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CHAPTER 10: TENSEGRITY

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10.1 Introduction.

Throughout this chapter, the word **tensegrity** is used in a broad sense, to describe any frameworks with cables, struts and bars. Over the last 30 years, such structures have appeared in many experimental designs, and been the subject of much speculation. Snelson (1973) describes his rigid masts built, during the 60's, with an envelope of cables and vertex–disjoint bars inside, forming discrete versions of a balloon. Emmerich (1966) describes other early structural experiments with such patterns. These tensegrity frameworks (tension + integrity) were popularized by Buckminster Fuller (1975) as lightweight rigid forms, often with fewer members than an infinitesimally rigid bar framework. For a more current engineering summary of these structures see Callidine (1978) and Motro (1983).

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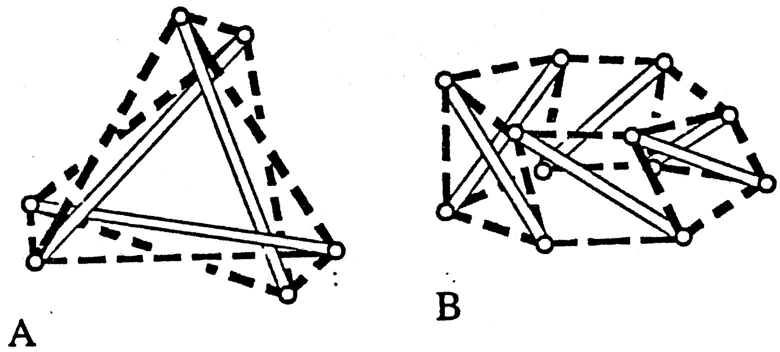

FIGURE 10.1.

Pugh (1975) gives detailed instructions for constructing some basic examples. In these models, the cables are pulled tight around fixed bars, forcing the configuration into a position of a minimum, or at least a local minimum, of energy. The resulting self–stress, with tension in the cables and compression in the bars gives a remarkable, but typically not infinitesimal, rigidity.

Working from such experimental evidence, Grünbaum&Shephard (1975) offered some stimulating conjectures about frameworks with cables, in the plane and in space. This class of examples and these specific conjectures motivated a series of mathematical studies of tensegrity frameworks over the last decade.

Chapters 2 and 3 presented the basic definitions of rigidity, infinitesimal rigidity and static rigidity for tensegrity frameworks. The basic result of chapter 2 showed that infinitesimal rigidity implies rigidity for tensegrity frameworks. However other basic facts, such as the equivalence of static and infinitesimal rigidity, were only provided for bar frameworks. In this chapter we will complete the basic pattern of results for infinitesimal rigidity and rigidity of tensegrity frameworks. In chapter 16 we will describe the more subtle global rigidity, and second–order rigidity for tensegrity frameworks, building upon these basic results. These results will also be the foundation for the results of chapter 15 on rigidity ans sphere packing.

Throughout this chapter, and chapter 16, we will encounter a principal theme: the intimate connection between the the mechanical rigidity of a tensegrity framework, and the patterns of the proper self–stresses of the framework, which are have only tension in the cables and compressions in the bars.

This connection appears implicitly in the equivalence of infinitesimal rigidity and statical rigidity for tensegrity frameworks (section 2). We draw out the explicit connection as a corollary which we call the "first–order stress test". The connection also forms an explicit part in the characterization of static rigidity of a tensegrity framework from the static rigidity of the underlying bar framework, with bars replacing all cables and struts, and the self-stresses of this underlying framework (section 3).

In section 4 this characterization is used to check the infinitesimal rigidity of a number of basic examples of tensegrity frameworks which are used throughout the rest of the book. The core of the techniques used involves tracing the sign pattern of the self-stresses in the new frameworks, when we modify the framework by: reversing cables and struts; projective transformations; exchange on two subframeworks; inserting crossing points etc.. These techniques for infinitesimal rigidity are essential for the effective use of rigidity in the analysis of the density of sphere packing, which is described in chapter 15.

In section 5, we show for a framework $G(\mathbf{p})$ at a fully regular point $\mathbf{p}$, rigidity is equivalent to infinitesimally rigid. Althought these infinitesimally rigid realizations form an open set in R^{vd}, the dependence on the sign of the self-stresses means they will not, in general, be dense in R^{vd}. We also show that for the special class of affine infinitesimal flexes, an infinitesimal flex is equivalent to a finite flex. To understand the statics of rigid, but infinitesimally flexible, tensegrity frameworks, we need to use energy functions which attain at least a local minimum at $G(\mathbf{p})$. These are introduced in section 7 to show that every rigid framework with at least one cable or strut has a proper self-stress. Such energy functions also form a basic theme of chapter 16.

10.2 Infinitesimal Rigidity of Tensegrity Frameworks.

For clarity, we partition the edges of our graph into three disjoint classes – the edges E_- for cables, E_0 for bars and E_+ for struts, creating a **signed graph** $G=(V;E_-,E_0,E_+)$. In our figures, cables are indicated by dashed lines, struts by double thin lines, and bars by regular thick lines (Figure 10.2). Some authors allow the cables and struts to overlap, and replace each bar by a cable and a strut on the edge (see section xx.7). We will stick with our convention throughout this chapter.

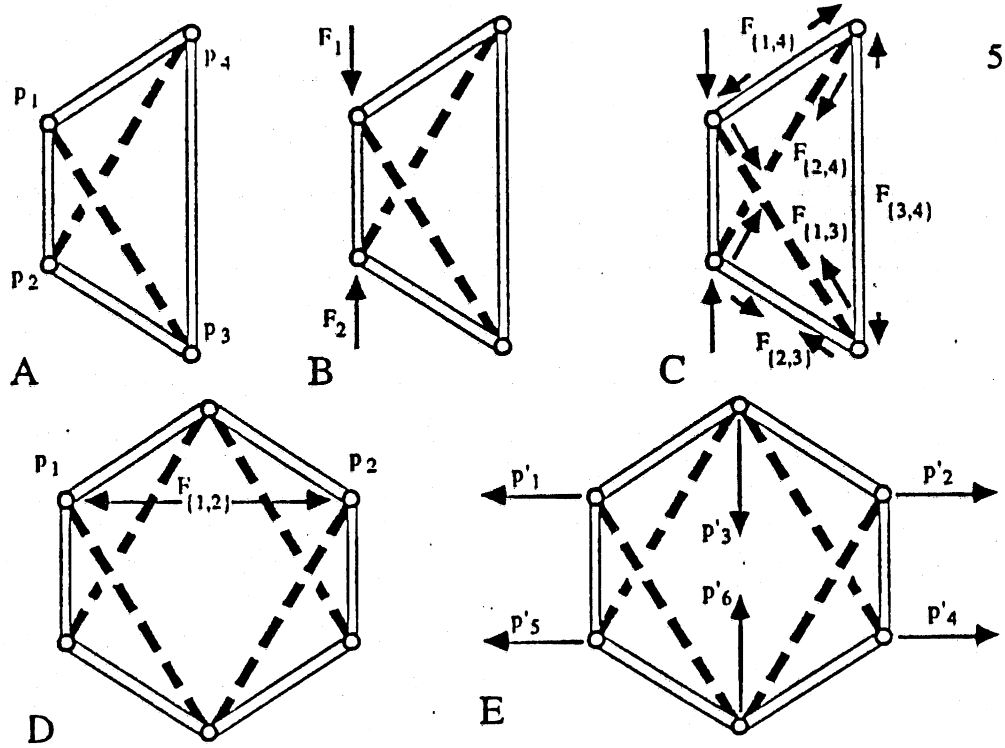

FIGURE 10.2.

We recall some basic definitions from chapters 2 and 3.

DEFINITION 10.1: A **tensegrity framework** in d-space $G(p)$ is a signed graph $(V;E_{-},E_0,E_+)$, and an assignment $p \in R^{dv}$ such that $p_i \neq p_j$ if $\{i,j\} \in E = E_{-} \cup E_0 \cup E_+$. The members in E_{-} are **cables**, the members in E° are **bars** and the members in E_+ are **struts.**

An **infinitesimal flex** of a tensegrity framework $G(p)$ is an assignment $p': V \rightarrow R^d$, of velocities $p'(v_i) = p'_i$ to the joints, such that for each edge $\{i,j\} \in E$ (Figure 2E)

$$(p_j - p_i)(p'_j - p'_i) \leq 0 \quad \text{for cables} \quad \{i,j\} \in E_{-}$$

$$(p_i - p_j)(p'_j - p'_i) = 0 \quad \text{for bars} \quad \{i,j\} \in E_0$$

$$(p_i - p_j)(p'_j - p'_i) \geq 0 \quad \text{for struts} \quad \{i,j\} \in E_+.$$

An infinitesimal flex p' is **trivial** if every there is a skew symmetric matrix S and a vector t, such that $p'_i = Sp_i + t$, for all vertices v_i.

A tensegrity framework $G(\mathbf{p})$ is **infinitesimally rigid** if every infinitesimal flex is trivial, and **infinitesimally flexible** otherwise.

An **equilibrium load** on a tensegrity framework $G(\mathbf{p})$ is an assignment $\mathbf{F}:V\rightarrow R^d$, or $\mathbf{F}=(\ldots,\mathbf{F}_i,\ldots)\in R^{dv}$ such that for each trivial infinitesimal flex $\mathbf{p}'=(\ldots,\mathbf{p}_i,\ldots)\in R^{dv}$, $\mathbf{F}.\mathbf{p}'=\mathbf{0}$. A **resolution** of an equilbrium load $\mathbf{F}=(\ldots,\mathbf{F}_i,\ldots)\in R^{dv}$ by a tensegrity framework $G(\mathbf{p})$ is an assignment of scalars to the edges $\omega\in R^e$, such that $\omega_{ij}\geq 0$ if $\{i,j\}\in E_-$ and $\omega_{ij}\leq 0$ if $\{i,j\}\in E_+$, and for each vertex i (Figure 2B,C):

$$\Sigma\omega_{ij}\,(\mathbf{p}_j\text{-}\mathbf{p}_i) + \mathbf{F}_i = \mathbf{0} \quad \text{(sum over } j \text{ with } \{i,j\}\in E= E_-\cup E_0\cup E_+).$$

or

$$\Sigma\omega_{ij}\,\mathbf{F}_{ij} + \mathbf{F}= \mathbf{0} \quad \text{(sum over } \{i,j\}\in E \text{).}$$

A tensegrity framework is **statically rigid** if every equilibrium load is resolvable.

A **self-stress** on a tensegrity framework is a resolution of the zero equilibrium load $\mathbf{0}=(\ldots,0,\ldots)$ - i.e. an assignment ω of scalars to the edges such that $\omega_{ij}\geq 0$ if $\{i,j\}\in E_-$ and $\omega_{ij}\leq 0$ if $\{i,j\}\in E_+$, and for each vertex i:

$$\Sigma\omega_{ij}\,(\mathbf{p}_j\text{-}\mathbf{p}_i) = \mathbf{0} \quad \text{(sum over } j \text{ with } \{i,j\}\in E= E_-\cup E_0\cup E_+).$$

A self-stress is **proper** if some $\omega_{ij}\neq 0$. A self-stress is **strict** if $\omega_{ij}>0$ for all $\{i,j\}\in E_-$ and $\omega_{ij}<0$ for all $\{i,j\}\in E_+$.

As expected, infinitesimal rigidity and static rigidity coincide for tensegrity frameworks. We will make particular use of the special equilibrium loads

$$\mathbf{F}_{ij}=(0,\ldots,0,\mathbf{p}_j\text{-}\mathbf{p}_i,0,\ldots,0,\mathbf{p}_i\text{-}\mathbf{p}_j,0,\ldots,0)$$

for any pair of joints in the frameworks. Intuitively the framework $G(\mathbf{p})$ does not resolve the load $\mathbf{F}_{jk}$ if and only if the framework can move, at least infinitesimally, to decrease this distance when the force is applied (Figure 10.2 D,E).

