# AN ATTACK ON RIGIDITY, II

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## §1: Introduction:

This is a continuation of Part I, where we have set up the notation and defined the fundamental formulae for our attack on rigidity. We are interested in the continuous rigidity of polyhedra in $R^3$, and we have specialized to the case of a suspension. Our main goal here is to show that if a polygonal suspension is embedded it is rigid.

We briefly review the notation. N, S, the north and south poles, are the suspension points. $p_1, p_2, \ldots p_n$ are the points on the equator in cyclic order.

$$e_j = N - p_j \ , \ e_j' = p_j - S \ , \ j = 1,2,\ldots n \ .$$
$$R = e_j + e_j' = N - S \ .$$
$$x = R \cdot R \ , \ z_j = R \cdot e_j = \frac{1}{2}(x + e_j \cdot e_j - e_j' \cdot e_j') \ .$$

$\mathcal{C} = \{(1,2),(2,3)\ldots(n,1)\}$ is the collection of indices of pairs of adjacent vertices (for notational convenience).

$$e_{jk} = e_j - e_k = e_k' - e_j' \ , \ (j,k) \in \mathcal{C} \ , \ \text{are edges of the equator.}$$

$$y_{jk} = |R| \, [e_j, e_k, R] i \ , \ (j,k) \in \mathcal{C}, y_{jk}^2 = -x \det \begin{pmatrix} e_j \cdot e_j & e_j \cdot e_k & z_j \\ e_j \cdot e_k & e_k \cdot e_k & z_k \\ z_j & z_k & x \end{pmatrix}$$

$$G_{jk} = x(e_j \cdot e_k) - z_j z_k + y_{jk} \ , \ (j,k) \in \mathcal{C}$$

$$H_j = |R \times e_j| \quad , \quad H_j^2 = x(e_j \cdot e_j) - z_j^2$$

$$F_{jk} = \frac{G_{jk}}{H_j H_k} = e^{i\theta_{jk}} \quad .$$

$\pi : \mathbb{R}^3 \longrightarrow \pi \mathbb{R}^3$ is the orthogonal projection onto the plane perpendicular to $R$, the reference vector.

$$\theta_{jk} = \text{angle from } \pi(e_j) \text{ to } \pi(e_k) \text{ in } \pi(\mathbb{R}^3) \ .$$

The basic equation is

$$\prod_{(j,k) \in \mathcal{C}} e^{F_{jk}} = 1 \ .$$

We present a brief table of contents for Part II.

## §2: Generalized Volume

To any polyhedron whose domain is an orientable 2 manifold, we associate a number which specializes to the ordinary notion of volume in case the polyhedron is embedded.

## §3: The Rigidity of Embedded Suspensions

Using §2 and the notation of Part I we calculate the volume of a suspension that flexes in a non-trivial way. It turns out to be 0, thus yielding a genuine rigidity theorem.

## §4: A Description of Flexible suspensions:

We give a fairly complete characterization of the solutions to the basic rigidity equation defined in Part I. It uses some basic results from algebraic geometry about the group operation on a non-singular cubic, as well as a flow graph to describe the roots

of the rigidity equation.  We also describe how to build some flexible octahedra (with ruler and compass say ).

## §5:  Applications and Conjectures:

We describe what we hope will be some ideas useful to structural engineering, as well as algebraic geometry.  We also pose a few cogent conjectures.

## §2. Generalized Volume

Associated to any polyhedron (not just a suspension), whose domain is an orientable 2-manifold, is a number that will turn out to be the volume enclosed by the polyhedron in case it is embedded.

Let $P$ be the polyhedron with vertices $p_1, p_2, \cdots$ . If $P$ is orientable and $\langle p_j, p_k, p_\ell \rangle$ determines a 2-simplex of $P$ , we can say $j, k, \ell$ agrees or disagrees with the orientation on $P$ . Note the orientation is such that if $\langle p_j, p_k, p_\ell \rangle$ and $\langle p_k, p_j, p_m \rangle$ are two 2-simplices of $P$ with a common face, then they both agree or both disagree with the orientation on $P$ . (In fact any even permutation of the indices does not change the orientation and any odd permutation does change the orientation.).

It is well known and easy to prove that the volume of the tetrahedron spanned by $O$ (the origin), $p_j, p_k, p_\ell$ is

$$\pm \frac{1}{6} \det(p_j, p_k, p_\ell) = \frac{1}{6} [p_j, p_k, p_\ell] \quad \text{the scalar triple product.}$$

Thus we define

$$(2.1) \qquad V(P) = \frac{1}{6} \Sigma [p_j, p_k, p_\ell] \quad ,$$

the generalized volume, where the sum is taken over all 2-simplices $\langle p_j, p_k, p_\ell \rangle$ such that they agree with the orientation of $P$ . (If we change the orientation of $P$ we change the sign of $V(P)$.)

Lemma 1: If $P$ is embedded, and the orientation is chosen appropriately, then $V(P)$ is the volume of the region enclosed by $P$ .

Proof: From the discussion before (2.1) we see that each summand of $V(P)$, $\frac{1}{6} [p_j, p_k, p_\ell]$, is just the volume of the cone over $\langle p_j, p_k, p_\ell \rangle$ with the sign chosen so that, if the normal pointing

away from the solid enclosed by the surface is on the opposite side of the plane determined by $p_j, p_k, p_\ell$, then the sign is +, and the sign is - in the opposite case. Note that this choice of sign is compatible with the orientation determined by the embedding and outward normal.

