Draft June 1987

16 Global and Second–Order Rigidity of Tensegrity Frameworks

Walter Whiteley*

16.1. Introduction.

Previous chapters presented a number of techniques for verifying the infinitesimal rigidity of both bar frameworks and more general tensegrity frameworks. These results included the correspondence between infinitesimal rigidity and rigidity for various classes of realizations, such as regular points, or affine motions. However many interesting structures are rigid, but not infinitesimally rigid. For example, chapter 4 introduced second–order rigidity for triangulations of convex surfaces built as bar frameworks. We now return to this general problem: how can we analyse the general rigidity of tensegrity frameworks, and how can we generate new rigid structures?

The experimental rigid structures described in the introduction to chapter 10 pose these problems in an explicit form. Grunbaum&Shephard (1976) gave some explicit conjectures drawn from this evidence, and Connelly (1982) gave some initial results using the global minima of a simple energy function. Recently Connelly&Whiteley (1987) combined this work and some recent engineering speculations [Pelligrino&Calledine (1986)] to give an

* Work supported, in part, by grants from NSERC Canada and FCAR Québec.

extended theory for the second–order rigidity of tensegrity frameworks.

In this chapter we will summarize and connect these results. The basic theme of this chapter is the interaction between deformations of the framework (general displacements, infinitesimal flexes or second–order flexes) and the self-stresses of the framework. We emphasize that this connection is purely geometric. Both the self-stresses and the deformations are properties of the signed graph and the positions of the joints.

This interaction between self-stresses and deformations can be recorded by energy functions. The self-stress is used to define an energy function. If this energy function has a sufficiently strong local, or global minimum under the allowed deformations, we show that the framework has a corresponding rigidity. In the extreme case of second–order rigidity, a number of energy functions may be used simultaneously to cover all the infinitesimal motions.

The theory remains essentially incomplete. We face the embarasing fact that all of the available methods do not demonstrate the rigidity of many of the original experimental structures in space. The analysis of these structures remain a major task for the students of rigidity.

Usually Not correct. Almost all these structures can be handled.

16.2. Global Rigidity for Spider Webs.

In chapter 10 we introduced energy functions to show that a rigid tensegrity framework, with at least one cable or strut, has a proper self–stress. The basic method extracted at least local minima of the energy from the rigidity of the framework.

For some self-stresses on tensegrity frameworks there is a simple energy function which has a global minimum at the realization. If the other realizations with this same minimum value are sufficently restricted, this guarantees that the framework is **globally rigid** or **uniquely embedded**, i.e every $G(\mathbf{q})$ which is dominated by $G(\mathbf{p})$ is congruent to $G(\mathbf{p})$. We illustrate this process with an example for pinned frameworks.

Recall from chapter 2 that a **pinned tensegrity framework** $(G(\mathbf{p}),V_o)$ is a tensegrity framework $G(\mathbf{p})$ and a subset $V_o \subset V$ of **fixed vertices.** A **flex** of the pinned tensegrity framework $(G(\mathbf{p}),V_o)$ in d–space is a continuous path $p(t)$ with $\mathbf{p}_j(t)=\mathbf{p}_j$ for all $j \in V_o$. A **self–stress** on a pinned tensegrity framework $(G(\mathbf{p}),V_o)$ is an assignment of scalars to the edges $(\ldots,\omega_{ij},\ldots)$ such that $\omega_{ij} \geq 0$ if $(i,j) \in E_-$ and $\omega_{ij} \leq 0$ if $(i,j) \in E_+$, and an equilibrium holds for each vertex v_i not in V_o.

The tensegrity frameworks of the following theorem are called **spider webs**, since they are threads (tensed cables) anchored to pinned joints, and the web of a spider has the form described (and the corresponding stability). For simplicitly we assume that the framework has no zero length bars

THEOREM 16.1. Connelly (1982) Given a pinned tensegrity framework $(G(\mathbf{p}),V_o)$ with a connected graph, only cables and a strict self–stress ω then $(G(\mathbf{p}),V_o)$ is globally rigid.

Proof: We define the simple quadratic function on $G(\mathbf{p}+\mathbf{p}')$, for **pinned displacements** $\mathbf{p}'$, with $\mathbf{p}'_i=\mathbf{0}$ for all pinned vertices in V_o:

$$H(\mathbf{p}') = \Sigma\omega_{ij}(\mathbf{p}_j+\mathbf{p}'_j-(\mathbf{p}_i+\mathbf{p}'_i))^2 \quad \text{(sum over } \{i,j\}\in E\text{)}$$

As a quadratic function in the $\mathbf{p}'_i$, this has the gradient

$$\nabla H(\mathbf{p}') = 2(\ldots,\Sigma_j\omega_{ij}(\mathbf{p}_j-\mathbf{p}_i)\cdot(\mathbf{p}'_j-\mathbf{p}'_i),\ldots).$$

Since ω is a self–stress on $G(\mathbf{p})$, we have $\nabla E(\mathbf{0})= \mathbf{0}$.

