (5.2) and thus must stay constant during the flex (since $\epsilon(s)$ is constant). Thus the suspension of the whole interval $[s,s'']$ must remain rigid. If there is at most one point on the equator for which $X = 0$, then the whole suspension must be rigid; i.e. all the flexed suspensions are congruent. If more than two $X(s)$'s are 0, then it is clear that Σ is not an immersion at either N or S. Thus we have:

<u>Proposition 1</u>: If a piecewise C^1 suspension, Σ, is immersed and $|N-S|$ is held fixed, then Σ is rigid.

In light of this we shall assume that x is not constant during a flex (and in fact x parametrizes the flex).

THE ROOTS OF $X\cdot X$

As with §4 we can flex part of Σ to gain information about the parameters. In this case we see that $X = 0$ only when Y is parallel to R and this happens only when

$$x = (\sqrt{\ell_S} - \sqrt{\ell_N})^2 = r_1 \quad \text{or} \quad (\sqrt{\ell_S} + \sqrt{\ell_N})^2 = r_2.$$

Thus from (5.4)

$$X \cdot X = \frac{1}{4x}(x - r_1)(x - r_2), \tag{5.7}$$

which can be verified directly.

Thus if Σ flexes, then x must at least be between $\max_s r_1(s)$ and $\min_s r_2(s)$. So we may assume $X \neq 0$ during the flex, if x varies at all.

Note also that (5.7) defines a function of x for all complex x except r_1, r_2. We shall next apply a similar procedure to $(X \cdot X)(\dot{X} \cdot \dot{X}) - (X \cdot \dot{X})^2$ and then define $\omega(x)$ for almost all complex x and then apply a technique similar to §4 to compute $\omega(x)$, if Σ flexes with a variable x.

THE ROOTS OF $(X \cdot X)(\dot{X} \cdot \dot{X}) - (X \cdot \dot{X})^2$

These are analogous to the branch points of §4. However, if Y is not piecewise linear, we do not have the suspension of a line segment to flex, as in §4, in order to find the zeros. Instead, we extend the tangent to the curve and then flex the suspension of the tangent line. From this we observe that there are always two roots to the above suspension lying between r_1 and r_2.

Define $E(x) = (X \cdot X)(\dot{X} \cdot \dot{X}) - (X \cdot \dot{X})^2$. Then from (5.4), (5.5), (5.6),

$$E(x) = \frac{1}{4}x + a + \frac{b}{x}, \tag{5.8}$$

where

$$a = -\frac{1}{4}\dot{\ell}_S \dot{\ell}_N + \frac{1}{2}(\ell_S + \ell_N)$$

$$b = -\frac{1}{4}(\ell_S-\ell_N)^2 - \frac{1}{8}(\ell_S+\ell_N)(\dot{\ell}_N-\dot{\ell}_S)^2 + \frac{1}{8}(\ell_S-\ell_N)(\dot{\ell}_S^2-\dot{\ell}_N^2).$$

Thus $x\,E(x)$ is a quadratic polynomial with two roots, which we want to show are distinct and real. Once again this could presumably be done with the explicit description in (5.7) (plus a few simple properties that would characterize those ℓ_N, ℓ_S's coming from suspensions) but we shall indicate a geometric process which shows this rather easily.

Fix s', and consider the tangent line through $Y(s')$. Let $Y_T(s)$ denote this tangent line, so that $Y_T(s') = Y(s')$, $\dot{Y}_T(s') = \dot{Y}(s')$. Let $\ell_{NT}(s) = |N-Y_T(s)|^2$, $\ell_{ST}(s) = |S-Y_T(s)|^2$. It is easy to check that

$$\ell_{PT}(s) = \ell_P(s') - \frac{1}{4}\dot{\ell}_P(s')^2 + (s-s' + \frac{1}{2}\dot{\ell}(s'))^2, \quad P = N,S.$$

($\ell_{PT}(s') = \ell_P(s')$, $\dot{\ell}_{PT}(s') = \dot{\ell}_P(s')$, the above is easily seen to be the correct form for a line, and $R\cdot\dot{Y}$, $Y\cdot\dot{Y}$, $\dot{Y}\cdot\dot{Y}$, which determine $\dot{Y}$, are expressible in terms of $\ell_p, \dot{\ell}_p$, $P = N,S$.)