Then it is easy to see that the volume enclosed by P is the sum of the volumes of the cones in some subdivision, where the signs are chosen + or - as above. Since V(P) is invariant under subdivision the result follows.//
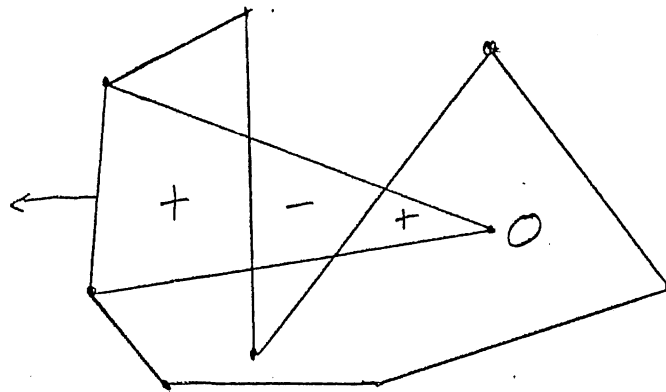


Figure 1.

Remarks:

--It is easy to check, in any case, whether P is embedded or not, that V(P) is independent of where the origin is chosen, if P is orientable. Also $V(P) \neq 0$ if P is immersed and bounds an immersed 3-manifold. In fact V(P) turns out to be the volume of the immersed 3-manifold.

--In the differentiable case, we can perform a similar analysis, and it is easy to show that the volume enclosed by a surface S is

$$\frac{1}{3} \int_S (N \cdot X) \, dA \quad ,$$

where $X = X(u,v)$ describes the surface, $N$ is the outward pointing normal, and $dA$ is the area differiential.

## Volume of Suspensions:

We specialize the above to the case of a suspension, and put it in the context of the notation of Part I.

Let us regard $S$, the south pole, as the origin. Then $R = N$, and

$$[p_j, p_k, N] = [N - p_j, N - p_k, R] = [e_j, e_k, R]$$
$$= \frac{y_{jk}}{|R|i} = \frac{y_{jk}}{\sqrt{x} \, i}$$

Thus if $P$ is a suspension (with an appropriate orientation),

$$(2.2) \qquad V(P) = \frac{1}{\sqrt{x} \, i} \sum_{(j,k) \in \mathcal{C}} y_{jk}$$

This is the key result used in §3.

## §3.  The Rigidity of Embedded Suspensions.

### Equivalence Classes of $y_{jk}$'s

Recall from I.4.3 that

$$y_{jk}^2 = \frac{1}{4} e_{jk} \cdot e_{jk} \ x \ (x-b_{jk}')(x-b_{jk}) \quad ,$$

where $b_{jk}' < b_{jk}$ are real (and $x$ is between them when $y_{jk}$ represents an actual polyhedron).

We say $y_{j_1 k_1}$ is __equivalent__ to $y_{j_2 k_2}$ (more precisely we should say $(j_1, k_1)$ is equivalent to $(j_2, k_2)$) iff $b_{j_1 k_1}' = b_{j_2 k_2}'$ and $b_{j_1 k_1} = b_{j_2 k_2}$ . The idea now is to split the fundamental rigidity (or flexing) equation into several other equations, each corresponding to an equivalence class.

__Lemma 2:__ Let $P$ be a polyhedron that flexes with variable $x$ , and let $\mathcal{C}_0 \subset \mathcal{C}$ be a subset corresponding an equivalence class described above.  Then,

$$(3.1) \quad \prod_{(j,k) \in \mathcal{C}_0} (Q_{jk} + y_{jk}) = \prod_{(j,k) \in \mathcal{C}_0} (Q_{jk} - y_{jk}) \quad .$$

is an identity in $x$ ,
where $Q_{jk} + y_{jk} = G_{jk}$ , and $Q_{jk} = x(e_j \cdot e_k) - z_j z_k$ .

__Proof:__ Let $b', b$ correspond to the $b_{jk}'$'s and $b_{jk}$'s in the definition of $\mathcal{C}_0$ . Let $\mathcal{C}_1 \subset \mathcal{C}$ be the subset corresponding to those $y_{jk}'$s such that $b_{jk} = b$ (disregarding $b_{jk}'$) .

We have from (I.3.5) that

$$\prod_{(j,k)\in\mathcal{C}} e^{(Q_{jk}+y_{jk})} = \prod_{(j,k)\in\mathcal{C}} e^{H_j^2}$$

and $H_j^2$ is a quadratic function of $x$. There are several ways now to see that

(3.2)
$$\prod_{(j,k)\in\mathcal{C}} e^{(Q_{j,k}+y_{jk})} = \prod_{(j,k)\in\mathcal{C}} e^{(Q_{jk}-y_{jk})} .$$

For instance $e^{i\theta_{jk}} = \dfrac{Q_{jk}+y_{jk}}{H_j H_k}$, and $e^{-i\theta_{jk}} = \dfrac{Q_{jk}-y_{jk}}{H_j H_k}$

Thus (3.2) follows from $\displaystyle\prod_{(j,k)\in\mathcal{C}} e^{i\theta_{jk}} = 1 = \prod_{(j,k)\in\mathcal{C}} e^{-i\theta_{jk}}$ .

(3.2) is an identity in $x$, where we regard $Q_{jk}$ and $y_{jk}$ as analytic functions of $x$. $Q_{jk}$ is a nice quadratic function of $x$ with no poles or branch points, and $y_{jk}$ has no poles and only two branch points at $b'_{jk}$ and $b_{jk}$. Thus if we start at any $x$ and proceed around a path that loops once around $b$ and not around any of the other $b_{jk}$'s or $b'_{jk}$'s (unless they are equal to $b$), then only the sign of the $y_{jk}$'s corresponding to $\mathcal{C}_1$ will change on both sides of (3.2). Thus

(3.3)
$$e^{-\prod_{\mathcal{C}_1}(Q_{jk}+y_{jk})} e^{\prod_{\mathcal{C}_1}(Q_{jk}-y_{jk})} = e^{-\prod_{\mathcal{C}_1}(Q_{jk}-y_{jk})} e^{\prod_{\mathcal{C}_1}(Q_{jk}+y_{jk})} .$$

