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CHAPTER 1

INTRODUCTION

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1.1. Introduction.

What do the words in the title **The Geometry of Rigid Structures** mean, at least in an informal, intuitive way? What are some of the fundamental concepts and questions of the subject? Where do they arise and why are they of interest? What are some historical lines of development and current trends of the geometry of rigid structures? How are these reflected in this book? These questions are our concerns in this introductory chapter.

Precise definitions of the basic concepts appear in Chapter 2 but these fundamental notions are easy to describe informally. For the most part, the structures that concern us are *frameworks* which consist of bars of fixed length and joints which connect some of the ends of the bars. The connecting joints allow the angles between bars to change freely (as if the bars were connected by universal joints). For example, Figure 1.1 show two very simple frameworks. The book treats a (perhaps bewildering) assortment of meanings for rigidity, but two of these notions of rigidity lie at the heart of the subject. The easiest of the two to describe is known simply as *rigidity* and means the absence of (relative) motion in the framework. For instance, the triangular framework shown in Figure 1.1(b) is rigid since the lengths of its three bars (and the ways they are joined) determine the relative positions of the three joints by the side-side-side congruence theorem from plane Euclidean geometry. On the other hand, the bent arm framework shown in Figure 1.1(a) is not rigid (and thus called *flexible*) since it can open or close with the unattached ends of the two bars moving away from or toward one another. Thus, as these examples indicate, the question of rigidity deals with certain points in Euclidean space (the joints) which are constrained by specified conditions (the bars have fixed length). And the conclusions we reach regarding rigidity will be mathematics theorems of a geometrical nature.

(a) (b)

Figure 1.1

An equally fundamental notion is the tangential or linearized version of rigidity, known as "infinitesimal rigidity". In the study of rigidity the basic constraints require that the lengths (or the squares of lengths) of the bars remain constant. Thus if $\mathbf{p}(t)$ and $\mathbf{q}(t)$ represent the location in Euclidean space of the joints at the end of a bar at time t, we require that the square

$$|\mathbf{p}(t) - \mathbf{q}(t)|^2 = (\mathbf{p}(t) - \mathbf{q}(t)) \cdot (\mathbf{p}(t) - \mathbf{q}(t))$$

of the Euclidean distance from $\mathbf{p}(t)$ to $\mathbf{q}(t)$ remains constant as t varies. Here " $\cdot$ " denotes the standard Euclidean inner product and "$|\cdots|$" denotes length. One way to think about infinitesimal rigidity is that now the lengths of bars are allowed to vary, and the basic constraints of infinitesimal rigidity require that the initial rate of change of the square of the length of each bar is zero. This amounts to conditions of the form

$$(\mathbf{p}(0) - \mathbf{q}(0)) \cdot (\mathbf{p}'(0) - \mathbf{q}'(0)) = 0$$

for the bars of the framework. In other words, one attempts to assign a nontrivial velocity vector $\mathbf{p}'$ to each joint $\mathbf{p}$ of the framework in such a way that

$$(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p}' - \mathbf{q}') = 0 \tag{1.1}$$

for every bar $\mathbf{pq}$ of the framework (see Figure 1.2). Here the meaning of nontrivial is that the velocity vectors are not given by tangent vectors at the joints to some movement of the whole framework as a rigid body. To avoid a detailed discussion of the issue of trivial velocity assignments, for the remainder of this chapter we consider only planar frameworks for which two joints joined by a bar are thought of as fixed (where, of course, we assign zero velocity to these two joints). Such a framework is *infinitesimally rigid* if $(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p}' - \mathbf{q}') = 0$ for every bar $\mathbf{pq}$ of the framework implies every $\mathbf{p}' = 0$.