For any nontrivial pinned displacement $\mathbf{p}'\neq\mathbf{0}$, $H(t\mathbf{p}')$ gives a simple quadratic function in t, which approaches ∞ as t approaches $\pm\infty$. Some free joint will go to infinity as t approaches $\pm\infty$. Since all members are cables, an edge can only have tension if it is connected through the graph to pinned vertices. Since there is a strict self-stress on a connected graph, each free vertex is connected to some fixed vertex in the graph. Therefore some cable in the path to the fixed vertex in V_0 will become unbounded. This gives an unbounded energy $E(t\mathbf{p}')\geq\omega_{ij}(\mathbf{p}_j+\mathbf{p}'_j-(\mathbf{p}_i+\mathbf{p}'_i))^2$. This quadratic function has its unique minimum at its critical point. Since the derivative at $t=0$ is $\nabla H(\mathbf{0}).(\mathbf{p}')=\mathbf{0}.\mathbf{p}'=0$, the minimum occurs at $t=0$. This shows that $\mathbf{0}$ gives the unique minimum for $H(\mathbf{p}')$.

Assume $G(\mathbf{q}) < G(\mathbf{p})$. This means that the cables can only be shorter. Setting $\mathbf{p}'=\mathbf{q}-\mathbf{p}$, this gives a smaller energy with $E(\mathbf{q}-\mathbf{p})\leq E(\mathbf{0})$. By the previous argument, we have a unique minimum at $\mathbf{q}-\mathbf{p}=\mathbf{0}$, or $\mathbf{q}=\mathbf{p}$. We conclude that $G(\mathbf{p})$ is globally rigid. []

REMARK 16.2. The reverse of these spider webs, formed with all struts in compression, will have a local maximum of the energy $E(\mathbf{p}')$ at $\mathbf{0}$. In chapter 15, we saw that such frameworks are rigid if and only if they are infinitesimally rigid.

EXERCISES.

16.1. Show that if a tensegrity framework $G(\mathbf{p})$ is globally rigid, and a new joint $\mathbf{p}_0$ is added in the interior of a cable, then the modified framework is also globally rigid.

16.2. Show that a pinned framework with at least one free vertex, and only struts attached to the pinned joints, cannot be globally rigid.

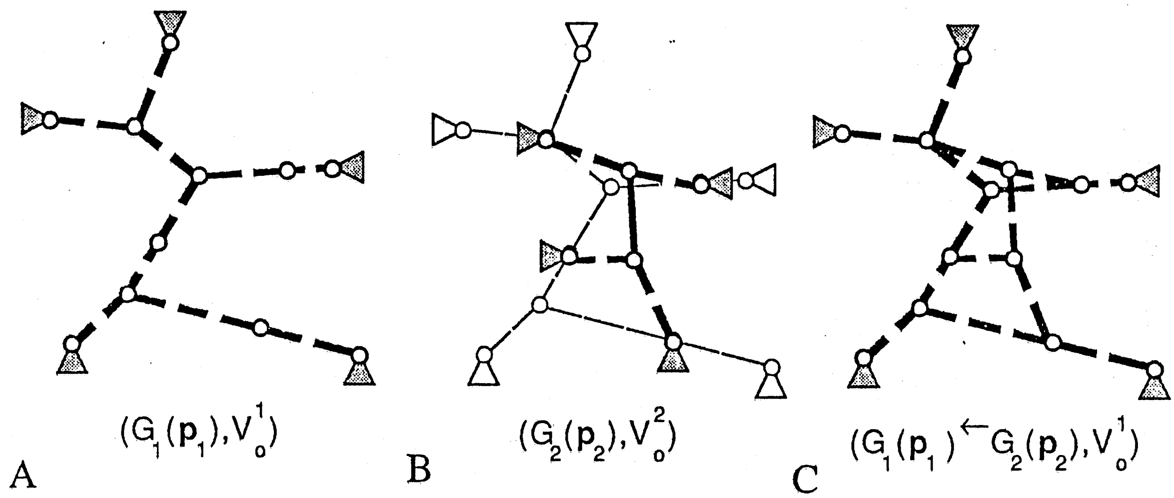


FIGURE 16.1

16.3. Assume $(G(\mathbf{p}_1),V^1{}_o)$ and $(G(\mathbf{p}_2),V^2{}_o)$ are globally rigid tensegrity frameworks with disjoint members, such that $\mathbf{p}_2(V^2{}_o)$ is contained in $\mathbf{p}_1(V^1)$ (Figure 16.1A,B). Let $(G(\mathbf{p}_1)\leftarrow G(\mathbf{p}_2),V^1{}_o)$ denote the pinned framework formed by combining the joints and members of the two frameworks, and keeping the pinned joints of $V^1{}_o$ (Figure 16.1C). Show that $(G(\mathbf{p}_1)\leftarrow G(\mathbf{p}_2),V^1{}_o)$ is globally rigid.

16.4. If $G(\mathbf{p})$ is a rigid pinned framework with only cables, show that:

(i) there is a sequence of subframeworks $G_i(\mathbf{p}_i)$ and pinnings $V^i{}_o$ such that each $(G_i(\mathbf{p}_i),V^i{}_o)$ has a strict self–stress, each $\mathbf{p}_{i+1}(V^{i+1}{}_o)$ is contained in $\mathbf{p}_i(V^i)$, and $(G(\mathbf{p}),V_o)=(G(\mathbf{p}_1)\leftarrow G(\mathbf{p}_2)\ldots\leftarrow G(\mathbf{p}_k),V^1{}_o)$;

(ii) $G(\mathbf{p})$ is globally rigid.