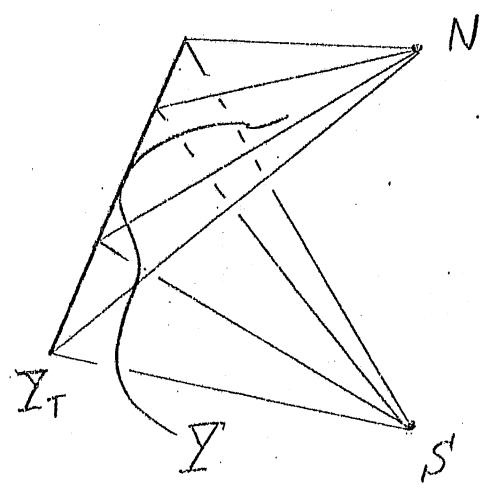


Figure 6

Thus we see that $E(x)$ at s' is the same for Y as it is for Y_T, for all x. If we flex N, S, Y_T as in §4, we see that $E(x)$ has two real roots between the roots of $X \cdot X$. Call the roots of $xE(x)$, $b_1(s) < b_2(s)$ ($b_1 = b_2$ only when N or S is on Y). Then

$$E(x) = -\frac{1}{4x}(x-b_1)(x-b_2),$$

where $r_1 \leq b_1 < b_2 \leq r_2$. Thus,

$$\max_{s\in[s_j,s_{j+1}]} b_1(s) \leq x \leq \min_{s\in[s_j,s_{j+1}]} b_2(x). \tag{5.9}$$

This gives a fairly complete algebraic description of the integrand of (5.3).

C^1 SUSPENSIONS AND LOCAL RIGIDITY

Suppose Σ is a C^1 suspension and we are worried about C^1 rigidity. We assume $N \neq S$. We consider three cases:

<u>Case 1</u> (Butterfly wings): For $s' \neq s''$, $Y(s')$, $Y(s'')$ is on the line through N and S, and $\dot{Y}(s')$, $\dot{Y}(s'')$ are parallel to R. Here it is easy to see that Σ flexes with x constant (in the C^1 sense), by fixing the suspension from s' to s'' and rotating the suspension from s'' to s' (regarding s', s'' as points on an oriented circle).

<u>Case 2</u>: There is a point s', where $Y(s') = N$ or S. By the above we then know that $b_1(s') = b_2(s')$ and $E(x) \leq 0$ with equality for only one value of x. Thus if Σ flexes it is with x constant.

Case 3: There is a point s', where in every neighborhood of s', $U_{s'}$, the sign of $\epsilon(s)$ is not constant. (E.g. $\theta_{s's''}$ has a relative max on min at s'), $X(s) \neq 0$.

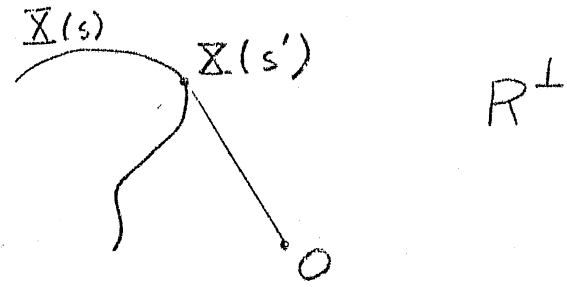


Figure 7

Here we show that Σ is locally rigid at s', in the C^1 sense. Clearly, if x is constant, the suspension of a neighborhood of x is rigid.

The hypothesis insures that $\dot{X}(s')$ is parallel to $X(s')$. Thus $E(x) = 0$ at s'. Thus $x = b_1(s')$ or $x = b_2(s')$. Let $x = x_0$ be the starting point for the flex. By the above we assume that x varies during the flex. Since $\epsilon(s)$ changes sign in a neighborhood of s', $U_{s'}$, if x_1 is chosen close enough to x (on the correct side), $\epsilon(s)$ will still change sign in $U_{s'}$ for x_1. Thus there is another point s'' in $U_{s'}$ such that Case 3 holds for s'' and x_1. But this implies that $x_1 = b_1(s'')$ or $x_1 = b_2(s'')$ which is impossible since then x_0 would then be on the "wrong side" of x_1 (I.e. x_0 violates (5.9)). Thus x must remain constant during a flex, and a neighborhood of s' is rigid in the C^1 sense.

Note how strongly the C^1 nature of things is used here.

We summarize:

Theorem 2: Let Σ be a C^1 suspension. If either

(a) $|N-S|$ is held fixed, and $X(s) = 0$, $\dot{X}(s) = 0$ for at most one s. (Case 1 does not occur),

or

(b) $X(s) \neq 0$ for all s and $\omega = 0$, then Σ is C^1 rigid.