THEOREM 10.2. Roth&Whiteley (1981) A tensegrity framework in d-space is infinitesimally rigid if and only if it is statically rigid.

Proof. Given a tensegrity framework $G(\mathbf{p})$, we define the set of *basic loads*

$$F^+ = \{-F_{ij} \mid \{i,j\} \in E_- \cup E_0\} \cup \{F_{ij} \mid \{i,j\} \in E_0 \cup E_+\}$$

The equilibrium loads resolved by the framework can be written as positive linear combinations of these basic loads:

$$L = -\Sigma_s \omega_s F_s = -\Sigma_{\{h,i\} \in E_+ \cup E_0} |\omega_{hi}| F_{hi} - \Sigma_{\{j,k\} \in E_- \cup E_0} \omega_{jk} F_{jk} = \Sigma \alpha_k F_k$$

where all the α_k in the final sum are positive. In the language of convexity theory, these positive linear combinations of F^+ form a closed convex cone in R^{vd}:

$$L^r = \{ L \mid L = \Sigma_k \alpha_k F_k , \alpha > 0 \}$$

(The limit of such positive linear combinations is another positive combination, and any convex combination $\beta L^1 + \alpha L^2$, $\alpha, \beta \geq 0$, is another positive combination.)

We also have a dual cone

$$L^* = \{ u \in R^{vd} \mid u.L \leq 0 \text{ for all } L \in L^r\}$$

$$= \{ u \in R^{vd} \mid u.F_{ij} \geq 0 \text{ for } (i,j) \in E_-, \ u.F_{ij} = 0 \text{ for } (i,j) \in E_0, \ u.F_{ij} \leq 0 \text{ for } (i,j) \in E_+\}$$

$$= \{ u \in R^{vd} \mid (p_i - p_j).(u_i - u_j) \leq 0 \text{ for } (i,j) \in E_-, \ (p_i - p_j).(u_i - u_j) = 0 \text{ for } (i,j) \in E_0, \ (p_i - p_j).(u_i - u_j) \geq 0 \text{ for } (i,j) \in E_+\} \quad (1)$$

This set is, of course, the infinitesimal motions $u = p'$ of $G(\mathbf{p})$. It is a simple exercise to check that this set and the subspace of trivial infinitesimal motions are closed convex cones, with the apex at the origin.

If the framework is not statically rigid, there is an equilibrium load $\mathbf{L}^o$ which is not in this convex cone. Therefore, by a standard result on convex sets [Rockafellar (1970)] applied to such cones through the origin, there is a hyperplane $\mathbf{p}'$ in R^{vd} such that $\mathbf{p}'.\mathbf{L}^r \leq 0$ for all loads in L^r, and $\mathbf{p}'.\mathbf{L}^o > 0$. Since each trivial motion satisfies $\mathbf{u}.\mathbf{L} \leq 0$ for all equilibrium loads, this $\mathbf{p}'$, with $\mathbf{p}'.\mathbf{L}^o > 0$, is a nontrivial infinitesimal motion. We conclude that the framework is not infinitesimally rigid.

Conversely, if the framework is not infinitesimally rigid, by equation 1 there is an nontrivial infinitesimal motion $\mathbf{p}'$, with $\mathbf{p}'.\mathbf{L} \leq 0$ for all loads $\mathbf{L} \in F^+$. There is a hyperplane $\mathbf{L}^o$ which separates this motion from the closed convex cone of trivial motions: $\mathbf{p}'.\mathbf{L}^o > 0$ but $\mathbf{u}.\mathbf{L}^o \leq 0$ for all trivial infinitesimal motions. Since both $\mathbf{u}$ and $-\mathbf{u}$ are trivial motions, we have $\mathbf{u}.\mathbf{L}^o = 0$ for all trivial infinitesimal motions. Thus $\mathbf{L}^o$ is an equilibrium load, and $\mathbf{p}'$ separates $\mathbf{L}^o$ from the resolved loads. We conclude that $G(\mathbf{p})$ is not statically rigid ▯

In this proof, we have implicitly shown that the set of infinitesimal motions of any tensegrity framework, and the set of resolvable loads are dual cones. At the center of this duality is a connection between the edges which are changed by infinitesimal motions and the edges which participate in proper self-stresses. We call the follwing corollary the **first-order stress test.** In chapter 16 we will present an extension which we call the second-order stress test.

COROLLARY 10.3. $G(p)$ has a self-stress with $\omega_{jk}\neq 0$, for an edge $(j,k)\in E_-\cup E_+$, if and only if every infinitesimal motion p' of $G(p)$ satisfies $(p_j-p_k)\cdot(p'_j-p'_k)=0$.

Proof. (i) Assume that $\omega_{hi}>0$, for a cable $(h,i)\in E_-$. For any infinitesmal motion p', and any proper self-stress ω we have: $\Sigma\omega_{jk}(p_j-p_k)=\mathbf{0}$. On the other hand we have:

$\omega_{jk}(p_j-p_k)\cdot(p'_j-p'_k)\leq 0$ for each cable;

$\omega_{jk}(p_j-p_k)\cdot(p'_j-p'_k)\leq 0$ for each strut;

and $\omega_{jk}(p_j-p_k)\cdot(p'_j-p'_k)=0$ for each bar.

Thus $\Sigma\omega_{jk}(p_j-p_k)\cdot(p'_j-p'_k)<0$ if $\omega_{hi}(p_h-p_i)\cdot(p'_h-p'_i)<0$. Since $\omega_{hi}>0$, we conclude that $(p_h-p_i)\cdot(p'_h-p'_i)=0$.

A similar argument works for struts.

Conversely, assume that for some cable (j,k) $\omega_{jk}=0$ for all self-stresses of $G(p)$. This means that F_{jk} is not in the convex cone L^r. By the proof of Theorem 10.2, there is an infinitesimal motion p_{jk}' which makes $p_{jk}'\cdot F_{jk}>0$, or $(p_j-p_k)\cdot(p_{jk}{}'_j-p_{jk}{}'_k)<0$. ∎

One form in which we use this stress test is to use infinitesimal motions to change the lengths of all unstressed members of a tensegrity framework. We call the set:

$A=\{\ (j,k)\in E_-\cup E_+ \mid \omega_{jk}=0$ for all proper self-stresses of $G(p)\ \}$

the **open** **members** of tensegrity framework

COROLLARY 10.4. There is an infinitesimal motion p' of $G(p)$ such that $(p_j-p_k)\cdot(p'_j-p'_k)<0$ for each open cable, and with $(p_j-p_k)\cdot(p'_j-p'_k)>0$ for each open strut.

Proof. By Corollary 10.3 we have such an infinitesimal motion for each cable in A and a similar infinitesimal motion $\mathbf{p}_{jk}'$ for each strut in E^*, $(\mathbf{p}_j-\mathbf{p}_k)\cdot(\mathbf{p}_{jk}{}'_j-\mathbf{p}_{jk}{}'_k)>0$ or $\mathbf{p}_{jk}'.\mathbf{F}_{jk}<0$. If we add up all these infinitesimal motions over A, we have, for each cable:

$$\mathbf{F}_{jk}\cdot(\Sigma_{(h,i)\in E^*}(\mathbf{p}'_{hi})) = \mathbf{F}_{jk}\cdot\mathbf{p}_{jk}'+\Sigma_{(h,i)\neq(j,k)}\mathbf{F}_{jk}\cdot\mathbf{p}_{hi}' > 0+0 = 0.$$

A similar inequality holds for each strut, and we are finished.[]

10.3. From Bar Frameworks to Tensegrity Frameworks.

There is a simple connection between the static rigidity of a tensegrity framework $G(\mathbf{p})$ and the static rigidity and self–stresses of the **induced bar framework** $\underline{G}(\mathbf{p})$ on the same joints $\mathbf{p}$ and the underlying unsigned graph $\underline{G}=(V;E)$ (i.e. all the members become bars).

THEOREM 10.5. Roth&Whiteley (1981) For a tensegrity framework $G(\mathbf{p})$, $G=(V;E_-,E_0,E_+)$, the following are equivalent:

(i) $G(\mathbf{p})$ is statically rigid;

(ii) $G(\mathbf{p})$ has a strict self-stress and the induced bar framework $\underline{G}(\mathbf{p})$ is statically rigid.

Proof: Assume that $G(\mathbf{p})$ is statically rigid. Since $\underline{G}(\mathbf{p})$ has the same equilibrium loads, and carries all the resolutions of $G(\mathbf{p})$, it is also statically rigid. For any edge $\{i,j\}\in E_-$, the load $-\mathbf{F}_{ij}$ is resolved by the tensegrity framework, using "proper scalars" ${}^{hi}\omega_{jk}$: $-\mathbf{F}_{ij}+\Sigma\,{}^{hi}\omega_{jk}\,\mathbf{F}_{jk}=\mathbf{0}$. This defines a proper self-stress on the tensegrity framework which is non–zero on this member. Similarly, for any edge $\{h,i\}\in E_+$, the load $\mathbf{F}_{hi}$ is resolved by the tensegrity framework, defining a proper self-stress $\mathbf{F}_{hi}+\Sigma_{jk}\,{}^{hi}\omega_{jk}\mathbf{F}_{jk}=\mathbf{0}$. The sum of these self-stresses will be the strict self-stress on the tensegrity framework.

Conversely, assume that G(p) has a strict self-stress β_{jk} and the induced bar framework $\underline{G}$(p) is statically rigid. Any equilibrium load F has a resolution $F+\Sigma\omega_{jk}F_{jk} = 0$ on the bar framework. If we add a positive multiple of the strict self-stress to this, we have

$$F+\Sigma\omega_{jk}F_{jk} + \alpha\Sigma\omega_{jk}F_{jk} = 0 \text{ or } F+\Sigma(\omega_{jk}+\alpha\beta_{jk})F_{jk}= 0.$$

For a sufficiently large α, each $(\omega_{jk}+\alpha\beta_{jk})$ will have the sign of β_{jk}. This gives a resolution of the load by the tensegrity framework, as required. □

This characterization is used in two directions:

(i) Given a tensegrity framework, we check the infinitesimal rigidity of $\underline{G}$(p), and then prove that there is a strict self-stress.

(ii) Given an infinitesimally rigid bar frameworks with extra members and a nontrivial self-stress, we define the cables and struts of a tensegrity framework by the signs of the self-stress.

Recall that a circuit is a framework which is a minimal self–stressed subframework, and will have a self-stress which is non-zero on every member. Therefore any generic realization of a generically rigid circuit will have a single self–stress which is non–zero on all members. If this self–stress is used to assign signs to the members, we have a statically rigid tensegrity framework.

EXAMPLE 10.6. By Laman's theorem (Theorem 7.xx), generic plane circuits are characterized by $e=2v-2$ and $e'\leq 2v'-3$ for proper subsets E' of edges. Simple examples include the bipartite graph $K_{3,4}$ and $K_{3,3}$ plus one bar (see chapter 7 and see Exercise 10.4). In generic position for some pattern of cables and struts following the single stress, these generic circuits will be infintesimaly rigid with $e_-+e_+ = 2v-2$.

In general, an infinitesimally rigid plane tensegrity framework with at least one cable or strut must have $e \geq 2v-2$. If all members are cables and struts in an infinitesimally rigid tensegrity framework, and $e_-+e_+ = 2v-2$, we must have $e' \leq 2v'-3$ for proper subsets of edges, and the framework is a circuit realized with a single strict self-stress.