Show the converse

16.5. If a planar graph G is 3-connected in a vertex sense, a **face polygon** is a polygon such that the removal of the vertices and edges of this polygon leaves the remaining edges connected. Assume that a single face polygon is pinned, in order, as a convex polygon in the plane and all other vertices of G are free in the plane, and define the energy function:

$H(\mathbf{p})= \Sigma_{ij}\omega_{ij}(\mathbf{p}_j-\mathbf{p}_i)^2$ (sum over all other edges of the graph) for arbitrary positive constants ω_{ij}.

(i) Show that $H(\mathbf{p})$ has a unique global minimum at which the ω_{ij} are the scalars of a spider web stresss;

(ii) Show that at this minimum the framework is planar, with no crossing members, no zero–length members and no overlapping members (Tutte (1964)).

(iii) Show that if the pinned face is a triangle, the equilibrium position is the orthogonal projection of a convex polyhedron in 3–space, using the results of chapter 12.

(iv) Show that any 3-connected planar graph either has a triangular face, or the plane dual has a triangle (i.e. the graph has a 3-valent vertex), using Euler's formula;

(v) Use parts (iii) and (iv) to give a proof of Steinitz's Theorem (Grünbaum (1967)):

Every 3-connected planar graph can be realized as the edges (1–skeleton) of a convex polyhedron in 3-space.

16.3. Stress Matrices and Global Rigidity

For an unpinned tensegrity framework we have a similar **stress energy form**

$$H(\mathbf{p}) = \Sigma\omega_{ij}(\mathbf{p}_j - \mathbf{p}_i)^2 \quad \text{(sum over } \{i,j\} \in E\text{)}$$

for a tensegrity framework $G(\mathbf{p})$ with a strict self-stress ω. Clearly if $G(\mathbf{p}) \geq G(\mathbf{q})$ then $H(\mathbf{p}) \geq H(\mathbf{q})$.

If this energy function is has a global minimum at $G(\mathbf{p})$, the rigidity depends on the set of $G(\mathbf{q})$ with the same minimum value. We search for this minimum at critical points of the form. Recall that a **critical point** of the stress energy form is a point $\mathbf{q}$ such that

$$\nabla H(\mathbf{q}) = 2(\ldots, \Sigma_j\omega_{ij}(\mathbf{q}_j - \mathbf{q}_i), \ldots) = \mathbf{0}.$$

Note that this means that ω is a self–stress on $G(\mathbf{q})$ for every critical point $\mathbf{q}$, and every point $\mathbf{q}$ with this self–stress is a critical point.

PROPOSITION 16.3. The stress energy form $H(\mathbf{q})$ and its gradient $\nabla H(\mathbf{q})$ have the following properties:

(i) $\nabla H(\mathbf{q})$ is a linear function of $\mathbf{q}$;

(ii) $\nabla H(\mathbf{p}) = \mathbf{0}$;

(iii) $H(\mathbf{q}) = 0$ for every critical point;

(iv) The critical points of $H(\mathbf{q})$ are invariant under affine linear transformations.

Proof: (i) This is clear from the fact that $\nabla H(\mathbf{q})=2(\ldots,\Sigma_j\omega_{ij}(\mathbf{q}_j-\mathbf{q}_i),\ldots)$.

(ii) $\nabla H(\mathbf{p})=2(\ldots,\Sigma_j\omega_{ij}(\mathbf{p}_j-\mathbf{p}_i),\ldots) = (\ldots,\mathbf{0},\ldots)$ since ω is a proper self–stress of $G(\mathbf{p})$.

(iii) For any critical point $\mathbf{q}$, the real valued function $g(t)=H(t\mathbf{q})=t^2H(\mathbf{q})$ has the derivative $g'(t)=\nabla H(t\mathbf{q}).\mathbf{q}=t\nabla H_1(\mathbf{q}).\mathbf{q}=\mathbf{0}$ for all t. Therefore $g(t)$ is a constant $g(t)=g(0)=0$ so $H(\mathbf{q})=0$.

(iv) A general affine transformation has the form $T(\mathbf{p}_i)=L\mathbf{p}_i+\mathbf{b}$, where L is a matrix and $\mathbf{b}$ is a constant vector.

$$\nabla H(T(\mathbf{q})) = 2(\ldots,\Sigma_j\omega_{ij}(T(\mathbf{q}_j)-T(\mathbf{q}_i)),\ldots) = 2(\ldots,\Sigma_j\omega_{ij}(L\mathbf{q}_j-L\mathbf{q}_i),\ldots)$$
$$= 2(\ldots,L(\Sigma_j\omega_{ij}(\mathbf{q}_j-\mathbf{q}_i)),\ldots) =2(\ldots,L(\mathbf{0}),\ldots) = \mathbf{0}.$$

Note that this means that $H(T(\mathbf{q})) = 0$ by (iii). □

The energy form can be written with a symmetric matrix $\underline{\Omega}$ as $H(\mathbf{q})=\mathbf{q}^T\underline{\Omega}\mathbf{q}$. If this matrix $\underline{\Omega}$ is positive semidefinite ($\mathbf{q}^T\underline{\Omega}\mathbf{q}\geq 0$ for all $\mathbf{q}$) then the critical points are the null space of the matrix: $\underline{\Omega}\mathbf{q}=0$. The previous proposition shows that this null space must contain the affine images of the the original critical point $\mathbf{p}$. If the null space is small enough, we show that the only possible flexes are affine motions.