Proof: For the proof of (a) we must only ammend Proposition 1 to consider the case when $X(s) = 0$, $\dot{X}(s) \neq 0$. But here it is clear that if we call $\theta_{s'}$ the argument of $\frac{d}{ds}(\frac{X}{|X|})$, so $\theta_{s''s'} = \theta_{s''} - \theta_{s'}$, then $\lim_{s' \to s^+} \theta_{s'} = \pi - \lim_{s' \to s^-} \theta_{s'}$. So it is possible to compute $\theta_{s's''}$ on both sides of s, and we may apply the proof of Proposition 1.

For (b) we simply observe that, if $\omega = 0$, then either Case 3 applies to at least two points, $\frac{X}{|X|}(s)$ is constant (in which case Σ is clearly rigid), or there are two points $s' < s''$ where $\frac{X}{|X|}(s)$ is constant for $s' \leq s \leq s''$, and $\epsilon(s)$ changes sign in $(s'-\delta, s''+\delta)$ for arbitrarily small δ. In this last case it is clear that an argument similar to the one used in Case 3 will apply to show Σ is rigid (over $(s'-\delta, s''+\delta)$).

It is interesting to compare this situation with that described by Sabitov for infinitesimal bendings of "corrugated" surfaces [6].

THE WINDING NUMBER THEOREM

Suppose $E(x)$ and $X \cdot X$ are both non-zero in the interval $[s', s'']$. Then using (5.2), (5.4), (5.5), (5.6) we regard $\theta_{s's''} = \theta_{s's''}(x)$ as a function of x for any complex x except for real x in the intervals $(-\infty, \max_s b_1(s)]$, $[\min_s b_2(s), \infty)$. It is easy to check that this defines $\theta_{s's''}(x)$ as in analytic functions of x.

Suppose Σ is a piecewise C^1 suspension that flexes with x variable. By (5.9) we see that for

$$s_j \leq s \leq s_{j+1}, \quad \max_{s\in[s_j,s_{j+1}]} b_1(s) < x < \min_{s\in[s_j,s_{j+1}]} b_2(x), E(x) \neq 0$$

$X\cdot X \neq 0$, so $\epsilon(s)$ is of one sign. Thus the above paragraph applies to $\theta_{s_j\, s_{j+1}}(x)$. Let A be a path in the domain defined above for x that starts at a real point and goes to ∞. Let $\theta_{s_j s_{j+1}}(\infty) = \lim_{x\to\infty} \theta_{s_j s_{j+1}}(x)$. From (5.4) and (5.8) we see that the integrand of (5.2) is of the order of $x^{-1/2}$ as $x \to \infty$. Thus $\theta_{s_j s_{j+1}}(\infty) = 0$. But from (5.3) we see that $2\pi\omega = \sum_j \theta_{s_j s_{j+1}}(x)$ is a constant for all x in the domain of θ's, since it is constant in a real interval. Thus $\omega = 0$. Summarizing:

<u>Theorem 3</u>: Let Σ be a piecewise C^1 suspension, that flexes with variable x. Then the winding number of the equator about the line through the north-south poles, ω, is zero.

Note Theorem 3 specializes to Theorem 1 in case Σ is piecewise linear.

<u>Corollary 2</u>: If Σ is a C^1 suspension that is immersed, then Σ is C^1 rigid.

<u>Proof</u>: If Σ flexes with x constant, the only way this can happen is with the butterfly wings example of Case 1. But in that case Σ is clearly not an immersion. If Σ flexes with x variable, then we may assume $X(s) \neq 0$ for all s, as well as $E(x) \neq 0$. Theorem 3 implies $\omega = 0$, and Theorem 2(b) implies Σ is C^1 rigid, a contradiction.

SUBDIVISIONS AND AN EXAMPLE

It seems to us that the notion of a C^1 suspension is somewhat restrictive and that one should allow at least for a finite number of subdivisions of the equator to discuss rigidity. Loosely speaking, if $\epsilon(s)$ changed sign at s', and s' is not one of the subdivision points, then Σ becomes rigid at s' because x is prevented from moving. If we subdivide at s', the situation changes and Σ may or may not become flexible. Thus it is reasonable to require that at every such s' where $\epsilon(s)$ changes sign we have a subdivision (some $s_j = s'$). (However, for a given Y it may happen that there are infinitely many such s''s and so it may not be possible to describe Y in this way as a finite piecewise C^1 suspension.) Note this more or less answers the conjecture of [2]. It is clear that the Y described there will flex, if we subdivide at all s''s, and will not flex otherwise.

If one were to build a paper model of the Case 3 situation and forced the model to flex (building the model only in a neighborhood of s'), a crease would suddenly appear and it would bend along that crease.