The pattern of cables and struts in a plane tensegrity framework can require $e > 2v-2$ to keep infinitesimal rigidity. Figure 10.3 shows a few examples. It is a simple exercise to create **minimal** infinitesimally rigid plane tensegrity frameworks: frameworks which are not rigid if even one member is removed, with $e_-+e_+=3v-6$ (Figure B). This construction extends to any number of joints in general position. If we put the joints in special position, this count can be raised to $e_-+e_+= 4v-10$. Figure C shows an example with $v=5$, but this can be extended with any number of new joints at the position **p**.

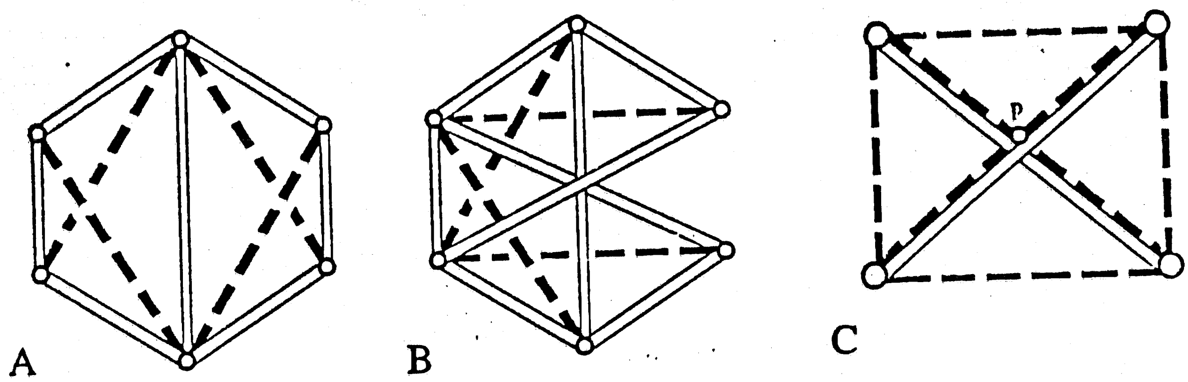

FIGURE 10.3.

If a minimal infinitesimally rigid plane tensegrity framework has its joints in generic position, and every proper stress is non–zero at each vertex, then $e=2v-3$ (see Exercise 10.8)

In chapter 2, the following upper bound was given for the number of members in a tensegrity framework:

$$e_- + e_+ + 2e_0 \leq 4v-6.$$

If the there are no bars, this gives the upper bound $e_- + e_+ \leq 4v-6$. We **conjecture** that the examples of Figure 10.4 B, with $e_- + e_+ = 4v-10$ give the sharp upper bound for minimal infinitesimally rigid tensegrity frameworks. Further, we **conjecture** that for vertices in general position $e_- + e_+ = 3v-6$ is the sharp upper bound.

Finally, we note that for infinitesimally rigid plane tensegrity frameworks with no bars, the sharp minimum value for e_+, or for e_-, is 2, as illustrated by the examples of Figure 3 B.

There is no characterization of generic circuits in 3-space (see Tay 1986)). However, examples of generic infinitesimally rigid circiuts include $K_{5,5}$ and $K_{4,7}$ (see chapter 7) and any 4-connected triangulated sphere, with one added diagonal (see Figures 10.6, 10.8 and Whiteley (1985a)).

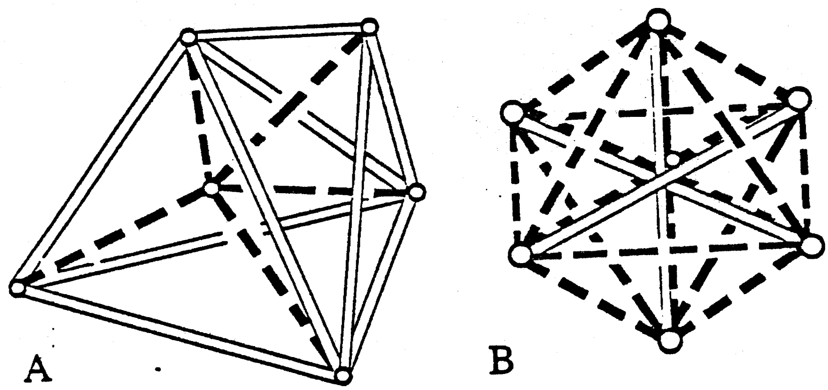

FIGURE 10.4.

For minimal infinitesimally rigid tensegrity frameworks in 3–space with at least one cable, we have $e \geq 3v-5$. Modifying the plane examples, we can create minimal infinitesimally rigid tensegrity frameworks with $e=4v-10$, with the joints in general position (Figure 10.4 A), and $e=6v-21$ with the joints in special position (Figure B). These should be compared with the upper bound: $e_- + e_+ + 2e_0 \leq 6v-12$ given in chapter 2.

Finally we note that in 3-space there are generic circuits which are not infinitesimally rigid (recall the two bananas of Example 2.xx). In generic realizations, these frameworks have a nontrivial self-stress, which leads to a tensegrity mechanism - a tensegrity framework which moves as a mechanism with a constant self-stress throughout the range of motion.

EXERCISES.

10.1. In a tensegrity framework on the line, an **oriented polygon** is a closed polygon such that all members traversed right (left) are struts and all members traversed left (right) are cables. Show that a tensegrity framework on the line with no bars is infinitesimally rigid if and only if every pair of joints is contained in an oriented polygon (compare with section 5.7).

10.2. If a minimal infinitesimally rigid plane tensegrity framework has its joints in generic position, and every proper stress is non-zero at each vertex, show that e=2v-3 e=2v-2 (?) should specialize to some examples of this!

10.3. Generating Infinitesimally Rigid Tensegrity Frameworks

In section (?) we describe some standard examples of infinitesimally rigid tensegrity frameworks and explore methods for transforming or combining infinitesimally rigid tensegrity frameworks into new infinitesimally rigid tensegrity frameworks.

EXAMPLE 10.7 Consider the plane convex quadrilateral, with two diagonals (Figure 10.5). As a bar framework, this is clearly statically rigid, Since e=2v-2, it contains a self-stress . At each corner, there are 3 edges with distinct directions, so the local equilibrium is unique at this corner, up to a scalar. The local equilibrium gives the two boundary edges one sign and the interior edge the opposite sign. If we choose one boundary edge to have tension, the rest of the boundary is tension and the interior members are compression (Figure B). This shows that the tensegrity framework with cables on the boundary, and struts in the interior is also statically, or infinitesimally rigid (Figure C). On the other hand, we can choose compression for one boundary edge, reversing the signs of all members in the self-stress, and prove the static rigidity of the tensegrity framework in Figure D.

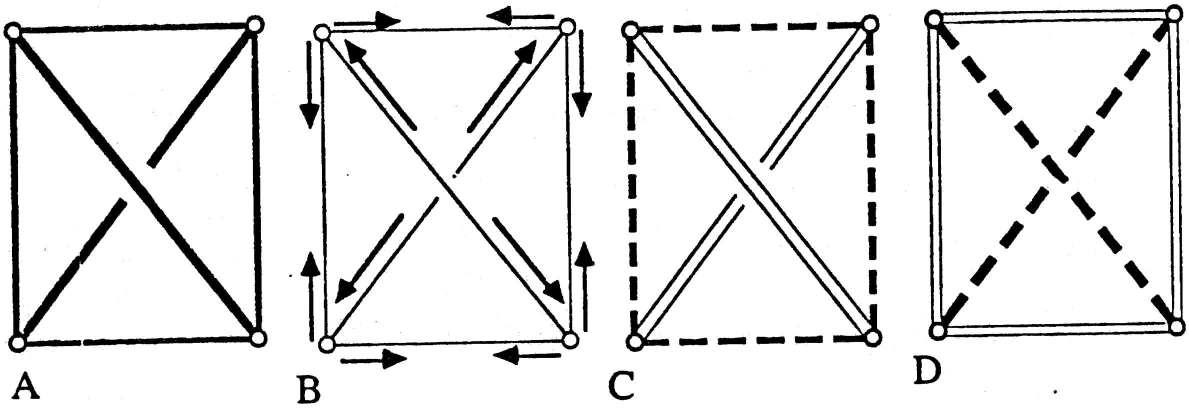

FIGURE 10.5.

This is an important Remark, should be Labeled

In general, the + and - signs on the edges of a proper self-stress at a vertex cannot be separated by a hyperplane. If such a hyperplane H did exist, with normal **n** towards the + side, the equilibrium $\mathbf{0}=(\Sigma\omega_{ij}(\mathbf{p}_j-\mathbf{p}_i)).\mathbf{n}$ would break up into the sum of two positive terms $\Sigma\omega_{ij}((\mathbf{p}_i-\mathbf{p}_j)).\mathbf{n}$ for the + and for the - sides, which is a contradiction.

REMARK 10.8. The **reverse** of a tensegrity framework $G(\mathbf{p})$, $G^-(\mathbf{p})$, is the tensegrity framework on the same joints, with the signed graph $G^-=(V;E_+,E_0,E_-)$ which interchanges cables and struts. $G(\mathbf{p})$ and $G^-(\mathbf{p})$ have the same underlying bar framework, and a strict self-stress ω on $G(\mathbf{p})$ reverses to a strict self-stress $-\omega$ on $G^-(\mathbf{p})$. Therefore a tensegrity framework $G(\mathbf{p})$ is statically rigid if and only if the reverse framework G⁻(**p**) is statically rigid. We will see below that this equivalence does not extend to general rigid frameworks. In fact, for some classes of tensegrity framework, G(**p**) and G⁻(**p**) are both rigid only if they are infinitesimally rigid.

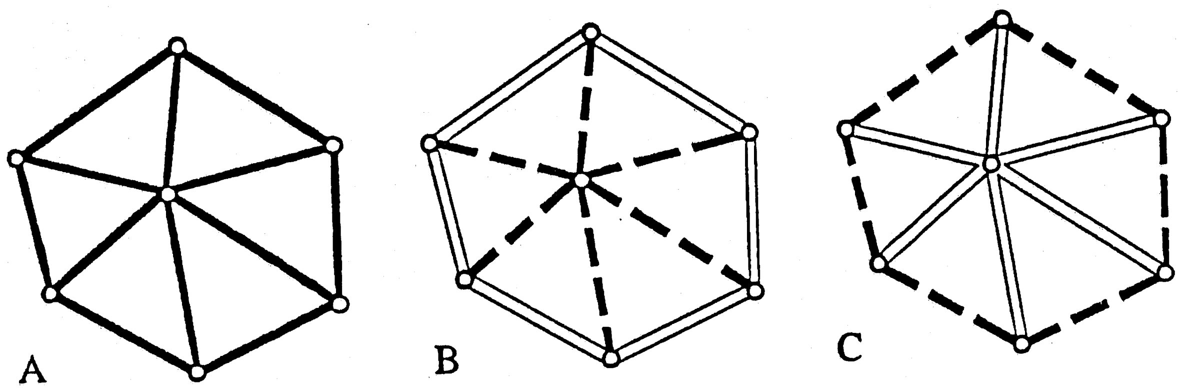

FIGURE 10.6.

EXAMPLE 10.9. Consider a **convex wheel** formed by a convex polygon and one interior vertex (Figure 10.6). This triangulated polygon is infinitesimally rigid as a bar framework. With $e=2v-2$, it has a one-space of nontrivial self-stresses. At the exterior vertices, the three edges require one sign for the boundary edges and the opposite sign for the interior edge. This gives a strict self-stress for all edges, proving the infinitesimal rigidity of the two tensegrity frameworks in Figures B and C.