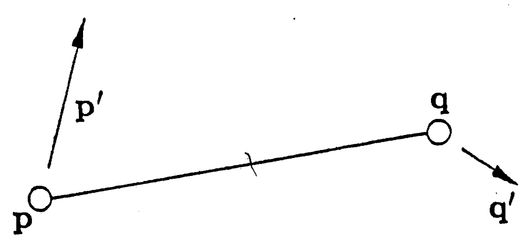

Figure 1.2

After fixing two joints joined by a bar in Figure 1.1, it is very easy to check that the triangular framework is infinitesimally rigid while the bent arm is not (and thus is said to be *infinitesimally flexible*). As one might expect, infinitesimally rigid frameworks are rigid since if a framework cannot move even "infinitesimally", it surely is not able to actually move. (Note that this rather dogmatic assertion is not a proof! Chapter 2 gives three proofs of the important fact that infinitesimal rigidity implies rigidity.) However, there exist rigid but not infinitesimally rigid frameworks so the two notions are not equivalent. Perhaps the simplest such framework is shown in Figure 1.3 where the two end joints have been fixed. (This example violates our convention that the two fixed joints be joined by a bar. One can arrange things so that the convention is honored by creating an infinitesimally rigid "truss" connecting the joints and fixing two adjacent joints in the truss. This is illustrated by the dashed lines in Figure 1.3.) However, the important issue is that a nonzero velocity vector satisfying Equation (1.1) can be assigned to the middle joint (as shown by the arrow in Figure 1.3) and thus the framework is infinitesimally flexible. But clearly this "infinitesimal motion" does not extend to an actual motion since the two bars have fixed length when rigidity is the issue. Hence the framework is rigid.

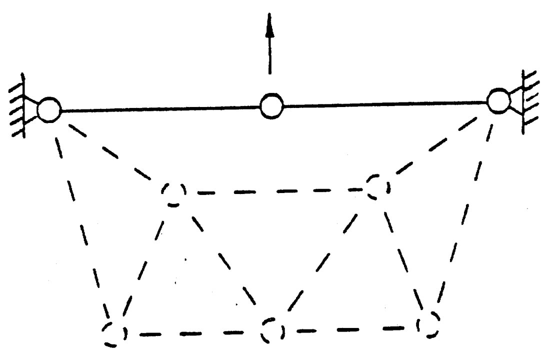

Figure 1.3

Figure 1.4 shows three frameworks that are combinatorially the same since each is a realization of the same graph – namely the graph $K_{3,3}$ which consists of two sets of three vertices with each vertex in one set is joined by an edge to all three vertices in the other set. Among other things, interest in $K_{3,3}$ stems from the fact that it has no triangles. The framework in (a) is flexible (and hence infinitesimally flexible), (b) is rigid but not infinitesimally rigid, while (c) is infinitesimally rigid. Incidentally, since our frameworks are really mathematical rather than physical objects, we steadfastly ignore any mechanical problems that may arise when bars cross or overlap one another. By considering first the (orthogonality) constraints imposed by Equation (1.1) on joints which are connected by bars to the two fixed joints, it is not difficult to see that (b) is infinitesimally flexible and (c) is infinitesimally rigid. Showing that (a) is flexible and (b) is rigid requires some geometrical ingenuity.