We recall two basic facts about positive semidefinite matrices. A positive semidefinite matrix M corresponds to a quadratic form $\mathbf{x}^TM\mathbf{x}$ which is the sum of squares. Therefore the sum of several positive semidefinite matrices is also positive semidefinite. Recall also that a matrix is positive semidefinite of rank r if there is a subset of r variables and for all $k\leq r$, the k by

k submatrix using the first k of these variables has positive determinant.

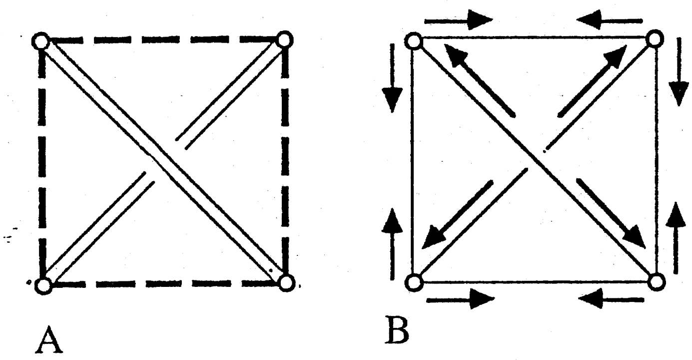
We condense the information in the matrix $\underline{\Omega}$ for $H(\mathbf{q})$. Expanding the energy function, and grouping terms, we find that:

the coefficient of the terms for $\mathbf{p}_i.\mathbf{p}_i = \Sigma_j\omega_{ij}$ (sum over j with $\{i,j\}\in E$)

the coefficient of the terms for $\mathbf{p}_i.\mathbf{p}_j = -\omega_{ij}$

Grouping the first coordinates of all points, we have a copy of the **stress matrix** introduced in chapter 3. Specifically, the stress matrix Ω for a self-stress ω is the symmetric matrix with entries $\Omega_{ij}=-\omega_{ij}$ for $i\neq j$, and $\Omega_{ii}= \Sigma_j\omega_{ij} = \Sigma_j\omega_{ji}$. If $Q(\mathbf{x}) = \mathbf{x}^T\Omega\mathbf{x}$ is the quadratic form of the stress matrix, then $H(\mathbf{q})=\Sigma_k Q(\mathbf{p}.\underline{\mathbf{e}}_k)$ where $\underline{\mathbf{e}}_k=(\mathbf{e}_k,\ldots,\mathbf{e}_k)$ (v copies) for the standard basis $(\ldots,\mathbf{e}_k,\ldots)$ of R^d. As a result H is positive semidefinite if and only if Ω is positive semidefinite. By the definition of Ω, the row and column sums of Ω are 0, so $(1,\ldots,1)$ is in the null space and the matrix Ω, and the **extended stress matrix** $\underline{\Omega}$, are never positive definite.

EXAMPLE 16.4. Consider the tensegrity square of Figure 16.2 A. A strict self–stress has coefficients 1 on the cables and -1 on the struts (Figure 16.2B). This gives the self–stress matrix of Figure 16.2C. It is a simple exercise to see that this is positive semidefinite of nullity 3. It also easy to see that any affine transformation of this framework produces a new self–stressed framework, with the same stress matrix.



$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

C

FIGURE 16.2

Some of the spider webs of the previous section convert to unpinned tensegrity frameworks in d–space which have a positive semidefinite stress matrix.

PROPOSITION 16.5. If framework $G(\mathbf{p})$ in d-space with a strict self–stress ω has a d-simplex $\mathbf{p}_1,\ldots,\mathbf{p}_{d+1}$, of struts, and all other members are cables, then the stress matrix Ω is positive semidefinite of nullity $d+1$.

Proof. Consider any displacement $\mathbf{q}$ of $G(\mathbf{p})$. By an affine motion A, we move $\mathbf{p}$ onto $\mathbf{q}$ with $A(\mathbf{p})_1=\mathbf{q}_1,\ldots,A(\mathbf{p})_{d+1}=\mathbf{q}_{d+1}$, so that $\mathbf{q}$ is a displacement of the $G(A(\mathbf{p}))$ with $A(\mathbf{p})_1,\ldots,A(\mathbf{p})_{d+1}$ pinned. Such an affine motion does not change the strict self-stress, so $G(A(\mathbf{p}))$ is a stressed spider web. If $\mathbf{q}-A(\mathbf{p})\neq\mathbf{0}$, then the spider web has $E(\mathbf{q})>E(A(\mathbf{p}))$. Since both realizations have the same length for the added struts, when we switch to the energy function H of the unpinned framework we still have $H(\mathbf{q})>H(A(\mathbf{p}))$ unless $\mathbf{q}=A(\mathbf{p})$.