Dividing (3.3) with (3.2) and cross-multiplying,

$$\left[ e^{\prod_{\mathcal{C}_1}(Q_{jk}+y_{jk})} \right]^2 = \left[ e^{\prod_{\mathcal{C}_1}(Q_{jk}-y_{jk})} \right]^2 .$$

Thus,

$$(3.4) \qquad \mathop{\Pi}_{\mathcal{C}_1} (Q_{jk} + y_{jk}) = \pm \mathop{\Pi}_{\mathcal{C}_1} (Q_{jk} - y_{jk}) .$$

If (3.4) holds with a minus sign, then we compute the coefficient of the highest power of $x$ on the left and right (i.e. divide both sides by $x^{2m}$, $m$ = number of elements in $\mathcal{C}_1$, and take the limit as $x \to \infty$ .) On the left it is $(-\frac{1}{4})^m$ and on the right it is $-(-\frac{1}{4})^m$ , since the leading coefficient of $Q_{jk}$ is $-\frac{1}{4}$ . Thus (3.4) must hold with a + sign.

We now repeat the above argument with (3.4) replacing (3.2), $\mathcal{C}_1$ replacing $\mathcal{C}$, $\mathcal{C}_0$ replacing $\mathcal{C}_1$ , and $b'$ replacing $b$ . (3.1) thus follows.//

## Angle Signs and Edge Lengths

We investigate the nature of $y_{jk}$'s more closely. If $P$ flexes with $x$ variable (3.1) (as well as (3.2) etc.) must hold in some real interval and thus holds for all complex $x$ on some appropriate Riemann surface, as was implicit above and discussed in Part I. When $x$ is restricted to that real interval, however, $y_{jk}$ is pure imaginary and we wish to distinguish when that imaginary part is + or - . So we define $\epsilon_{jk}$ = sign of $[e_j, e_k, R]$ in that flexing interval. Note that $\epsilon_{jk}$ is also the sign of $\theta_{jk}$, and if the orientation on $P$ is chosen correctly $\epsilon_{jk}$ is +1 or -1 as $P$ is convex or concave, respectively, at $e_{jk}$. (If $P$ were an embedding). In any case

$$(3.5) \qquad y_{jk} = \frac{1}{2}\epsilon_{jk}|e_{jk}| \sqrt{-x(x-b'_{jk})(x-b_{jk})}\, i \qquad ,$$

where the positive square root is chosen, and $x$ is in the flexing interval.

Angle Sign Edge Length Lemmas.

Lemma 3: Let $P$ be a flexible suspension, with $x$ variable, and let $\mathcal{C}_0$ represent an equivalence class of $y_{jk}$'s . Then

$$(3.6) \qquad \sum_{(j,k)\in\mathcal{C}_0} \epsilon_{jk} |e_{jk}| = 0 .$$

Proof: If we expand (3.1) of lemma 2 by the binomial theorem and collect the terms on one side, we obtain

$$(3.7) \qquad (2 \prod_{\mathcal{C}_0} Q_{jk}) \sum_{\mathcal{C}_0} \frac{y_{jk}}{Q_{jk}} + \dots = 0 ,$$

where the terms left out involve higher powers of the $y_{jk}$'s and lower powers of the $Q_{jk}$'s .

From (3.5) we see that the order of $x$ in each $y_{jk}$ is $3/2$ , and of course in $Q_{jk}$ it is 2. Thus from (3.5) we see that the coefficient of the highest power of $\sqrt{x}$ in (3.7) in a power series exansion about $\infty$ (i.e. we take the limit along a path in the upper half plane avoiding the branch points) is just

$$(-\tfrac{1}{4})^{m-1} \sum_{(j,k)\in\mathcal{C}_0} \epsilon_{jk} |e_{jk}| .$$

Thus (3.6) follows.//

One may regard (3.6) as a kind of infinitesimal non-rigidity at $\infty$ .

Lemma 4: With the same hypothesis as lemma 3, then

$$\sum_{(j,k)\in\mathcal{C}_0} y_{jk} = 0 , \quad \text{and thus} \quad \sum_{(j,k)\in\mathcal{C}} y_{jk} = 0 = V(P) .$$

is an identity in $x$ .

Proof: Let $b'$, $b$ be the $b'_{jk}$'s and $b_{jk}$'s , respectively, defining $\mathcal{C}_0$ . Then,

$$\sum_{\mathcal{C}_0} y_{jk} = \sum_{\mathcal{C}_0} \epsilon_{jk} |e_{jk}| \sqrt{-x(x-b')(x-b)} \, i$$

$$= \left( \sum_{\mathcal{C}_0} \epsilon_{jk} |e_{jk}| \right) \sqrt{-x(x-b')(x-b)} \, i = 0 \; //$$

Theorem 1: If $P$ is an embedded suspension, or is immersed bounding an immersed 3 manifold, then $P$ is rigid.

Proof: In either case $V(P) \neq 0$ . If $P$ flexes with $x$ fixed $P$ is an immersion at neither $N$ nor $S$, as was discussed in Part I.

## §4. A Description of Flexible Suspensions

We wish to investigate in more detail how flexible suspensions can exist in a non-trivial way, namely, when $x$ varies during the flex. This amounts to "solving" the basic rigidity equation which in turn hinges on the nature of the roots of each side of (3.2). The idea is to find enough conditions on the extrinsic and intrinsic parameters used to define $F_{jk}$ to allow us to build some non-trivial "flexors" (flexible polyhedra).

### The Roots

Recall

$$e^{i\theta_{jk}} = F_{jk} = \frac{Q_{jk}+y_{jk}}{H_j H_k} \quad , \quad \text{and}$$

the parameters used to define these variables depended only on lengths of the five edges, $e_j$, $e_k$, $e'_j$, $e'_k$, $e_{jk}$.