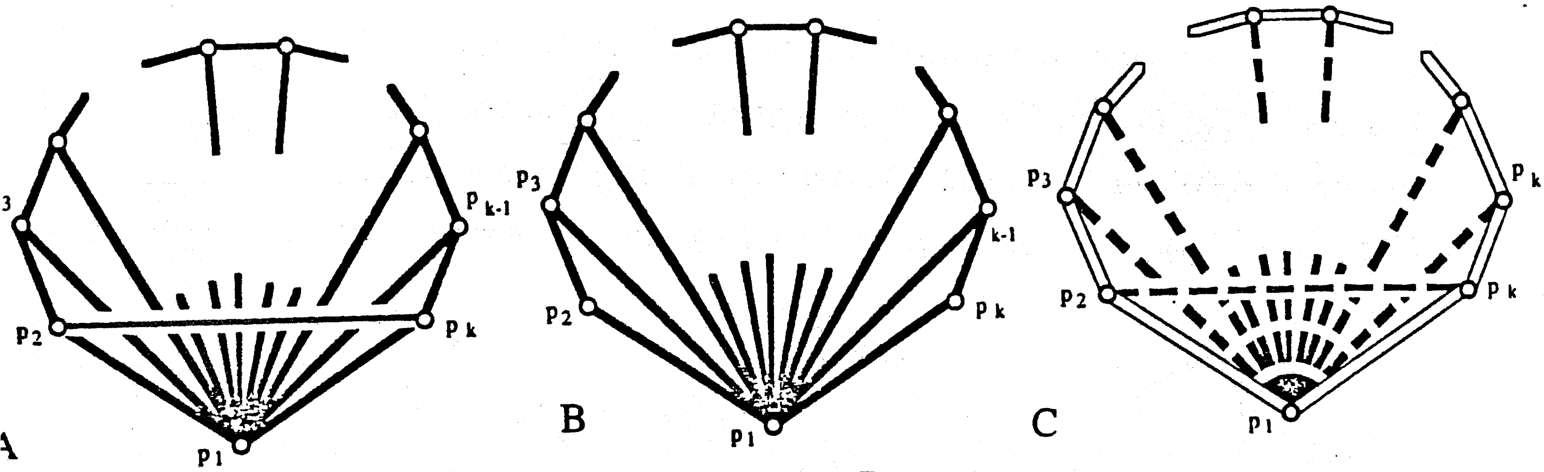

FIGURE 10.7.

EXAMPLE 10.10. Consider a **Grünbaum polygon** built on a convex k–gon of struts $k \geq 4$, with $k-3$ interior cables joining joint $\mathbf{p}_1$ to all non–adjacent joints, and a cable joining $\mathbf{p}_2$ and $\mathbf{p}_k$ (Figure 10.7). (For $k=4$ this is the convex quadrilateral with the two diagonals.) As before, with edge $\{2,k\}$ removed we have a triangulated polygon which is statically rigid as a bar framework (Figure B). With the added member, the framework has a nontrivial self–stress. At the 3-valent joints $\mathbf{p}_2,\ldots\mathbf{p}_k$ the separation property shows that the boundary edges have one sign (+) while all interior edges have the opposite sign (-), as required for the tensegrity framework of Figure C.

EXAMPLE 10.11. Consider the tensegrity framework in Figure 10.8A. Removing one bar leaves a 2-simple graph (Figure B), so $\underline{G}(\mathbf{p})$ is infinitesimally rigid. If we add the two self-stresses on subframeworks $G^1(\mathbf{q})$ and $G^2(\mathbf{r})$ shown in Figure C, then appropriate scalars give a cancellation on the common edge $\{1,2\}$, leaving a proper self-stress for $G(\mathbf{p})$. Thus $G(\mathbf{p})$ is infinitesimally rigid. We call $G(\mathbf{p})$ the **tensegrity exchange** of $G^1(\mathbf{q})$ and $G^2(\mathbf{r})$.

If we choose the different sign pattern of Figure 10.1, the tensegrity graph is not infinitesimally rigid, or even rigid, as shown in Figure 1D.

As this examples illustrates, the qualitative pattern of a convex polygon of struts with cables in the interior, which often appeared in the preceeding plane examples is not sufficient to guarantee rigidity. Chapter 16 will give an extensive analysis of such patterns.

This example also illustrates the following general technique.

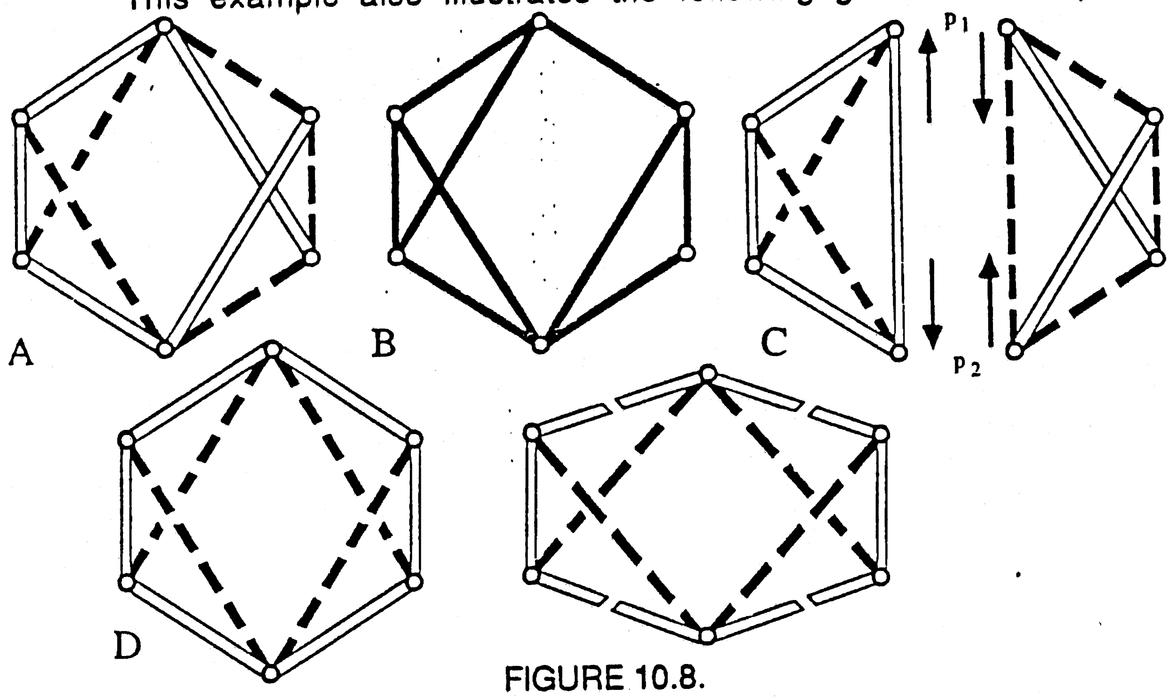

FIGURE 10.8.

PROPOSITION 10.12. Roth&Whiteley (1981) If $G^1(\mathbf{q})$ and $G^2(\mathbf{r})$ are two plane statically rigid tensegrity frameworks sharing at least two joints $\mathbf{p}_1$ and $\mathbf{p}_2$, with $(1,2) \in E^1_+$ and $(1,2) \in E^2_-$, and all other common members of $G^1(\mathbf{q})$ and $G^2(\mathbf{r})$ agree in sign, then the tensegrity framework $G(\mathbf{p})$ with $V = V^1 \cup V^2$, $E_- = E^1_- \cup E^2_- - \{(1,2)\}$,

$E_0 = E^1{}_0 \cup E^2{}_0$, $E_+ = E^1{}_+ \cup E^2{}_+ - \{(1,2)\}$ and with the same positions for the joints, is statically rigid in the plane.

Proof. Since the induced frameworks $\underline{G}^1(q)$ and $\underline{G}^2(r)$ are infinitesimally rigid with a nontrivial self-stress on (1,2), they are infinitesimally rigid with this bar removed. Therefore their join on at least the two joints, $\underline{G}(p)$, is also infinitesimally rigid (Chapter 3). If we add the strict self-stresses on $G^1(q)$ and $G^2(r)$ with $\omega^1{}_{12}=1$ and $\omega^2{}_{12}=-1$ we obtain a strict self-stress for $G(p)$.

□

EXAMPLE 10.13. Consider a tetrahedron with one interior vertex, forming the complete graph K_5 (Figure 10.9 A). With one edge removed, this is a 3–simple, isostatic bar framework (recall chapter 2). Therefore the full bar framework contains a 1–space of nontrivial self–stresses. At each exterior vertex, this self–stress must involve all 4 members, and no plane can separate the tension members from the compression members. Thus the 3 boundary members have one sign, and the interior member has the opposite sign. This boundary sign determines all other boundary signs as the same, and all interior members are the opposite. Thus the two frameworks in Figure B are statically rigid in 3-space.

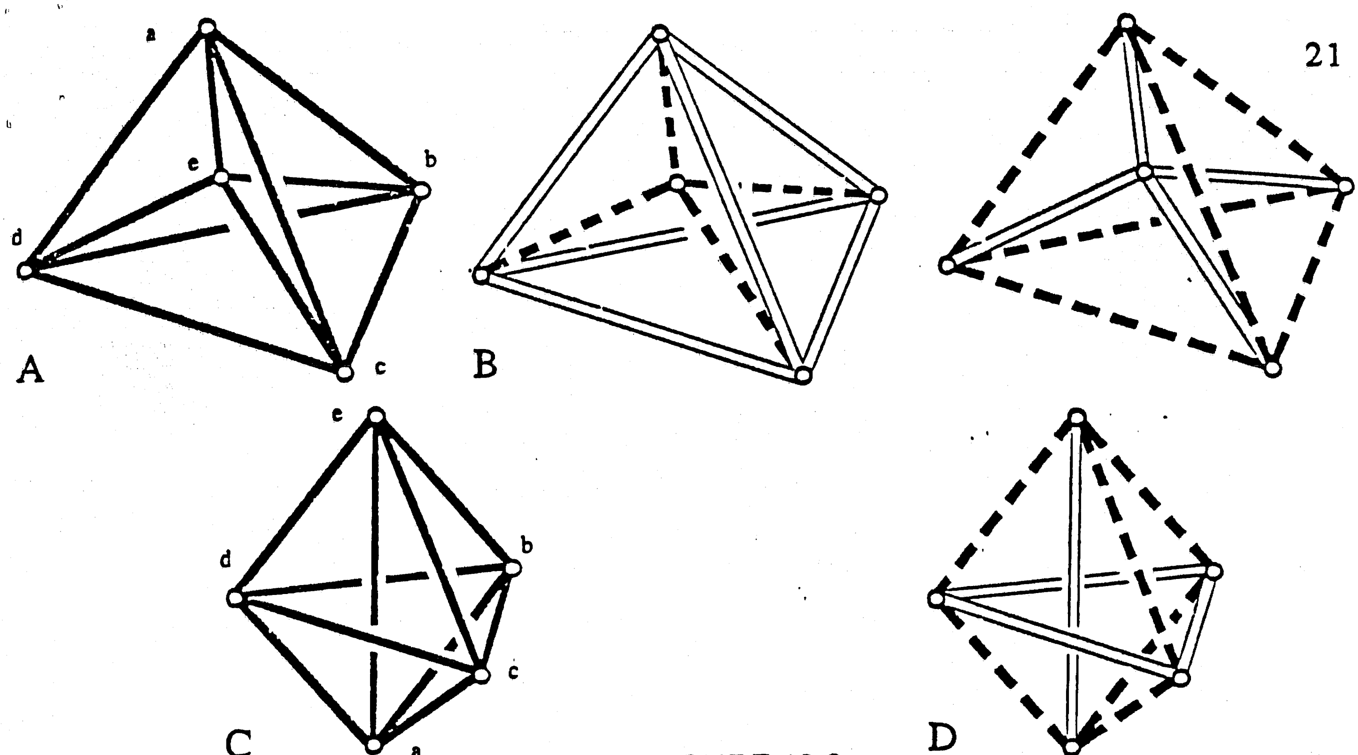

FIGURE 10.9.