Since $\mathbf{p}$ and $A(\mathbf{p})$ are both critical points of H, Proposition 16.3 guarantees that $H(\mathbf{p})=0=H(A(\mathbf{p}))$. We conclude that $H(\mathbf{q})>0$

unless $\mathbf{q}=A(\mathbf{p})$ for some affine motion A of $\mathbf{p}$. This means that $\underline{\Omega}$ is positive semidefinite with kernal equal to the space of affine infinitesimal motions. Since the affine motions form a space of dimension $d(d+1)$ and $\underline{\Omega}$ is d copies of Ω disjoint columns, we conclude that Ω is positive semidefinite of nullity $d(d+1)/d = (d+1)$. ▯

We offer a geometric interpretation of the nullity of Ω, which is used to check the nullity for some tensegrity frameworks.

PROPOSITION 16.6. Assume $G(\mathbf{p})$ in R^d has a self-stress ω and the affine span of $\mathbf{p}_i$ in R^d is a k–dimensional space K.

(i) the nullity of $\Omega(\omega)$ is $\geq k+1$.

(ii) if $d\geq$ (nullity of $\Omega(\omega)$)-1, then there is a $G(\mathbf{q})$ in R^d with $\mathbf{q}$ projecting to $\mathbf{p}$ orthogonally to K, such that ω is a self-stress of $G(\mathbf{q})$ and the dimension of the affine span of $\mathbf{q}$ is (nullity of $\Omega(\omega)$)-1.

Proof. Recall from chapter 3 that Ω is a solution to the equations $\underline{\mathbf{P}}\Omega=\mathbf{0}$, where $\underline{\mathbf{P}}$ is the matrix of affine coordinates of the joints $\mathbf{p}_i$. Conversely, since Ω is symmetric, $\Omega\underline{\mathbf{P}}^{tr}=\mathbf{0}$ is also the condition that $G(\mathbf{p})$ is a framework with ω as a self-stress.

Since the $\mathbf{p}_i$ have an affine span of dimension k, we know that $\underline{\mathbf{P}}$ is a matrix of dimension k+1. Since $\Omega\underline{\mathbf{P}}^{tr}=\mathbf{0}$, we conclude that Ω has a null space of dimension $\geq k+1$.

For convenience in the converse, assume that d=(nullity of Ω)−1=m-1 and that the affine space of the $\mathbf{p}_i$ is R^k inside R^d. We add rows to $\underline{\mathbf{P}}$ to form an m by v matrix $\underline{\mathbf{Q}}^{tr}$ which is basis for the null space of Ω. Deleting the row $(1,\ldots,1)$ we have a set of points $\mathbf{q}_i$ such that ω is a self–stress on $G(\mathbf{q})$. Moreover $\mathbf{q}_i=(p_{i1},\ldots,p_{ik},q_{i(k+1)},\ldots,q_{i(m-1)})$. We conclude that $\mathbf{q}$ projects orthogonally onto $\mathbf{p}$, as required. □

For example, the tensegrity square of Figure 16.2 A has nullity 3, so this self-stress cannot be realized in a larger space than the plane. Conversely, any of the triangles with interior spider webs of Proposition16.5 cannot be realized with this self–stress in a larger space than the plane, so the stress matrix must have nullity 3. The next proposition points out a specific advantage of a positive semidefinite stress matrix.

PROPOSITION 16.7. If a tensegrity framework $G(\mathbf{p})$ whose vertices affinely span R^d has a proper non–zero self–stress ω with a positive semidefinite stress matrix Ω of nullity $d+1$ then all $G(\mathbf{q})$ dominated by $G(\mathbf{p})$ are affinely equivalent to $G(\mathbf{p})$.

Proof. Let $G(\mathbf{p}')$ be dominated by $G(\mathbf{p})$. Since the energy form is positive semidefinite, $H(\mathbf{p}')\geq 0=H(\mathbf{p})$. If the inequality is strict, some cable is longer, or some strut is shorter, which is impossible. Therefore $H(\mathbf{p}')=0=H(\mathbf{p})$. Therefore $H(\mathbf{q})\geq 0=H(\mathbf{p}')$ for all other $\mathbf{q}$, and $\mathbf{p}'$ is a local minimum. Therefore $\nabla H(\mathbf{p})=\mathbf{0}$ and $G(\mathbf{p}')$ is in equilibrium for ω.

The affine maps T generate a subspace of the null space of $\underline{\Omega}$ of dimension $d(d+1)$. Since $\underline{\Omega}$ is d column disjoint copies of Ω, the entire null space also has dimension $d(d+1)$. Thus $\mathbf{p}'=T(\mathbf{p})$ for some affine map. □

Recall that affine images $G(T(\mathbf{p}))$ are not, in general, dominated by $G(\mathbf{p})$, and a tensegrity framework dominates a nontrivial affine image if and only if it has a flex which is a path of affine images. We will find it easy to check whether a framework has such affine flexes, so that a positive semidefinite stress matrix will give quick information on the rigidity, of the framework.