The Case 3 situation perhaps can be illuminated by the following example: Let Y be a curve in a plane perpendicular to $N-S = R$, between N and S, and let s' be a Case 3 point, where $\theta_{s''s}$ has a minimum at s'. In a neighborhood of s' for $s \geq s'$ reflect only that part of Σ about the plane through $N, S, Y(s')$ to obtain a new curve $\hat{Y}$.

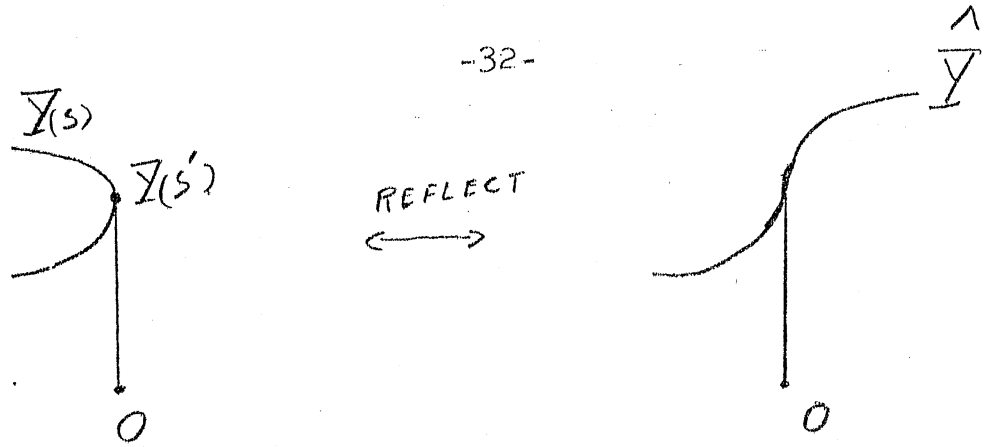


Figure 8

Then this new suspension flexes as x is decreased. However the tangent $\dot{\hat{Y}}(s') = \dot{X}(s')$ is no longer parallel to $X(s')$. So if we reflect back through the plane through $Y(s')$, N,S, $\dot{Y}(s')$ is not defined. So s' starts a crease in Y.

After Flexing

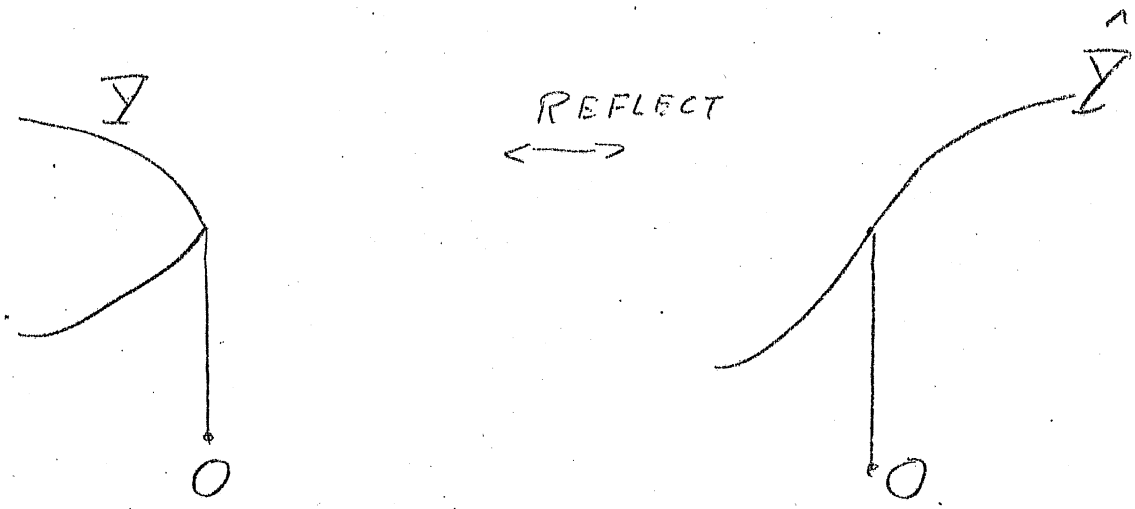


Figure 9.

§6. Remarks and Comments.

So far we have not proved that any polyhedron (even an octahedron) is rigid for all embeddings. As is the case in the classical theory we must put conditions on the embedding or map. This is dissatisfying, but for the cases we consider (polygonal suspensions) we have generalized considerably the classical result for convex polyhedra, assuming we are only interested in rigidity.