EXAMPLE 10.14. Consider a triangular bipyramid, with an added member from the top to the bottom (Figure 10.9C). Without the added member this triangulated convex sphere is isostatic, by chapter 4. With the added member in compression, there is a single self–stress. At the top and bottom vertices the three boundary members must be in tension, since we can never separate the two signs of a self–stress by a plane. At each vertex of the triangular waist, the self–stress must show 4 sign changes to avoid separation (recall the index lemma from chapter 4). The equator is therefore in compression. This proves the infinitesimal rigidity of the tensegrity framework in Figure D.

We can move from Example 10.14 to Example 10.13 by a projectiove transformation, where the new plane at infinity passed horizontally below joint **a**. As we saw in chapter 3, this projective transformation does not preserve infinitesimal rigidity, unless we switch cables and struts which are cut by the image of the plane at infinity, or equivalently, were cut by the plane being sent to infinity.

DEFINITION 10.15. Given a tensegrity framework $G(\mathbf{p})$, a **permissible projective transformation** is a projective transformation T of d-space which takes all points $\mathbf{p}_i$ to finite points $T(\mathbf{p}_i)=\mathbf{q}_i$. For a permissible projective transformation T, the **projection of the tensegrity framework** is $T(G(\mathbf{p}))=TG(\mathbf{q})$, where $TG=(V;TE_-,E_0,TE_+)$ with TE_-,TE_+ created from E_-,E_+ by replacing every cable (i,j) of G (respectively, strut (i,j) of G), for which the line segment $[\mathbf{q}_i,\mathbf{q}_j]$ intersects the image of the hyperplane at infinity under T, by a strut (respectively, cable).

THEOREM 10.16. Roth&Whiteley (1981) A tensegrity framework $G(\mathbf{p})$ is statically rigid if and only if each permissible projection $TG(\mathbf{q})$ is statically rigid.

Proof. By Theorem 3.xx , $\underline{G}(\mathbf{p})$ is infinitesimally rigid if and only if $\underline{T(G}(\mathbf{q}))= \underline{G}(T(\mathbf{p}))$ is infinitesimally rigid. It remains to recall the correspondence of the strict self–stresses which was presented in chapter 3.

We recall that the projective transformation T is the composition of two maps on the affine coordinates for the points, where we replace $\mathbf{a}=(x_1,\ldots,x_n)$ by $\hat{\mathbf{a}}=(x_1,\ldots,x_n,1)$. We first have an invertible linear transformation $A: (x_1,\ldots,x_n,1) \Rightarrow (y_1,\ldots,y_n,y_{n+1})$; followed by a homogeneous multiplication D, taking:

$$D(y_1,\ldots,y_n,y_{n+1})= (1/y_{n+1})(y_1,\ldots,y_n,y_{n+1})= (T(x_1,\ldots,x_n),1).$$

For each joint p_i let α_i be the divisor used in D. We saw in chapter 3 that a self-stress $(\ldots,\omega_{ij},\ldots)$ on $\underline{G}(p)$ goes to a self-stress $(\ldots,\alpha_i\alpha_j\omega_{ij},\ldots)$ on $\underline{G}(T(p))=\underline{TG}(T(p))$. Now ω_{ij} and $\alpha_i\alpha_j\omega_{ij}$ have the same sign unless the new plane at infinity cut the line segment p_jp_k. In this case we have $\alpha_i\alpha_j<0$, and we have switched the member in TG. □

As this result empasizes, the infinitesimally rigid realizations of a signed graph G, with many cables and struts and few bars, will not be dense in R^{dv}. We recall the following simpler result from chapter 3.

PROPOSITION 10.17. For any signed graph G, the set $\{ p \in R^{dv} \mid G(p)$ is an infinitesimally rigid tensegrity framework in $R^d\}$ is an open set in R^{dv}.

Theorem 10.5 shows that the boundaries of this open region arise in two ways:

(i) The induced framework $\underline{G}(p)$ becomes infinitesimally flexible, on an algebraic variety defined by minors of the rigidity matrix of $\underline{G}(p)$ (see chapter 11 for these "pure conditions").

(ii) Some cables or struts drop out of the strict self-stress.

Figure 10.10B,C illustrates such boundary positions for the pattern in Figure A. If we pass "through" this boundary, the self-stress returns to all members, but the sign switch on the boundary members (Figure D,E). Some preliminary study of these sign switches appears in White&Whiteley (1983), but the general pattern has not been characterized (see Figure 10.10 and chapter 11).

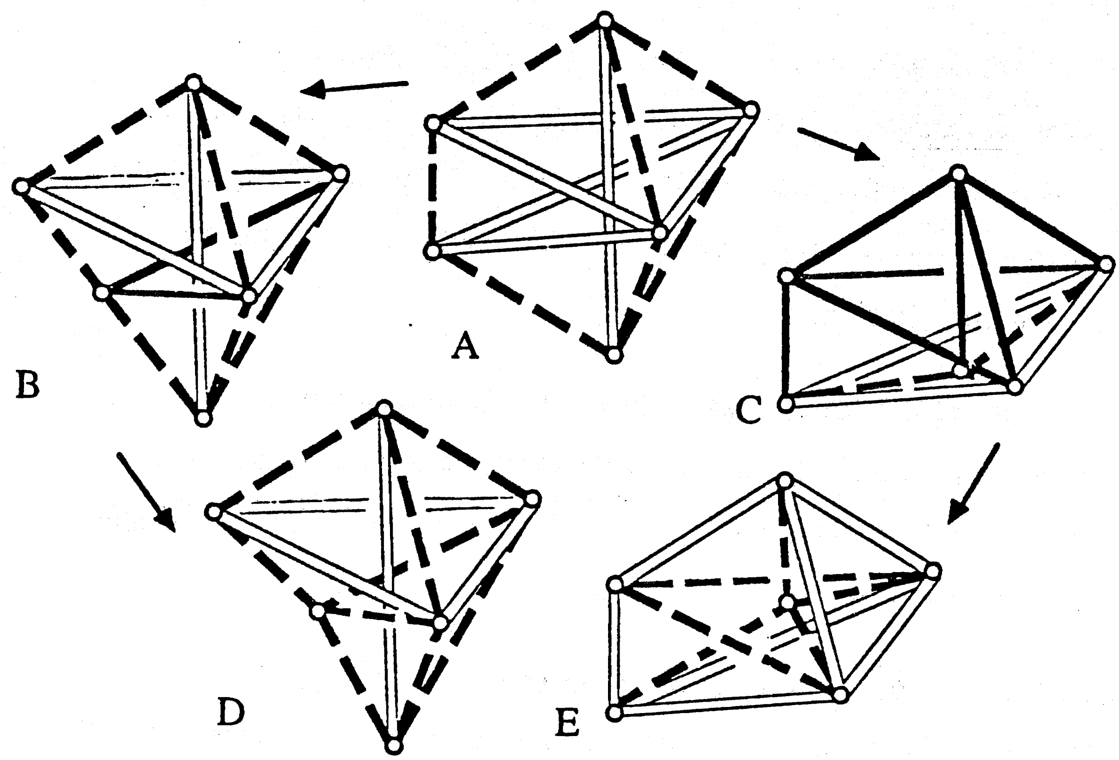

FIGURE 10.10.

We close with the spatial analogue of the convex wheel.

EXAMPLE 10.18. If a convex triangulated sphere is built with cables, and an interior joint is added with struts to all of the original joints, the tensegrity framework is statically rigid. The static rigidity $\underline{G}(\mathbf{p})$ comes from Cauchy's theorem for bar frameworks plus the insertion of a non-coplanar 3-valent joint. The proper self–stress will follow from Maxwell's theorem in chapter 12. (For the tetrahdron, this is Example 10.13.)

If the vertices are in general position (no 4 in a plane) in this example, it seems that this is an example of a minimal tensegrity where each proper stress is $\neq 0$ at each vertex, but $e \neq 3v-5$.

EXERCISES

10.3. Prove that a convex k-gon of struts, $k \geq 4$, with all joints $\mathbf{p}_i$, $i > 2$, joined by a cable to one of $\mathbf{p}_1$ or $\mathbf{p}_2$ is statically rigid (Figure 10.11A) [Roth&Whiteley (1981)].

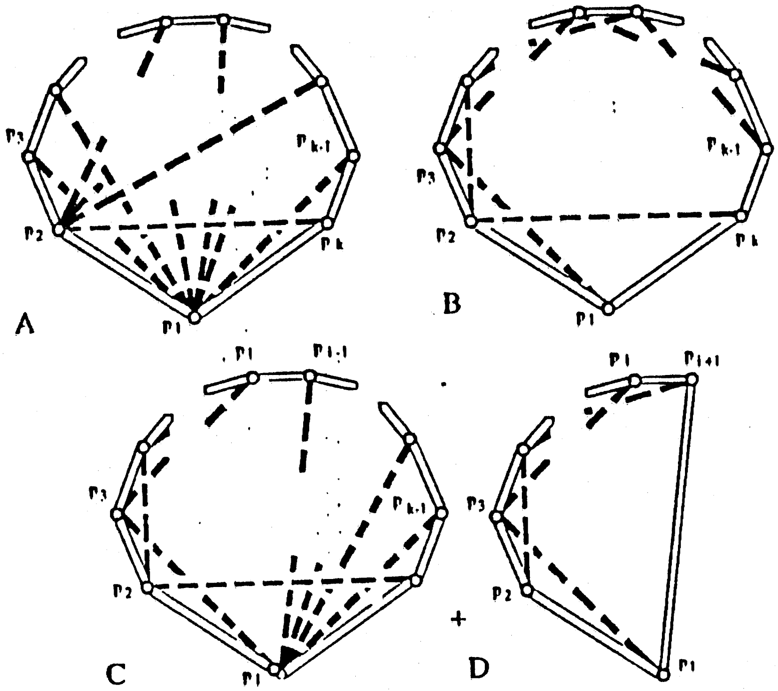

FIGURE 10.11

10.4. Prove the static rigidity of the **Cauchy polygons** (Figure 10.11B) by an inductive sequence of exchanges of the mixed polygons of Figure C and the smaller Cauchy polygons of Figure D [Roth&Whiteley (1981)].

10.5. If a plane infinitesimally rigid tensegrity framework is modified by adding a new joint at the interior crossing point of two members, splitting these members and preserving the signs (Figure 10.12A), show that the new tensegrity framework is also infinitesimally rigid.