(4.1) $$(Q_{jk}+y_{jk})(Q_{jk}-y_{jk}) = H_j^2 H_k^2 = \frac{1}{8}(x-r'_j)(x-r_j)(x-r'_k)(x-r_k)$$

from I.4.2, and the section on roots of Part I.

We see that the four roots in (4.1) are entirely arbitrary up to the conditions imposed on them that all the $r'_j$'s be smaller than the smallest $r_j$. Also it is easy to see that the four roots of (4.1) determine $|e_j|, |e'_j|, |e_k|, |e'_k|$ (but one does not know the order) by say

$$|e_j| = \frac{1}{2}(\sqrt{r_j} + \sqrt{r'_j}) \quad , \quad |e'_j| = \frac{1}{2}|\sqrt{r_j} - \sqrt{r'_j}| \quad ,$$

with $e_j$ and $e'_j$ possibly switched.

The other parameters used to define the factors on the left of (4.1) are $b_{jk}$ and $b_{jk}$, and implicitly we shall discuss their relationship to the $r_j$'s later.

The importance of (4.1) is that it says that each factor $Q_{jk} + y_{jk} = G_{jk}$ has exactly four roots (counting roots with multiplicities) on the Riemann surface it defines. In fact if we have a root one sheet of the Reimann surface it is not a root on the other sheet but is a root for the other factor (except at a branch point of course), on the other sheet.

## Standard Form for the Factors

The basic equation we shall deal with is (3.1) for each equivalence class $C_0$. If (3.1) holds for each equivalence class then (3.2) holds for all of $C$. This in turn, in view of (4.1), implies that $\prod_C F_{jk} = \pm 1$, and as before this must hold with $+ 1$ to be an identity in $x$. This is the basic rigidity equation, which in a sense says that the suspension stays closed up as $x$ varies. Thus, if we can devise polyhedra such that (3.1) holds for each $C_0$, we will have flexors.

Thus we now consider a fixed $C_0$, with $b', b$ as before. We define $y = \sqrt{-x(x-b')(x-b)}\, i$ so that

$$(4.2) \qquad y^2 = x(x-b')(x-b) .$$

In view of §3, the roots of $G_{jk}$ are the intersections of the curve defined by (4.2) and the quadratic

$$(4.3) \qquad y = \frac{2Q_{jk}}{\epsilon_{jk}|e_{jk}|}$$

This is the way we describe the roots.

## The Symmetry of the Roots.

Note $y_{jk} = \frac{1}{2} \epsilon_{jk} |e_{jk}| y$ . Thus we may regard both sides of
(3.1) as polynomials in $x$ and $y$ and a root as a pair $(x,y)$ .
Then (3.1) simply says that $(x_0, y_0)$ is a root of the left side
say, if and only if $(x_0, -y_0)$ is also a root. This in turn says
that the intersections defined by the curve (4.2) and all the
curves defined by (4.3) are symmetric about the x-axis. Also it
is not hard to see that if we have quadratics defined by (4.3)
and the intersections are symmetric about the x-axis, then (3.1)
holds.

One might be tempted into guessing that the symmetry condition
implies that the quadratic factors of (3.1) cancell, but in fact
this does not necessarily happen.

## The Non-Singular Cubic

We now have the problem of how to describe in reasonably
general terms how one creates factors with the symmetry condition
of above.

Fortunately, however, the non-singular cubic, of which (3.2)
is an example, has a long and illustrious history. Much is known
about this curve, not the least of which is that it is an abelian
variety. It turns out that it is possible to define a group
operation on the curve in very natural way. Namely, we can choose
any point and call it $O$ . We shall choose $O$ to be the point

at $\infty$ on the y axis. Then if $Q_1$, $Q_2$, $Q_3$ are three distinct points on the intersection of a line with the curve, or two of the $Q_j$'s are equal and the line is tangent to the curve there, the group is defined by the condition $Q_1 + Q_2 + Q_3 = 0$. If Q is on the curve, -Q is the reflection of Q about the x-axis. It is well known that this in fact defines an Abelian group (see Walker [9]). Over the complex numbers this group is the torus $S^1 \oplus S^1$, and in our case over the reals (with the zero, $\infty$, thrown in) it becomes a subgroup isomorphic to $Z_2 \oplus S^1$.
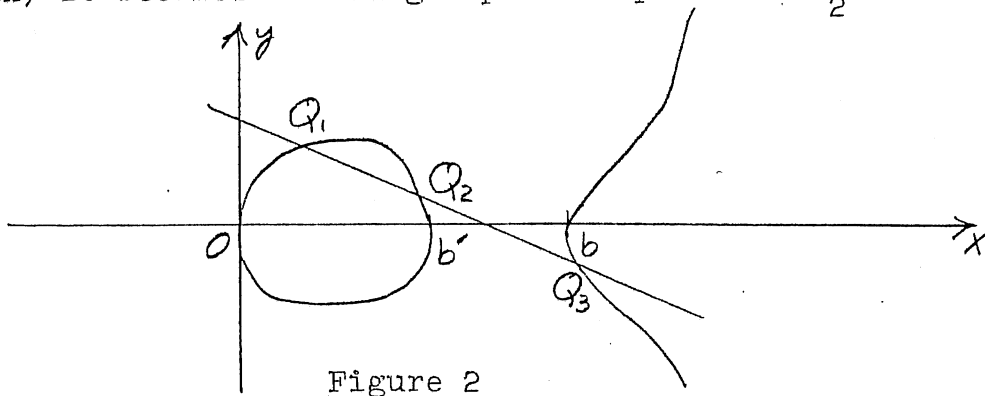


Figure 2

## Elliptic Functions

We also remark that (4.2) is also an elliptic curve in the sense that elliptic functions can be used to parametrize it. In fact it is almost in the standard form that is classically used for the Wierstrass $\mathcal{P}$ function. There, if we define the curve

$$(4.4) \qquad y^2 = 4x^3 - g_2 x - g_3 \; ,$$

it is satisfied by $y = \mathcal{P}'(z)$, $x = \mathcal{P}(z)$. Thus $y = \dfrac{\mathcal{P}'(z)}{2}$, $x = \mathcal{P}(z) + \dfrac{b+b'}{2}$, for appropriate $g_2$, and $g_3$ defining $\mathcal{P}$, parametrizes (4.2), and is in fact a group homomorphism. $\mathcal{P}$ is doubly periodic and its fundamental parallelogram in our case is

a rectangle with the image of the top side and middle line the real part of the curve. (see Lang [6] or Du Val [7]). Thus in what follows the description could just as easily be carried out upstairs in the complex plane.