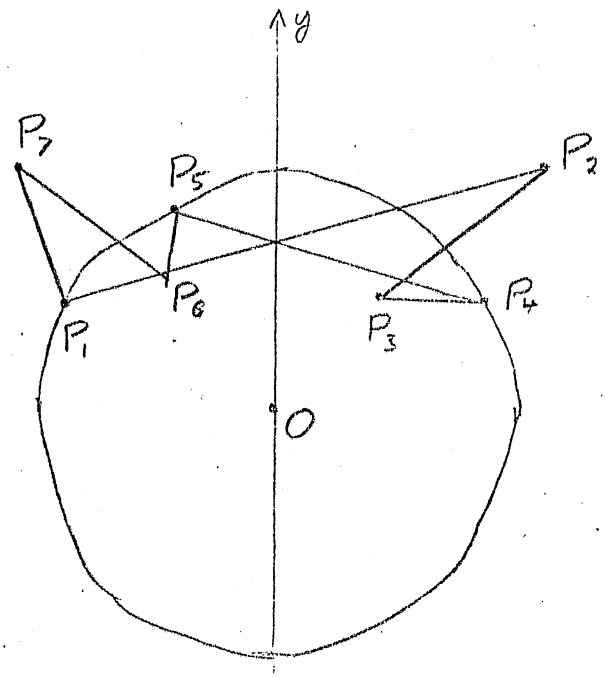
THE TANGENT WINDING NUMBER

Before we finish Part I, we think it is appropriate to mention a few results (without proof) and ideas that may be helpful even if they seem to lead to deadends.

The notion of a tangent winding number for a piecewise smooth curve in the plane is well known. (It is just the degree of the tangent vector thought of as a map of S^1 to S^1, with due consideration made for jumps at the vertices, see [9].) If Σ is a suspension, let τ denote the tangent winding number of the projection of the equator into the plane perpendicular to the line through the poles.

Theorem: If Σ is an immersion $\tau = \pm 1$.

Thus if we could show that, if Σ flexes (say with variable x) then $\tau \neq \pm 1$, then we would have a rigidity theorem for all immersed suspensions. Unfortunately, this conjecture is false even for the (PL) orthogonal suspensions of [2]. The following is a picture of a curve in the plane that satisfies condition F, and thus its orthogonal suspension flexes, but $\tau = 1$.



P_7, P_2

$P_1 P_4$ symmetric about y-axis

OP_7 bisects P_1OP_5

$\angle P_2P_3P_4 = \angle P_7P_6P_5$

$|OP_6| \equiv |OP_3|$

$P_7P_5P_4$ colinear

Figure 10

If one is given to looking for counterexamples to the rigidity conjecture, one must at least look at a class of suspensions where a typical example may look like the suspension represented by Figure 10.

IDEAS FOR THE GENERAL PROBLEM

Clearly something more is needed for the general rigidity problem for polyhedra. However, we believe that the equations described in §3 could prove the start for a complete proof. Unfortunately, at this time, it seems difficult even to provide appropriate conjectures. The best we can do now is to describe how the equations might be modified to suit the general situation, and a vague outline that surely has to be modified.

First, choose a point-call it the origin, O - that is in a clever way an appropriate function of the p_j's.

Next consider the normalization map $n:P \to S^2$ defined by $n(p) = \frac{p}{|p|}$. We wish to compute the degree of n from the intrinsic and extrinsic information. Namely let $x_j = |p_j|^2$; and let p_j, p_k, p_ℓ be a triangle (2-simplex) of P. Let $\theta_{j:k,\ell}$ be the angle from $n(p_k)$ to $n(p_\ell)$ with respect to $n(p_j)$ along geodesics, in S^2. Then from the analysis in §3

$$e^{i\theta_{j:k,\ell}} = \frac{(p_k \cdot p_\ell)(p_j \cdot p_j) - (p_j \cdot p_k)(p_j \cdot p_\ell) + |p_j|[p_k, p_\ell, p_j]i}{|p_j \times p_k|\ |p_j \times p_\ell|}$$

where all the variables can be written in terms of x_j, x_k, x_ℓ and the lengths of the edges. Now the signed area of the spherical triangle $n(p_j), n(p_k), n(p_\ell)$ is $\theta_{j:k,\ell} + \theta_{k:\ell,j} + \theta_{\ell:j,k} - \pi = \theta[j,k,\ell]$. Thus $e^{i\theta[j,k,\ell]}$ can be written in terms of the x_j's and the length of the edges. Thus we obtain a formula similar to that of (3.4). We also obtain a formula corresponding to each p_j as before by taking $R = p_j$. Together these formulae should give some intrinsic and extrinsic information about P, if P flexes. The extrinsic information should say something about when the degree of n is 0 (for helpful choices of the origin). Hopefully this would be enough to determine rigidity.

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