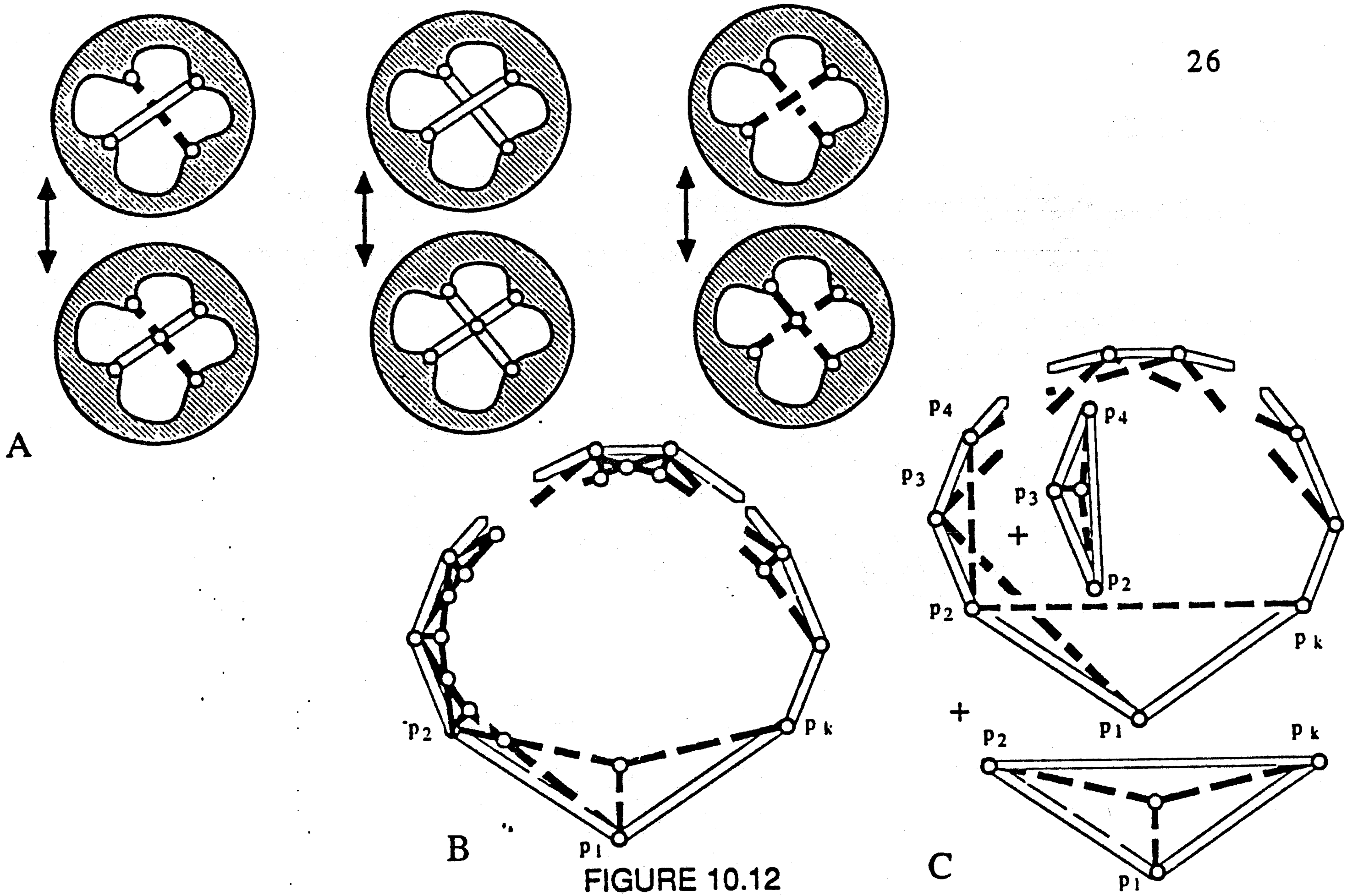

FIGURE 10.12

10.6. Prove that the **altered Cauchy polygons** of Figure 10.12B are infinitesimally rigid, using the exchanges in Figure C, and the previous exercises [Connelly (1982)]. Note that the added vertices can be arbitrarily close to the boundary polygon.

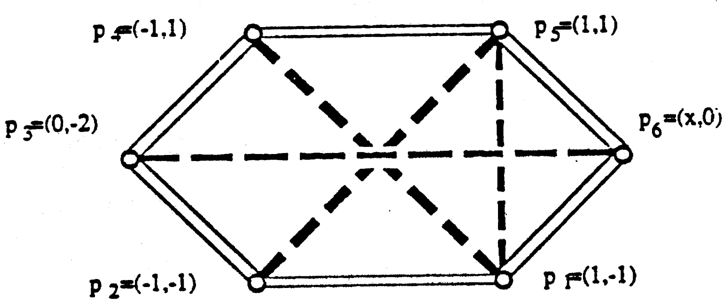

FIGURE 10.13

10.7. For the realizations of $K_{3,3}$ +one bar in plane, shown in Figure 10.13, show that the framework is infinitesimally rigid for $1<x<2$, and not infinitesimally rigid for $x\geq 2$.

10.8. Extend Alexandrov's theorem, Theorem 4.x to show the static rigidity of any convex polyhedron built with bars on the natural edges and all facial diagonals as cables [Connelly (1980), Whiteley (1984)].

10.5. Finite Rigidity and Infinitesimal Rigidity.

For convenience we recall the basic terminology, and the three equivalent definitions of rigidity and flexibility from chapter 2. A tensegrity framework $G(\mathbf{p})$ **dominates** the tensegrity framework $G(\mathbf{q})$, written $G(\mathbf{p})\geq G(\mathbf{q})$, if:

$$|q_i-q_j|\leq|p_i-p_j| \text{ if } (i,j)\in E_-;\ |q_i-q_j|=|p_i-p_j| \text{ if } (i,j)\in E_0;$$
$$|q_i-q_j|\geq|p_i-p_j| \text{ if } (i,j)\in E_+.$$

A tensegrity framework $G(\mathbf{p})$ is **rigid** in R^d if any of these three equivalent conditions holds:

(i) there is an $\varepsilon>0$ such that if $G(\mathbf{p})>G(\mathbf{q})$ and $|\mathbf{p}-\mathbf{q}|<\varepsilon$ then $\mathbf{p}$ is congruent to $\mathbf{q}$;

or (ii) for every continuous path, or **continuous flex**, $\mathbf{p}(t)\in R^{vd}$, $\mathbf{p}(0)=\mathbf{p}$, such that $G(\mathbf{p})>G(\mathbf{p}(t))$ for all $0\leq t\leq 1$, then $\mathbf{p}$ is congruent to $\mathbf{p}(t)$ for all $0\leq t\leq 1$;

or (iii) for every analytic path, or **analytic flex**, $\mathbf{p}(t)\in R^{vd}$, $\mathbf{p}(0)=\mathbf{p}$, such that $G(\mathbf{p})>G(\mathbf{p}(t))$ for all $0\leq t\leq 1$, then $\mathbf{p}$ is congruent to $\mathbf{p}(t)$ for all $0\leq t\leq 1$.

The basic theorem of chapter 2 says:

THEOREM 10.19. If a tensegrity framework is infinitesimally rigid then it is rigid.

We recall that two of the three proofs given in chapter 2 actually applied to tensegrity frameworks.

The equivalence of rigidity and infinitesimal rigidity of bar frameworks at regular points also extends. For tensegrity frameworks we need a slightly stronger assumption. Recall that a point $\mathbf{p} \in R^{vd}$ is **regular** for a graph G if :

$$\text{rank } df_G(\mathbf{p}) = \max \{\text{rank } df_G(\mathbf{q}) \mid \mathbf{q} \in R^{vd}\}$$

A point $\mathbf{p} \in R^{vd}$ is **fully regular** for a signed graph G if **p** is regular for all subgraphs G' of $\underline{G}$.

THEOREM 10.20. (Roth&Whiteley 1981) A tensegrity framework G(**p**) at a fully regular point **p** is rigid if and only if G(**p**) is infinitesimally rigid.

Proof. Theorem 10.19 gives one direction.

Assume that G(**p**) is infinitesimally flexible for some fully regular point **p**. Each nontrivial infinitesimal flex **p'** of G(**p**) has $(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{p}'_i - \mathbf{p}'_j) \neq 0$ for some edges of G. Otherwise **p'** is an nontrivial infinitesimal motion of the bar framework $\underline{G}$(p), at a regular point **p**, which proves that $\underline{G}$(**p**), and G(**p**), are both flexible.

If $(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{p}'_i - \mathbf{p}'_j) \neq 0$ for all edges of G, we "integrate" **p'** as the nontrivial flex of G(**p**):

$$\mathbf{p}(t) = \mathbf{p} + t\mathbf{p}' = (\mathbf{p}_1 + t\mathbf{p}'_1, \ldots, \mathbf{p}_v + t\mathbf{p}'_v).$$

Clearly $\mathbf{p}(0) = \mathbf{p}$, and the derivative of $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2$ at $t=0$ is $2(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{p}'_i - \mathbf{p}'_j)$, which is non-zero of the correct sign. Thus for small t, $G(\mathbf{p}(t))$ is dominated by $G(\mathbf{p})$, and all members are changed in length. This is a nontrivial flex.

Assume $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}'_i-\mathbf{p}'_j)\neq 0$ for some edges of G, but $(\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}'_i-\mathbf{p}'_j)=0$ for the edges in a minimal set A. Since $\mathbf{p}$ is a regular point of G, df_A has a maximum rank and $f^{-1}(f_A(\mathbf{p}))$ is a manifold near $\mathbf{p}$ whose tangent space is $\ker(df_A(\mathbf{p}))$. Therefore $\mathbf{p}'\in\ker(df_A(\mathbf{p}))$ is tangent to a smooth path $\mathbf{p}(t)$ through $\mathbf{p}$ in $f^{-1}(f_A(\mathbf{p}))$. $|\mathbf{p}_i(t)-\mathbf{p}_j(t)|^2=|\mathbf{p}_i-\mathbf{p}_j|^2$ for all t and all $\{i,j\}\in A$, and $|\mathbf{p}_i(t)-\mathbf{p}_j(t)|^2<|\mathbf{p}_i-\mathbf{p}_j|^2$ (respectively $>$) for all $\{i,j\}\in E_--A$ (respectively E_+-A) for small positive t. This is the required flex. ▯

REMARK 10.21. By Corollary 10.4, the set A in this proof is the set of bars and the members in a maximal proper self-stress. Thus there is a flex at a fully regular point $\mathbf{p}$ which strictly changes the length of a cable or strut if this member is not covered by a strict self-stress.

A converse also holds: a member is open in all strict self-stresses at a fully regular point $\mathbf{p}$ if there is a flex which strictly changes the length of the cable or strut (i.e. $|\mathbf{p}_i(t)-\mathbf{p}_j(t)|\neq|\mathbf{p}_i(0)-\mathbf{p}_j(0)|$ for all $t>0$). Proof. Assume such a flex exists. For $t>0$, but close to 0, $(\mathbf{p}_i(t)-\mathbf{p}_j(t))\cdot(\mathbf{p}'_i(t)-\mathbf{p}'_j(t))\neq 0$. By the first-order stress test, this means that for $t>0$, but close to 0, the edge (i,j) has $\omega_{ij}=0$ for all proper self-stresses.

Since the set of fully regular points in open we can know that the point $\mathbf{p}(t)$ are also fully regular. At all fully regular points $\mathbf{q}$, we have the same sets of minimal dependences (circuits) in $\underline{G}(\mathbf{q})$. Moreover the coefficients of the self-stresses change continuously as we move along $\mathbf{p}(t)$. If there is a proper self-stress with $\omega_{ij}\neq 0$ at $\mathbf{p}(0)$, then such a self stress exists for $\mathbf{p}(t)$, $t>0$. This contradiction completes the proof. □

EXERCISES.

10.9 Show that joining two rigid tensegrity frameworks in k-space by identifying k+1 affinely independent joints creates a rigid framework.

10.10(i) Show that a tensegrity framework (with no zero length edges) is rigid on the line if and only if it is infinitesimally rigid.

(ii) Give an example of tensegrity framework which is rigid on the line, but flexible in the plane.

(iii) Give an example of a tensegrity framework on the line which is rigid in the plane, but flexible in 3-space.

10.11. Show that the tensegrity exchange of two plane-rigid tensegrity frameworks sharing two vertices a,b, (deleting an edge {a,b} which is a cable in one and a strut in the other), creates a rigid plane framework (an analogue of Proposition 10.11).

10.12. Give an example of a tensegrity framework $G(\mathbf{p})$ at a non-regular point which $\omega_{ij}=0$ for all proper self-stresses, but $G(\mathbf{p})$ has no flex changing the distance $||\mathbf{p}_i-\mathbf{p}_j||$.

10.6. Affine Flexes.

Beyond these fully regular points, there are other special patterns for which an nontrivial infinitesimal flex guarantees a flex. This is true for frameworks on the line with no zero length bars (Exercise 10.1). This was also true for the plane grids discussed in chapter 5. We will see that it holds for pinned frameworks with only struts (chapter 15) and for special patterns on a convex polygon (chapter 16). We present another basic example.