## The Quadratic

Our basic problem is to describe how unsymmetric quadratics can intersect the cubic (4.2) in such a way that the intersections are symmetric.

Let $y = \hat{Q}_{jk}$ be a quadratic curve, where $\hat{Q}_{jk}$ is a quadratic function of $x$. It is easy to see that this curve intersects (4.2) at four finite points (perhaps complex points in general, but in our case they are always real). It is also easy to see that if we homoginize the equations (complete everything to projective situation) that there is in fact a double root at $\infty$ (what we called the origin before) thus jiving with Bezout's theorem. Let $Q_1$, $Q_2$, $Q_3$, $Q_4$ be the four finite intersections of $y = \hat{Q}_{jk}$ with (4.2). Then by well known results of algebraic geometry (eg. theorem 9.2 of Walker [9]) we see that $Q_1 + Q_2 + Q_3 + Q_4 = 0$, and this condition is sufficient for the existance of such a $\hat{Q}_{jk}$ to intersect (4.2) at the four given points.

## The Conditions

We wish to write down a collection of conditions that must be satisfied if a suspension is to flex (with variable $x$). However, we need a certain amount of notation.

Let $\hat{Q}_{jk} = \dfrac{2\, Q_{jk}}{\epsilon_{jk}\,|e_{jk}|}$ be the quadratic of (4.3). We have

the four roots of (4.1) which serve as the intersection of (4.2) and (4.3), and we need a way of labeling them. For $Q_{jk}$ suppose $k = j + 1$ on $j = r$, $k = 1$. Then we label the four points $(x,y)$ on the curve (4.3) $Q'_{j-}, Q_{j-}, Q'_{k+}, Q_{k+}$ corresponding respectively to the $x$ values $r'_j$, $r_j$, $r'_k$, $r_k$. Also, for each point $Q$ on the curve let $\overline{Q}$ denote its $x$ coordinate.

If the $Q_{j\pm}$'s correspond to the roots of (3.1) they must satisfy the following conditions.

(A) $\overline{Q}'_{j-} = \overline{Q}'_{j+}$, $\overline{Q}_{j-} = \overline{Q}_{j+}$, for all $j = 1, 2 \ldots n$.

(B) $Q'_{j-} + Q_{j-} + Q'_{k+} + Q_{k+} = 0$ in the group corresponding to $(j,k) \in \mathcal{C}$.

(C) For any equivalence class $\mathcal{C}_0$, the collection of $Q$'s (counting multiplicities) is symmetric about the x-axis. (Also, the $Q'$'s are on the finite component and the $Q$'s on the infinite component)

(B) and (C) have been discussed above. (A) is simply the condition that $r_j$ and $r'_j$ depend only on $|e_j|$ and $|e'_j|$. Also it is handy to note that if $(k,j)$ and $(j,\ell)$ are in the same equivalence class (thus defining the same curve (4.2) and the same group), then (A) is just the condition that $Q'_{j-} = \pm Q'_{j+}$, $Q_{j-} = \pm Q_{j+}$.

Using the condition (A), (B), (C) it is possible to write down points on a curve (4.2) that would hopefully come from a non-trivial flexor. The following table describes such a situation where three points A, B, C are chosen on a curve (4.2), with say

A, B on the infinte component and C on the finite component.
(It is clear that $Q'_{j\pm}$ is on the finite component and $Q_{j\pm}$ is
on the infinite component). There is only one equivalence class
and $n = 4$ .

| $j$ | $Q_{j-1,-}$ | $Q_{j+}$ | $Q'_{j-1,-}$ | $Q'_{j+}$ |
|---|---|---|---|---|
| 1 | A | B | C | -A-B-C |
| 2 | B | -A | A + B + C | -2B-C |
| 3 | -A | -B | 2B + C | A-B-C |
| 4 | -B | A | -A + B + C | -C |

Table 1.

(The first and third columns are shifted to make it easier to verify
condition (B).)

## The Graph

The conditions above are sufficient to enable one to create
many non-trivial flexors, in particular, in the case of $n = 4$,
the octahedron. However, we would be less than honest if we did
not indicate how such tables as the one above were constructed.
In particular there is a graph associated to a non-trivial flexor
which considerably simplifies the construction of such tables.

We construct a graph, $G_{e_0}$, (a multigraph in the sense of
Harary [6]) corresponding to each equivalence class or group as
follows: The vertices of $G_{e_0}$ consist of the elements $(j,k) \in e_0$.
By property (C) there is a pairing between the roots, the
$Q_{j\pm}$'s and $Q'_{j\pm}$'s . Choose one such pairing. We say $(j_1, k_1)$

is adjacent to $(j_2, k_2)$ if one of the Q's for $(j_1, k_1)$, $Q'_{j_1-}$, $Q_{j_1-}$, $Q'_{k_1+}$, $Q_{k_1+}$ is paired with minus (in the group) one of the Q's for $(j_2, k_2)$.