We are interested in other examples of tensegrity frameworks which have positive semidefinite stress matrices of nullity $d+1$. Projective transformations are one way to create them.

THEOREM 16.8. If a self-stress ω on a tensegrity framework $G(\mathbf{p})$ has a positive semidefinite stress matrix Ω of nullity k, then the corresponding self-stress on the proper projective image, $TG(T(\mathbf{p}))$, with appropriate sign switches, has a positive semidefinite stress matrix $T(\Omega)$ of nullity k.

Proof. $T(\omega)$ multiplies each ω_{ij} by the scalars $\omega_i.\omega_j$, the weights of the images $T(\mathbf{p}_i)$. For the stress matrix Ω this has the effect of multiplying each row and column for vertex v_i by ω_i. Such a scalar multiplication cannot change the rank of the matrix. A symmetric matrix is positive semidefinite of rank n if, and only if there is a symmetric set of n rows and columns such that the n minors down the diagonal of this submatrix are >0. Since we multiplied both rows and columns symmetrically, these determinants remain >0 and $T(\Omega)$ is positive semidefinite of nullity k. ▯

What is $T(\omega)$, $T(\Omega)$

This is correct but needs more explicit discussion!

EXAMPLE 16.9. Consider the frameworks in Figure 16.3 A,B. These are both projective images of the tensegrity square of Example 16.2, so they have positive semidefinite stress matrices of nullity 3. If we combine these pieces in the tensegrity exchange of Figure 16.3.C, we have a new framework with a positive semidefinite stress matrix.

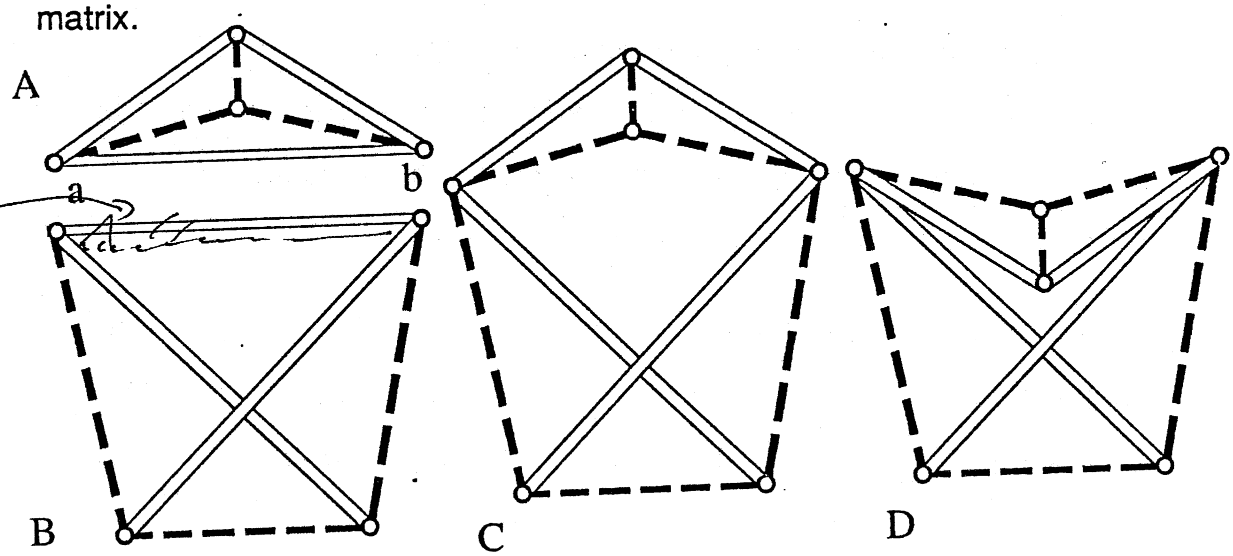


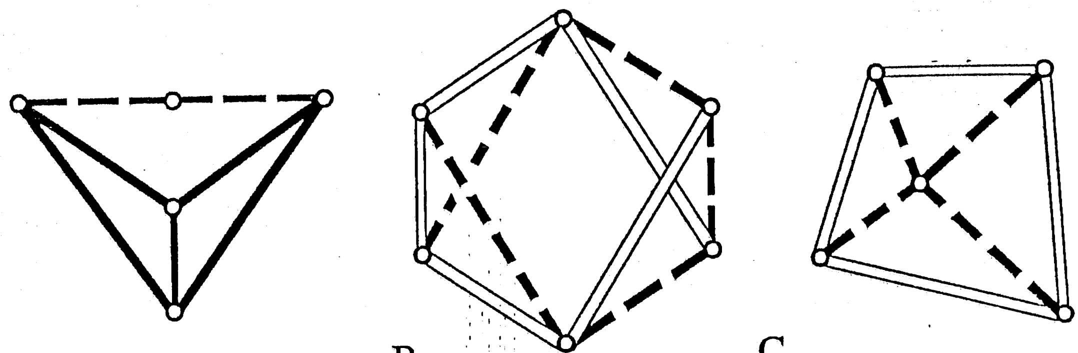
FIGURE 16.3.