THEOREM 10.22. (i) If a tensegrity framework has a nontrivial infinitesimal flex $\mathbf{p}'$ which is an affine motion $\mathbf{p}'_i = T(\mathbf{p}_i)$, then there is a nontrivial flex $\mathbf{p}(t)_i = T(t)\mathbf{p}_i$ composed of affine images of $\mathbf{p}$.

(ii) If a tensegrity framework $G(\mathbf{p})$ dominates an non–congruent affine image $G(\mathbf{q})$, $\mathbf{q}_i = T(\mathbf{p}_i)$, then there is a nontrivial affine flex $\mathbf{p}(t)_i = T(t)(\mathbf{p})$.

Proof. (i) Assume that $A\mathbf{p}_i$ is a nontrivial infinitesimal flex of $G(\mathbf{p})$. Since trivial infinitesimal flexes are represented by skew symmetric matrices (recall chapter 2), after a trival infinitesimal flex, we can assume A is symmetric. Moreover $(\mathbf{p}_j - \mathbf{p}_i)^T(\mathbf{p}'_j - \mathbf{p}'_i) = (\mathbf{p}_j - \mathbf{p}_i)^T A(\mathbf{p}_j - \mathbf{p}_i)$ has the correct sign for each member.

Any symmetric matrix can be made diagonal by an othogonal transformation: $D = O_1{}^T A O_1$. For small $t_0 > t \geq 0$, $tD+I$ is a diagonal matrix with positive entries. We can take square roots down the diagonal to create a diagonal matrix $C(t)$, $C(t)_{kk} = \sqrt{tD_{kk}+1}$, and we define $B(t) = O_1 C(t) O_1{}^T$. We claim that $\mathbf{p}_i(t) = B(t)\mathbf{p}_i$ is the desired path. For each member:

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$$(\mathbf{p}_j(t)-\mathbf{p}_i(t))^T(\mathbf{p}_j(t)-\mathbf{p}_i(t)) = (\mathbf{p}_j-\mathbf{p}_i)^T B(t)^T B(t)(\mathbf{p}_j-\mathbf{p}_i)$$
$$=(\mathbf{p}_j-\mathbf{p}_i)^T[tA+I](\mathbf{p}_j-\mathbf{p}_i) = t(\mathbf{p}_j-\mathbf{p}_i)^T A(\mathbf{p}_j-\mathbf{p}_i) +(\mathbf{p}_j-\mathbf{p}_i)^T(\mathbf{p}_j-\mathbf{p}_i).$$

Since $t(\mathbf{p}_j-\mathbf{p}_i)^T A(\mathbf{p}_j-\mathbf{p}_i)$ has the correct sign for each member, $t_0>t\geq 0$, $G(\mathbf{p}) > G(\mathbf{p}(t))$ for all t. A is nontrivial if and only if $A\neq 0$, which is equivalent to $\mathbf{p}(t)$ is not congruent to $\mathbf{p}$ for $t>0$.

(ii) We need only show that a non-congruent $G(\mathbf{q})<G(\mathbf{p})$, with $\mathbf{q}_i=B\mathbf{p}_i$ implies that $\mathbf{p}'_i = A\mathbf{p}_i=[B^TB-I]\mathbf{p}_i$ is a nontrivial affine infinitesimal flex. Retracing the calculation shows that:

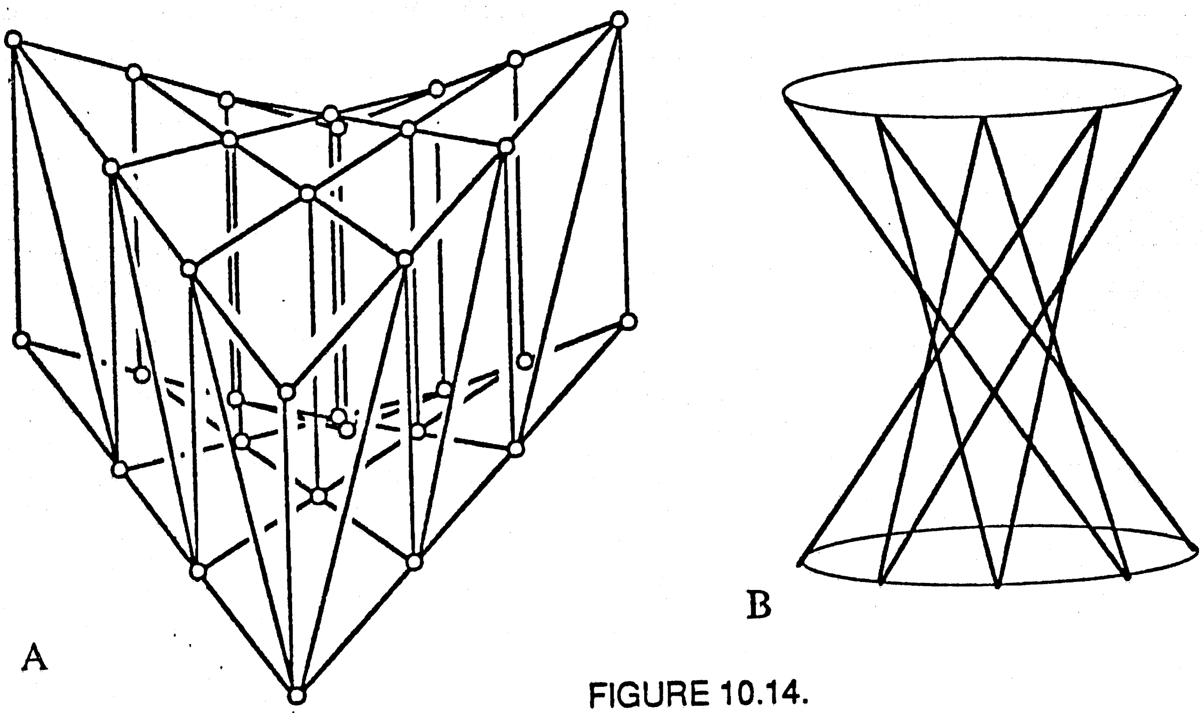
$$(\mathbf{p}_j-\mathbf{p}_i)^T A(\mathbf{p}_j-\mathbf{p}_i)=(\mathbf{p}_j-\mathbf{p}_i)^T[B^TB-I](\mathbf{p}_j-\mathbf{p}_i)$$
$$=(\mathbf{p}_j-\mathbf{p}_i)^T B^T B(\mathbf{p}_j-\mathbf{p}_i)-(\mathbf{p}_j-\mathbf{p}_i)^T(\mathbf{p}_j-\mathbf{p}_i)=|\mathbf{q}_j-\mathbf{q}_i|^2-|\mathbf{p}_j-\mathbf{p}_i|^2$$

Since $G(\mathbf{q})<G(\mathbf{p})$, $\mathbf{p}'_i = A\mathbf{p}_i$ is an infinitesimal motion. This will be nontrivial if, and only if $A=[B^TB-I] \neq 0$, which is true unless B is a congruence □

In taking square roots in this proof, we made ± choices for each diagonal entry. These choices amount to choosing a reflection of the dominated realization $G(\mathbf{q})$.

EXAMPLE 10.23 What do these affine mechanisms look like in 3–space? The affine transformation B will define a projective conic on the plane at infinity $\mathbf{u}^T B^T B\mathbf{u}=0$. The bars directions must lie on this conic, the cable directions must lie in one component cut by this conic (or on the conic) and all struts must lie in the other component.

If the conic at infinity is two lines, then all bars are parallel to two planes in space, and we have a simple mechanism, which is illustrated in Figure 10.14 A.



A B

FIGURE 10.14.

If the conic is a projective ellipse, then all bars at a joint must lie in a cone of directions to this ellipse. The resulting mechanism can be very complex in structure, but it will contain no triangles. Figure 10.14 B shows a simple example: two rulings of an hyperboloid . The affine mechanism preserves these straight lines, and moves to a new hyperboloid with the same axis. This is a classic example in 3-space [Larmor (1886)].

We know that infinitesimal motions are preserved by projective transformations. In chapter 6, we saw the projective image of these affine infinitesimal flexes, as special cases of extended bipartite frameworks.

EXERCISES.

10.13. Show that an affine transformation of the plane which is not a congruence preserves distances parallel to at most two lines.

10.14. Show that affine transformations of frameworks do not always preserve rigidity, by checking the following three facts:

(i) the bar framework of Figure 10.15A is flexible;

(ii) the bar framework of Figure 10.15B is affine equivalent to the framework of Figure A;

(iii) the bar framework of Figure 10.15B is rigid.

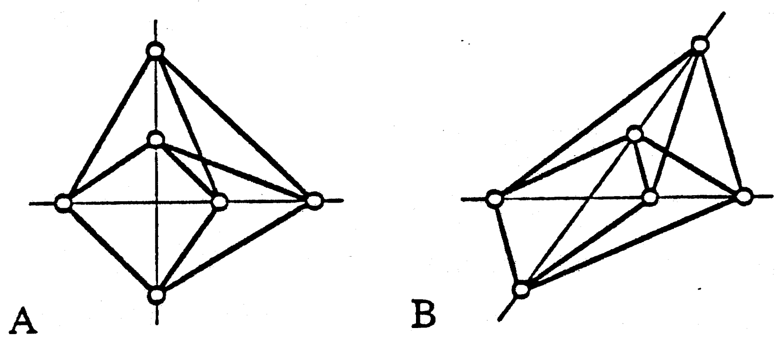


FIGURE 10.15.

10.15. Show that if $G(\mathbf{p})$ has an affine infinitesimal flex $\mathbf{p}'_i = A\mathbf{p}_i$, and T is an non–singular affine transformation of the space, then $\mathbf{q}'_i = T^{-1}A\mathbf{q}_i$ is an affine infinitesimal flex of $G(\mathbf{q})$ with $\mathbf{q}_i = T\mathbf{p}_i$. Show that $G(\mathbf{q})$ has a nontrivial affine flex only if $G(\mathbf{p})$ has a nontrivial affine flex.

10.16. Show that no plane tensegrity framework with a proper self–stress and no collinear polygons can have a nontrivial affine flex.

10.6. Proper Self–stresses and Energy

Any rigid tensegrity framework at a regular point has a strict self–stress. This fails for some rigid tensegrity frameworks at singular points. Figure 10.16A shows a rigid plane framework which does not have a strict self-stress. We will prove that this failure is very local (Figure B) and all rigid tensegrity frameworks with at least one cable or strut have a proper self-stress (Figure C), which may be zero on some members.

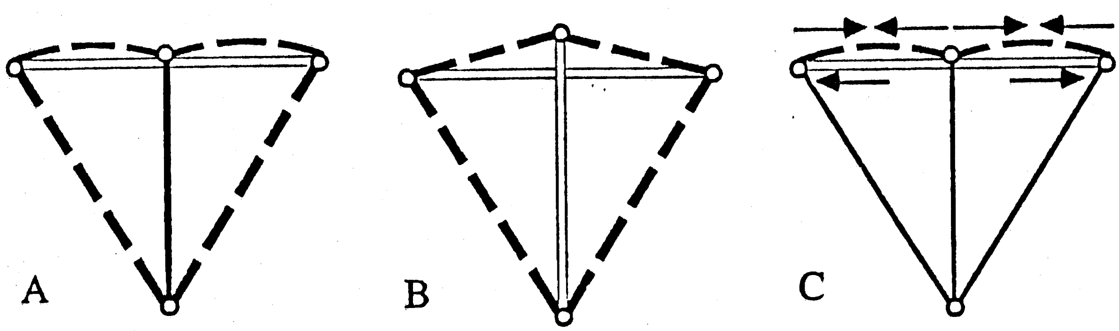


FIGURE 10.16

To prove this result, and to prepare for the chapter 16, we introduce energy functions on the tensegrity framework. For each member we write L_{ij} for the length $|\mathbf{p}_j - \mathbf{p}_i|$. Any rigid bar framework $G(\mathbf{p})$ gives a local minimum at $\mathbf{p}$ for the simple energy function:

$$H(\mathbf{q}) = \Sigma\ (|\mathbf{q}_j - \mathbf{q}_i|^2 - |\mathbf{p}_j - \mathbf{p}_i|^2\ )^2 = (|\mathbf{q}_j - \mathbf{q}_i|^2 - (L_{ij}))^2 \quad \text{(sum over } \{i,j\} \in E\text{)}$$

since rigidity is uniqueness within some open neighborhood, and any change of even one length increases the energy. Since this energy is constant for conguences classes, we can say that $H(\mathbf{q})$ has a strict minimum, modulo congruences.