We furthermore wish to define a flow on $G_{e_0}$ in the nature described in Berge [1]. Assign a direction to the edges of $G_{e_0}$ arbitrarily. If the direction of an edge is from $(j_1, k_1)$ to $(j_2, k_2)$, then the flow is X, if X is the value (thought of as in the group) of the $(j_1, k_1)$ Q that is paired with $(j_2, k_2)$. Note that condition (B) implies that the total flow into any vertex is zero.

Since each $(j, k)$ corresponds to four Q's, the degree (not counting direction) at each vertex is four. Notice, also, from the nature of the graph and the fact that (4.2) has two components each Q' is necessarily paired with a Q' and similarly for the Q's. This is because the Q''s are all on the finite component and the Q's are all on the infinite component. Thus the edges can be partitioned into two equal collections, corresponding to the Q''s and the Q's, and each vertex is adjacent to two edges of each type. Each collection of edges is called a two-factor and we call them F' and F respectively. Thus the graph obtained is simply a graph with two disjoint two-factors, that also has a non-trivial flow. Chapter 5 of Berge [1] gives a good discussion of how to construct all possible flows (where the flow is in any abelian group) in such a situation. We shall construct examples in a moment.

## The Octahedron

We at last have enough information to construct at least
all flexible octahedra. We assume that the edges do not overlap
yielding a trivial type of flexor.

There are three main types of flexible octahedron. The first
type, which we call planar, is constructed by taking a quadralateral
in the plane that has opposite edges equal, but crosses itself,
and then choosing the north and south pole in the plane of symmetry
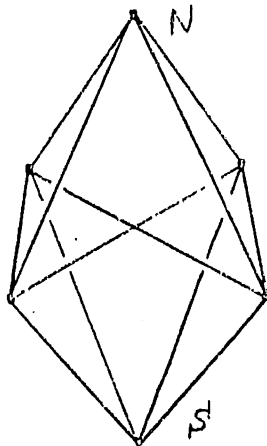through the crossing point. These were discribed in example 1 of [2].

Figure 3

The second type, which we call the symmetric flexors, have
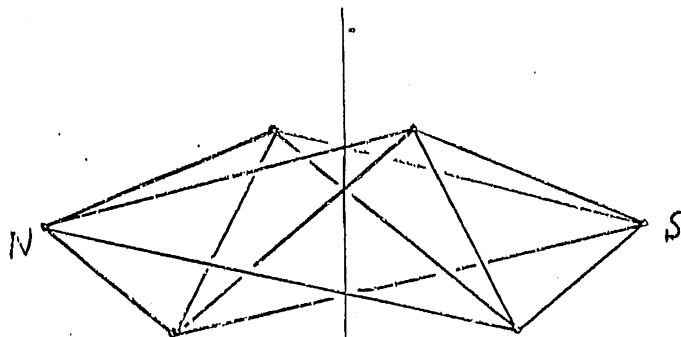all the opposite sides equal, but they are symmetric about some line.
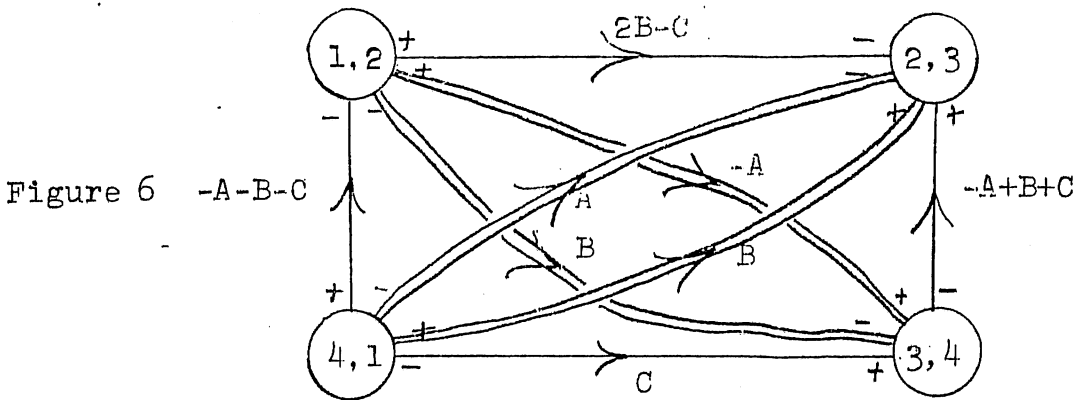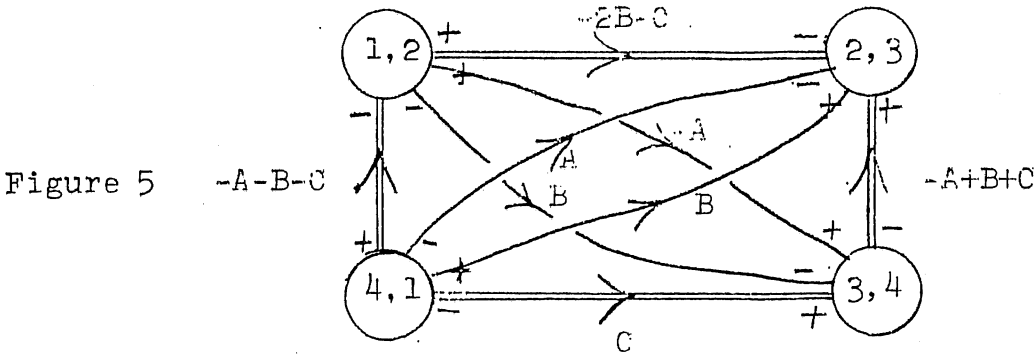
Figure 4

In both of these types, there are two classes, each consisting of two elements. In these cases the factors of (3.2) cancell in pairs.