However this framework is not globally rigid in the plane (Figure 16.3D). The stress matrix has nullity 4, and the lifted framework in 3–space has the flex consisting of rotating the one component about the hinge ab. The reader can check that this is an affine flex, and that the two realizations of FIgures 16.3 C,D are projections of affinely equivalent liftings on 3-space, as required by the theorems.

EXERCISES.

16.6. Show that the stress matrix for the framework G(**p**) in Figure 16.4A has nullity 4 by constructing a framework G(**q**) in 3–space.

I get nullity 3 ?



A B C

FIGURE 16.4.

16.7. (i) Show that the stress matrix for the framework of Figure 16.4B has nullity ~~3.~~ 4

(ii) By constructing a dominated framework of lower energy, show that the stress matrix is not positive semidefinite.

(iii) Show that the underlying bar framework is not globally rigid in the plane.

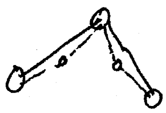
16.8. (i) Show that the underlying bar framework for Figure 16.4C is globally rigid in the plane.

(ii) Show that the tensgrity has a single strict self-stress, up to a positive scalar, and this self-stress has an indefinite stress matrix of nullity 3.

(iii) Show that the tensegrity framework is rigid, but not globally rigid, in the plane.

16.9. Show that if a rigid framework in d–space $G(\mathbf{p})$ has a proper self–stress ω which is non-zero at all vertices, with a positive semidefinite stress matrix Ω of nullity $d+1$, then $G(\mathbf{p})$ is globally rigid in all dimensions $\geq d$.

16.10. Give an example of a plane tensegrity framework $G(\mathbf{p})$ with a strict self–stress which is positive semidefinite of nullity 3, but $G(\mathbf{p})$ is not rigid in the plane.



16.11. (i) Show that for any polygon G, with vertices at algebraically independent positions $\mathbf{p}_1,\ldots,\mathbf{p}_v$ on the line, the bar framework $G(\mathbf{p})$ is globally rigid on the line.

(ii) Show that any polygon on the line, with no zero–length edges, has a self-stress of nullity 2.

(iii) Characterize the collinear tensegrity polygons which have a self-stress which is positive semidefinite.

(iv) Show that the polygons of part (iii) are the only tensegrity polygons which are globally rigid on the line.

16.12. Assume that $G_1(\mathbf{p}_1)$ and $G_2(\mathbf{p}_2)$ are frameworks with positive semidefinite stress matrices Ω_1 and Ω_2 of nullity $k_1+1=$ affine dimension $(\mathbf{p}_1)+1$, and $k_2+1=$ affine dimension $(\mathbf{p}_2)+1$ respectively. Let $k_{12}=$ affine dimension$(\mathbf{p}_1 \cap \mathbf{p}_2)$, and $G_{1\cup 2}(\mathbf{p}_1 \cup \mathbf{p}_2)$ be the framework formed by identifying common joints (assuming any common members have either the same sign or cancel on at most one common member in the combined stress of $\Omega_1+\Omega_2$).

(i) Show that $G_{1\cup 2}(\mathbf{p}_1 \cup \mathbf{p}_2)$ has a positive semidefinite stress matrix $\Omega_1+\Omega_2$;

(ii) Show that this stress matrix has nullity $k_1+k_2-k_{12}+1$;

(iii) Show that if $G_1(\mathbf{p}_1)$ and $G_2(\mathbf{p}_2)$ are globally rigid frameworks then $G_{1\cup 2}(\mathbf{p}_1 \cup \mathbf{p}_2)$ is globally rigid if and only if $k_{12}=\min(k_1,k_2)$.

16.4. Global Rigidity of Convex Polygons

We begin with a sample tensegrity pattern which is positive semidefinite, of nullity 3, on the vertices of any strictly–convex plane polygon.

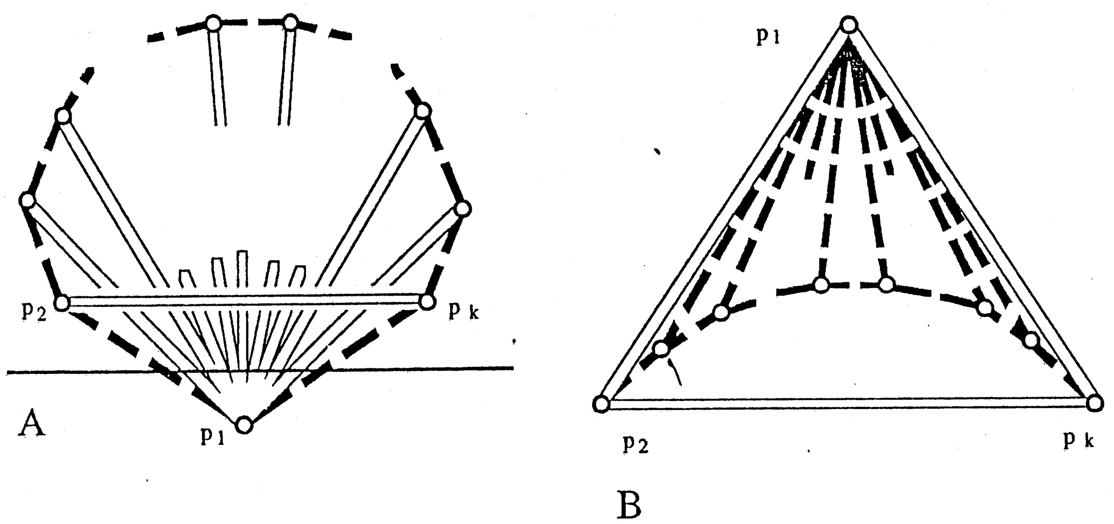


FIGURE 16.5

COROLLARY 16.10 Every Grünbaum polygon with cables on the exterior, struts on the interior (Figure 16.5A), has a positive semidefinite stress matrix of nullity 3.