Similarly, a rigid tensegrity framework $G(\mathbf{p})$ gives a local minimum of the energy function:

$$H(\mathbf{q}) = \tfrac{1}{2}\Sigma f_{ij}(|\mathbf{q}_j - \mathbf{q}_i|^2) \quad \text{(sum over } \{i,j\} \in E)$$

where $f_{ij}(|q_j-q_i|^2) = (|q_j-q_i|^2-(L_{ij}))^2$ for bars, and the function f_{ij} is modified to $f_{ij}(|q_j-q_i|^2) = 0$ for longer struts, with $|q_j-q_i|^2 \geq |p_j-p_i|^2$, and $f_{ij}(|q_j-q_i|^2) = 0$ for shorter cables, with $|q_j-q_i|^2 \leq |p_j-p_i|^2$ (Figure 10.17A). Again this is a strict minimum, modulo congruences.

In itself, this provides no new information. The trick is to modifiy the energy functions to get additional information about the framework.

For example, if we take a rigid framework, lengthen each strut and shorten the cables by a small amount, then release the framework to seek an equilibrium, we expect to find a nearby G(q) with a strict self-stress. The modified functions f_{ij} in the following proof (Figure 17B) represent this intuitive process.

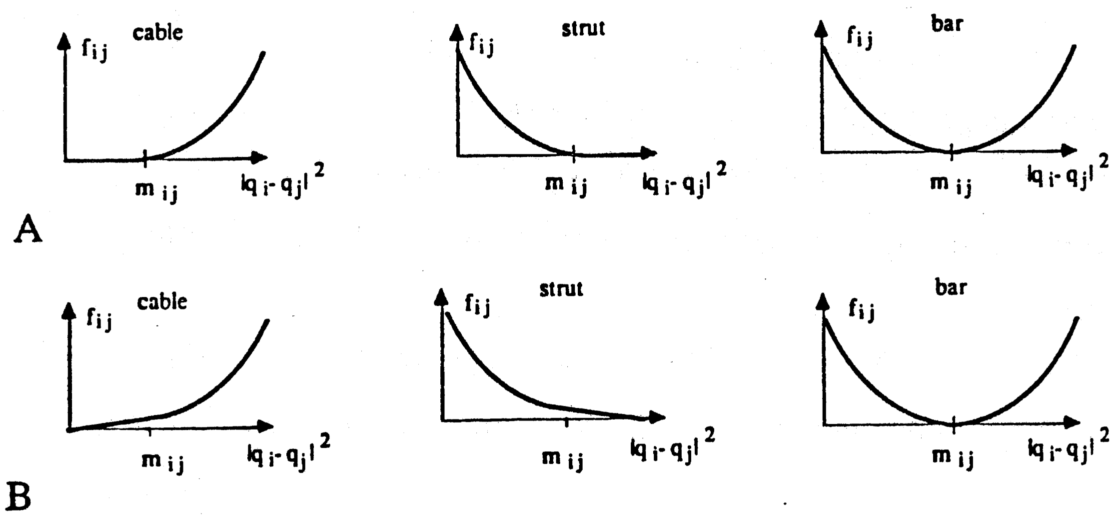


FIGURE 10.17.

LEMMA 10.24. Connelly (1982) Theorem 2. Let G(p) be a rigid tensegrity framework in d-space, then for each $\varepsilon>0$, there is a $q(\varepsilon) \in R^{vd}$ such that $|p-q(\varepsilon)|<\varepsilon$ and $G(q(\varepsilon))$ has a strict self-stress.

Proof. We modify the energy functions for cables and struts as shown roughly in Figure 10.16B, to form a sequence of functions $f(n)_{ij}$ converging uniformly to f_{ij}. Specifically, for $f(n)_{ij}$ on a cable we add $1/n$ to the quadratic and then splice on a straight line of slope $2/n$ in a C^1 manner on the left end (changing formulas at the point $(L_{ij}+1/n,\ (1/n)+(1/n)^2)$ where the two equations have the same slope). We do a similar operation for struts. If $|\mathbf{q}-\mathbf{p}|<\varepsilon$ for $\varepsilon<1$ then $|f(n)_{ij}(|\mathbf{q}_j-\mathbf{q}_i|^2)-f_{ij}(|\mathbf{q}_j-\mathbf{q}_i|^2)|<1/n$.

For simplicity, we pin down one vertex of the framework $\mathbf{p}_0$, to remove translations and assume that all $\mathbf{q}$ have the same fixed joint. We now tie down all congruences, by fixing directions, planes etc. for joints of $\mathbf{p}$, and keep the same directions, planes etc. fixed for $\mathbf{q}$, restricting $\mathbf{q}$ to the affine span of $\mathbf{p}$. We write this set of $\mathbf{q}$ as $F_\mathbf{p}$. The set $S_\varepsilon(\mathbf{p})=\{\mathbf{q}\in F_\mathbf{p} \mid |\mathbf{q}-\mathbf{p}|=\varepsilon\}$ is a compact set for each ε, since the restrictions of $F_\mathbf{p}$ are preserved by limits, and the set is bounded. Since $H(\mathbf{q})$ has a strict local minimum at $\mathbf{p}$ in this compact set $B_\varepsilon(\mathbf{p})=\{\mathbf{q}\in F_\mathbf{p} \mid |\mathbf{q}-\mathbf{p}|\leq\varepsilon\}$, for small ε: $H(S_\varepsilon)>0$ and there is a minimum value $\delta_\varepsilon>0$ for H on this sphere.

Consider the energy functions

$$H(n)(\mathbf{q}) = \tfrac{1}{2}\Sigma f(n)_{ij}(|\mathbf{q}_j-\mathbf{q}_i|^2) \text{ (sum over } \{i,j\}\in E).$$

For sufficiently large n, and $\varepsilon<1$, $|H(n)(\mathbf{q})-H(n)(\mathbf{q})| < (1/2)v/n<\delta_\varepsilon$ for all $\mathbf{q}$ in B_ε and $(1/2)\delta_\varepsilon>|H(n)(\mathbf{p})-H(\mathbf{p})|>H(n)(\mathbf{p})$. Since the $H(n)$ are continuous functions, there is a local minimum for $H(n)$ in the open set $N_\varepsilon(\mathbf{p})=\{\mathbf{q}\in F_\mathbf{p} \mid |\mathbf{q}-\mathbf{p}|<\varepsilon\}$. Since the $H(n)$ are differentiable, this local minimum guarantees a critical point $\mathbf{q}(n)$ with $|\mathbf{p}-\mathbf{q}(n)|<\varepsilon$). Taking derivatives d coordinates at a time, this gives

$$\mathbf{0} = \nabla H(n)(\mathbf{q}(n)) = (\ldots,\Sigma_j(f(n)_{ij})'(\mathbf{q}(n)_i-\mathbf{q}(n)_j),\ldots).$$

By our choice of functions $f(n)_{ij}$, $(f(n)_{ij})'>0$ for all cables and $(f(n)_{ij})'<0$ for all struts. This means that $(\ldots,(f(n)_{ij})',\ldots)$ is a strict self-stress for $G(\mathbf{q}(n))$, as required. □

Connelly (1982) gives a different proof of this theorem, using more topological arguments, rather than the explicit limits.

THEOREM 10.25 Connelly (1982) Theorem 3. Let $G(\mathbf{p})$ be a rigid framework with a cable or a strut. Then $G(\mathbf{p})$ has a proper non zero self-stress.

Proof: Lemma 10.24 guarantees for each rigid framework, and each $\varepsilon>0$, there is $\mathbf{q}(\varepsilon)$ such that $|\mathbf{q}-\mathbf{q}(\varepsilon)|<\varepsilon$ and $G(\mathbf{q}(\varepsilon))$ has a strict self-stress. Take a convergent sequence of such frameworks $G(\mathbf{q}(m))$, with strict self-stresses $\omega(m)$, approaching $G(\mathbf{p})$. We divide each $\omega(m)$ by $|\omega(m)|$, so the sequence stays on the unit sphere. Therefore a subsequence of these must converge to a nontrivial ω on $G(\mathbf{p})$. The limiting process preserves the equilibrium equations at the vertices and the non-strict inequalities of a proper self-stress. Therefore ω is the required proper self-stress on $G(\mathbf{p})$. □

Note that the self-stress on $G(\mathbf{p})$ may be zero on all cables and struts. We only know that it is nontrivial and proper.

By the results of chapter 2, any rigid bar framework which is not infinitesimally rigid also contains a proper self-stress. Can this self-stress be used to place some cables or struts into the framework, as we did for infinitesimally rigid bar frameworks? The following example shows that this sometimes fails.

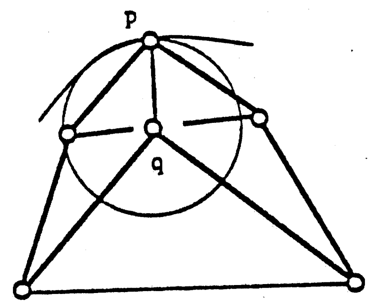


FIGURE 10.18

EXAMPLE 10.26. The framework of Figure 10.18 is rigid, with a self–stress, but the member **pq** cannot be replaced by a cable or a strut. With **pq** removed and the lower triangle pinned, the joint **q** traces out a 6th degree curve. We choose a position where the circle of curvature actually crosses this curve, and placed **p** at the center of this circle of curvature. Since the curve of **q** and the circle from **p** have the same tangent lines, there is an infinitesimal motion, and a nontrivial self–stress. If **pq** is a cable, the framework has a flex, moving to the inside of the circle of curvature. If **pq** is a strut, there is a flex moving to the outside of the circle of curvature. If **pq** is a bar, there is no flex and the framework is rigid.

This example has a second-order flex (recall chapter 4). The importance of this distinction will become clearer in chapter 16.

EXERCISES:

10.17. (i) Use energy functions to prove that for any rigid tensegrity framework G(**p**) and any external load F_{ij} there is a nearby position G(**q**(ε)) which resolves F_{ij} ← not the same load

$F_{ij}(p)$ or $F_{ij}(q(\varepsilon))$

(ii) Use energy functions to prove that for any rigid tensegrity framework $G(\mathbf{p})$ and any external load $\Sigma\alpha_{ij}\mathbf{F}_{ij}$ there is a nearby position $G(\mathbf{q}(\varepsilon))$ which resolves $\Sigma\alpha_{ij}\mathbf{F}_{ij}$.

same confusion here

10.18. Consider a **local projective transformation** T of a tensegrity framework $G(\mathbf{p})$, which does no cut any member of $G(\mathbf{p})$ by the new plane at infinity.

(i) Show that T takes a proper self–stress of $G(\mathbf{p})$ to a proper self–stress of $G(T\mathbf{p})$.

(ii) Show that if $G(\mathbf{p})$ is rigid then there is neighborhood U of $\mathbf{p}$ such that $T\mathbf{p} \in U$ implies $G(T\mathbf{p})$ is also rigid.

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