The third type, which we call flat flexors, have only one class and thus when N and S are at the maximum or minimum distance apart the whole flexor lies flat in a plane. These are first described by means of the associated graph as follows:

Figure 5
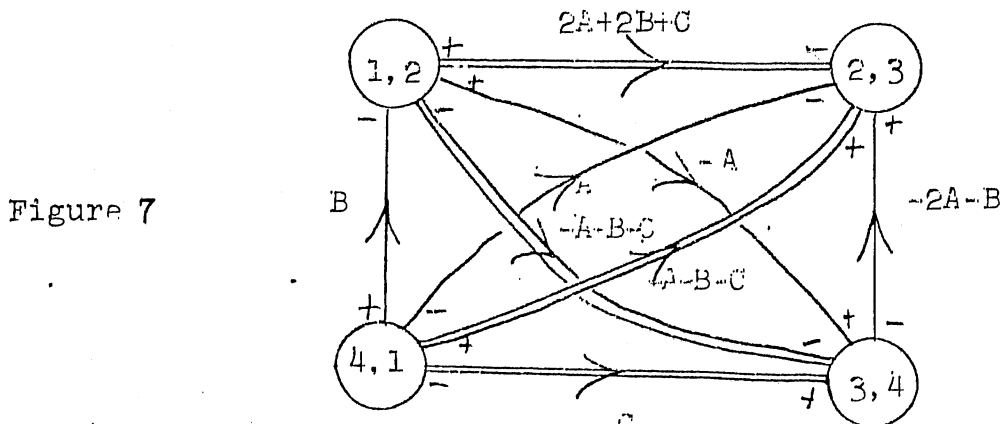
Figure 6

Figure 7

∥ is the F' factor, ╱ is the F factor.

Figure 5 generates table 1. Note that condition (A) puts additional constraints on the flow and is automatically incorporated in the above flows. With a bit of work, one can show that the three graphs above generate all possible flat flexors, and the three types exaust all possible octahedra which flex (non-trivially).

## Ruler and Compass Construction

For flat flexors we give a ruler and compass construction. (We thank R. Walker for suggesting this construction as a simplification of an earlier one).

Step 1: Choose two points $N$, $S$ in the plane, and draw a circle, $C$, (of not too large or small a radius) about $S$.

Step 2: Choose two points $p_1$, $p_3$ on an ellipse (or hyperbola) with foci at $N$, $S$. (i.e.

$$|p_1 - N| + |p_1 - S| = |p_3 - N| + |p_3 - S| ).$$

Step 3: Through $p_1$ draw the two lines $L_1$, $L_1'$ that reflect $N$ into $C$. (i.e. the circle through $N$ with center at $p_1$ intersects $C$ at $A$, $A'$. $L_1$, $L_1'$ are the perpendicular bisectors of $\overline{NA}$, $\overline{NA'}$ respectively.)

Step 4: Similarly, through $p_3$ draw two lines $L_3$, $L_3'$ that reflect $N$ into $C$.

Step 5: Call $p_2 = L_1 \cap L_3$, $p_4 = L_1' \cap L_3'$.

N, S are the north and south poles, $p_1, p_2, p_3, p_4$ the equator, for a flatened out flat flexor, if you have

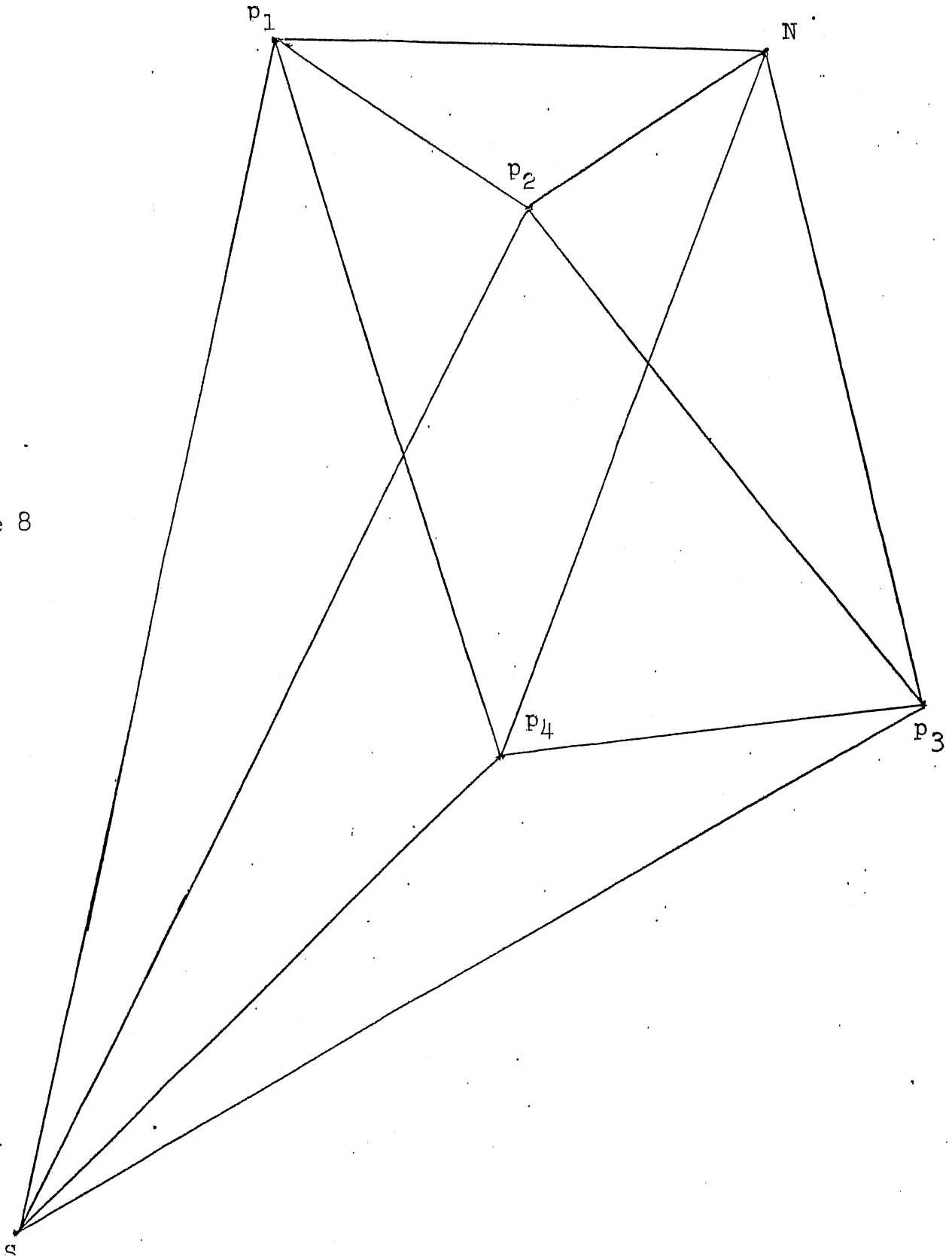not messed up the choices involved to complete the
construction above.