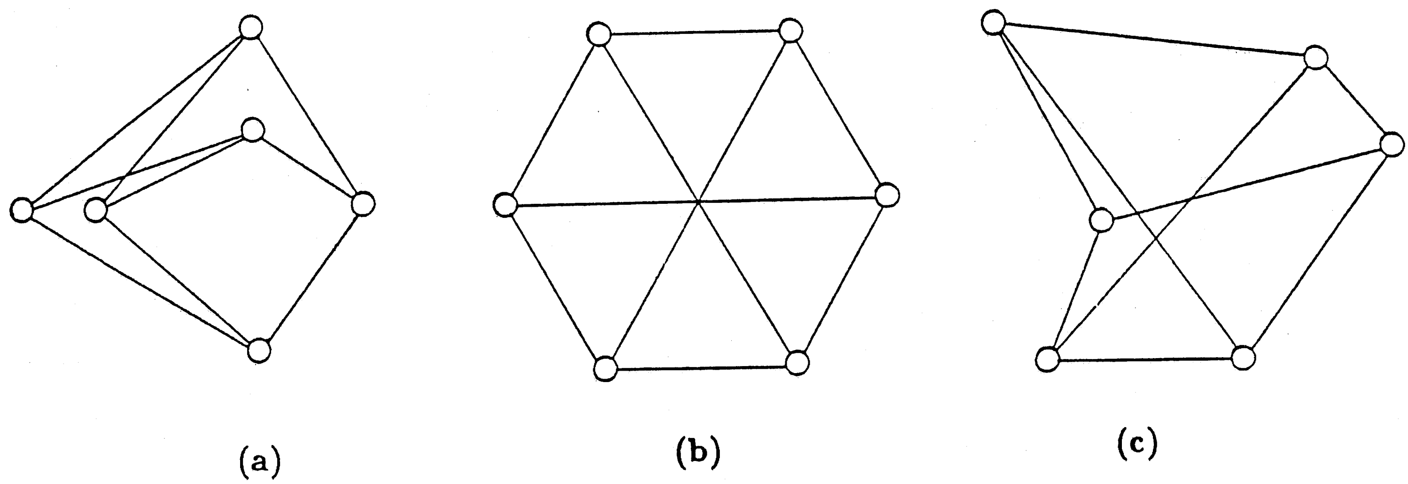

Figure 1.4

Exercise 1.1. Establish these claims.

Incidentally, doing this exercise will suggest to the reader that infinitesimal rigidity is a good deal more tractable mathematically than rigidity. Much of the book will reinforce this impression!

1.2. Historical Lines of Development.

Since the nineteenth century scientists, engineers, architects, and mathematicians have formally studied the geometry of rigid structures in a variety of contexts employing an incredible assortment of names – pin-jointed trusses, linkworks, mechanisms, frames, linkages, framed structures, and others. But the origins of the subject are surely much older than a few hundred years. As soon as our ancestors began to construct shelters and assemble tools, questions of rigidity and flexibility in a general sense, not tied to the specific context of frameworks, arise. One sees traces of an analytical approach in the works of Archimedes and Leonardo da Vinci. The appearance of the connecting-rod and crank mechanism late in the fourteenth century and the shift from the arch to the wooden truss in bridge-building and architecture during the sixteenth century are specific examples of important historical developments before the nineteenth century.

Nevertheless, the nineteenth century represents the veritable golden age of the geometry of rigid structures; the emergence of three distinct lines of development can be clearly discerned during this century. One arose out of advances in the technology of machines and mechanisms that accompanied the industrial revolution and led to progress in the understanding of flexible frameworks. Examples of the effects of the industrial revolution on the study of flexibility for frameworks range from the attempts of James Watt to find a mechanical device that moves in a straight line as part of his improvements on the steam engine to a visit of P.L. Chebyshev to England in 1852 which aroused his interest in the same problem. (Chebyshev polynomials are said to be a by-product of this quest for rectilinear motion!) Perhaps the nineteenth century culmination of this line of development was the publication in 1875 of the classic work in the kinematics of mechanisms, **Theoretische Kinematik**, by Franz Reuleaux.

Another line of development connects the phenomenal growth of railroads in Europe, Russia, and the United States during the nineteenth century to dramatic progress in the infinitesimal rigidity of frameworks. Railway bridges, fabricated from iron or wooden trusses, were at the center of these advances and the basic problem was the determination of the way in which an infinitesimally rigid framework resolves external forces (or loads) by the formation of forces of tension and compression in the bars of the framework. Such considerations lead directly to the notion of "static rigidity" for frameworks, a notion which turns out to be equivalent to infinitesimal rigidity. Graphical statics, which encompasses a variety of geometric techniques for determining these internal forces of tension and compression, was developed by Rankine, Culmann, and Cremona, all of whom combined engineering experience (often on the railroads) with geometrical insight. Moreover, there was a growing recognition during this time of the important role that projective geometry plays in infinitesimally (or statically) rigid frameworks. Our first glimpse of this connection is provided by Figure 1.4; the infinitesimal flexibility of the frameworks in (a) and (b) occurs because the joints of $K_{3,3}$ in these two realizations lie on conics. This line of development culminates in the work of James Clerk Maxwell on stresses, reciprocal figures, and projections of three-dimensional polyhedra.