Figure 8

## §5. Applications and Conjectures

### Structural Engineering

It is a little difficult to convince someone that what you have is vital to them when they are not even aware they need it. However, we believe that some of the ideas presented here could be useful in some aspects of structural engineering. Part of the problem, it seems to us, is that most of the structural engineering that deals with what we are concerned about (rigidity and flexibility) has to do with what they call geometric stability (see [4] for instance), which is what mathematicians call infinitesimal rigidity. This is because a geometrically stable structure is necessary for all the forces to be (more or less) calculated.

However, if a structure is in an unstable state, but still rigid, it takes infinite internal forces to maintain the structure. So what probably would happen is that the members would deform slightly to a geometrically stable structure in which the forces can be calculated. (For instance, any flat vertices introduced in a rigid structure would surely have this property). Such would be the case unless the structure were continuously non-rigid (flexible) in the first place, in which case the structure would probably fail.

On the other hand Gluck has shown that the chance of constructing such a structure at random is zero. (see [5]). But Gluck's results says nothing about being _close_ to a non-rigid structure. It may happen that a structure is built close (in the sense of the natural topology discussed by Gluck) to a non-rigid

structure, and then with only a <u>small</u> deformation of the members
start to follow the bending inherent in that non-rigid structure,
and possibly fail, or bend beyond specified limits, even though
mathematically this could not happen (since the edges are not
assumed to change length).

We propose that some of the ideas discussed above should
be helpful in investigating this problem. In particular, if we
have a precise idea of what the flexible structures are, we can
calculate how far away any particular structure is from the nearest
flexor. This, hopefully, would serve as a good enough buffer to
prevent the structure from bending beyond specified limits.

Although it is difficult in general to provide such precise
information, at least in the case of an octahedron more or less
complete information is available in §4. For such a structure,
we should be able to develope a good idea of how far we must be
from a flexor to be sure of no more than such and such a bending.

For more general structures, possibly the idea of generalized
volume may be useful to detect being close to a flexor, since the
generalized volume is a continuous function of the structure.
(Also, certain trivial flexors must be taken into account as well).
Presumably, the volume of a flexor in general is zero, unless
the structure flexes in a trivial way (which would be unlikely
to be built perhaps).

We feel that the above should be useful in some limited
circumstances, but in most practical instances such detailed
information is surely not needed.

## Algerbraic Geometry and Number Theory

One very interesting observation about what we have done is that most is purely formal algebraic manipulations of one sort or another. In fact, although the complex numbers were used as a completion of the reals in a non-trivial way, most could be done over a fairly arbitrary ring. (We must be allowed to divide by 2, and possibly 3.)

Formally our problem consists of "solving" the quadratic equations $(p_j-p_k) \cdot (p_j-p_k) = $ constant, for $j,k$ adjacent. The solution in general is complicated, but again at least in the case of an octahedron, it is fairly complete.

## Conjectures and Problems

There are many questions one can ask. We mention only a few.

1. If a polyhedron is flexible, when must its generalized volume be zero?
2. Even for the piecewise $C^1$ suspensions of Part I, if they flex in a non-trivial way, does the volume have to be zero?
3. How does one get information about the strong rigidity conjecture? Namely, even for suspensions, is an immersed suspension rigid? (This is true for an octahedron since it is immersed if and only if it is embedded, however.)

If one has a taste for number theory there are two questions that seem interesting.

4. If $p_1, p_2, \ldots$ represent the vertices of a polyhedron, what conditions on the lengths of the edges $e_{jk}$ will

insure that the quadratic equations $|p_j - p_k|^2 = |e_{jk}|^2$ will have a solution having the $p_j$'s with integer coordinates?

5. With $|e_1|, |e_2|, \ldots$ representing the edge lengths of a flexible polyhedron, when will the lengths be integers?

In a different direction we have the following conjecture:

6. Let $a: S^2 \longrightarrow S^2$ denote the antipodal map, and let $\rho: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denote reflection about the z-axis (i.e. $\rho(x,y,z) = (-x,-y,z)$). Let $f: S^2 \longrightarrow \mathbb{R}^3$ be a map (a polyhedron possibly) such that $fa = \rho f$. Then we conjecture that $f$ is flexible.

Clearly 6. is true when $S^2$ is a PL suspension and $f$ is an isometry, from the discussion earlier. If this conjecture were true and $f$ were an immersion the strong rigidity conjecture would be false.

However, such an $f$ cannot be an immersion as can be shown with a bit of work. (We would like to thank R. Livesay and I. Berstein for helping us see this).

Of course the big question is how can the results for suspensions here be generalized to more polyhedra. We fervently hope and believe that the ideas developed here will be useful.

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