However, the heritage of the subject lies not only in these intensely practical questions from statics and kinematics – there is also a purely geometrical line of development that passes from Euclid through Cauchy in the early nineteenth century to the present. The story begins with Euclid's definition of *equal and similar* (i.e., congruent) *solid figures* as those with corresponding faces congruent and arranged in the same way. Does this definition really guarantee that one of the solid figures can be made to coincide with the other by a distance-preserving transformation of three-space? It does *not* for nonconvex figures, but in 1813 Cauchy proved that convex polyhedra with congruent faces arranged in the same way are themselves congruent. While Cauchy's is really a uniqueness theorem, its implications for the rigidity of surfaces are immediate since it clearly gives the rigidity of convex polyhedral surfaces with rigid faces hinged along common edges. Cauchy's theorem has inspired analogous uniqueness and rigidity theorems for convex surfaces with varying degrees of smoothness, often involving techniques

Isn't rigidity itself a type of uniqueness

including not smoothness a la Alexandrov–Pogorelov's uniqueness thm.

reminiscent of Cauchy's. Even more recently, Fuller's geodesic domes and *tensegrity frameworks* (which, in addition to bars, allow *cables* providing an upper bound for the distance between a pair of joints and carrying only forces of tension) are examples of developments along this geometrical line. Note that replacing the two bars in Figure 1.3 by cables gives a rigid tensegrity framework, but replacing these bars by *struts* (which are compression members providing only a lower bound for the distance between pairs of joints) leads to a flexible tensegrity framework. It is interesting to note that recent applications of tensegrity frameworks to circle and sphere packing problems rely on tensegrity frameworks involving struts.

1.3. Current Trends.

Chapter 2 develops in detail (with many examples) the fundamental concepts of the geometry of rigid structures and, in a sense, the remainder of the book applies this basic theory. Three threads can be seen running through current work on the geometry of rigid structures and, as expected, these themes are reflected in the organization of this book. One is combinatorial, another projective, and the third deals with convexity. Of course, in some chapters there is considerable interplay between several themes and in some cases our classification seems (and is) rather arbitrary.

Why not put the subjects in the order they appear

The convexity theme, which obviously grows out of one of the historical trends, is most clearly represented by the chapter on the rigidity of frameworks given by convex surfaces. But, as was suggested in the discussion of the geometrical line of development, convexity plays a central role in much (although certainly not all) of the material in the tensegrity chapter. Perhaps less clearly this thread can also be seen running through the chapter on sphere and circle packing.

The projective ~~thread~~ theme, also the direct outgrowth of an historical trend, is obviously the focal point of the chapter on the projective geometry of frameworks. However, the chapter on frameworks given by complete bipartite graphs (of which $K_{3,3}$ is an example) is substantially projective as is the chapter on Maxwell's work on projected polyhedra and reciprocal diagrams.

The final theme, namely the combinatorial one, has gained prominence only since 1975. Thus some discussion of its nature seems in order. Given a graph, one can realize it as a framework in some Euclidean space in many ways. For instance, three planar frameworks with $K_{3,3}$ as their underlying graph appear in Figure 1.3. These frameworks show that it is *not* the case that every realization of a given graph in a given Euclidean space behaves the same from the point of view of, say, infinitesimal rigidity. Do "most" realizations behave the same? They do. Every graph has a typical or *generic* classification as infinitesimally rigid or infinitesimally flexible in each Euclidean space. In the case of $K_{3,3}$, one notices that the location of the joints in (a) and (b) of Figure 1.3 seem very carefully contrived and one suspects (correctly) that, in some sense, (c) represents the usual behavior of $K_{3,3}$ in the plane. Thus $K_{3,3}$ is said to be generically rigid in the plane. The chapter on generic rigidity seeks purely combinatorial or graph-theoretic characterizations of this generic classification. The chapter on grids deals with frameworks that exhibit a high degree of regularity (such as grids of squares or cubes) and provides graph-theoretic characterizations of infinitesimally rigid grids. Incidentally, matroid theory appears (rears its ugly head?) in both these chapters. The chapter on polyhedral combinatorics deals with applications of the rigidity of frameworks to the study of combinatorial and geometric properties of d-polytopes.