Proof. In Example 10.8 we saw that the Grünbaum polygon has a proper self–stress. If we apply a projective transformation, as indicated in Figure 16.5B, we create a spider web suspended in a triangle. By Proposition 16.4 this has a positive semidefinite stress matrix of nullity 3. We project back to the Grünbaum polygon, which now has a positive semidefinite stress matrix of nullity 3, by Theorem 16.7. ∎

THEOREM 16.11. Connelly (1982) Given a tensegrity framework on the vertices of a convex polygon, with cables on the boundary, struts on the interior, and a proper self-stress ω, then the stress matrix is positive semidefinite with nullity 3.

Proof. The nullity of Ω is at least 3, since it is realized in the plane. If the nullity is greater than 3, it is the projection of a $G(\underline{p})$ in R^3, with the same self-stress ω. Let $\underline{e}$ be a diagonal edge between upper faces of the convex hull of $G(\underline{p})$. Take a vertical plane P through $\underline{e}$ which creates a cut-set of the edges in a self-stressed framework (Figure 16.6A). By Proposition 2.xx, the forces from the edges on the left of P are in equilibrium. All forces at the two vertices of $\underline{e}$ intersect this line, and any other forces must be from struts crossing below $\underline{e}$ (Figure 16.6A). Such lower forces, all pushing right on H, cannot be in equilibrium with forces through the line of $\underline{e}$. (Note that this force would "flatten out" $G(\underline{p})$ along this hinge and decreases the energy). We conclude that there are only zero forces below the line.

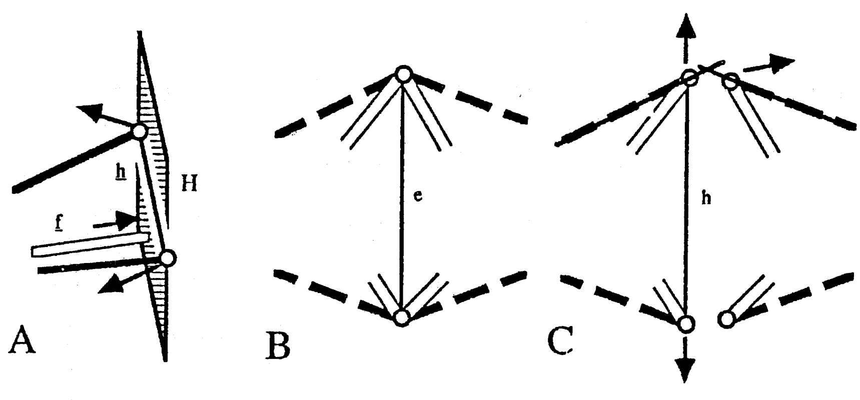


FIGURE 16.6.

After vertical projection into $G(\mathbf{p})$, we have a diagonal e, with no crossing struts in compression (Figure 16.6B). Again, e cuts a self–stressed framework, so the net forces on the left, at the two ends of e, must be in equilibrium. These two forces must be collinear, along e, and by the signs of the members, must point outward (Figure 16.6C). However the force at the top end from the left must also reach an equilibrium with the forces at this vertex from the right. Since this force is also outwards, it cannot be in equilibrium with an outward force along e (Figure 16.6C). We conclude that no such $G(\mathbf{p})$ exists and nullity of Ω is 3.

We now show that Ω is positive semidefinite. Let Ω^* be the reduced stress matrix of the Grünbaum polygon on the same vertices. The reduced stress matrix $\Omega(t)=(1-t)\Omega+t\Omega^*$ $0\leq t\leq 1$ gives a continuous family of self–stresses on the graph with cables on the boundary and struts on the interior edges of the Grünbaum polygon and $G(\mathbf{p})$. By the above, each $\Omega(t)$ has nullity 3. We checked that $\Omega^*=\Omega(1)$ is positive semidefinite. If we restrict $\Omega(t)$ to the complement of the constant null-space, then $\Omega(1)$ is positive definite, and the rank is constant, so $\Omega(t)$ is positive definite on this subspace for all t. We conclude that $\Omega(0)=\Omega$ is positive semidefinite on the whole space. □

COROLLARY 16.12. Connelly (1982) Given a tensegrity framework on the vertices of a convex polygon, with cables on the boundary, struts on the interior, and a non-zero proper self–stress ω, then $G(p)$ is uniquely embedded and rigid in R^d for all $d\geq 2$.

Proof. Since Ω is positive semidefinite of nullity 3, Proposition 16.4 guarantees that frameworks $G(\mathbf{q})$ dominated by $G(\mathbf{p})$ are affine images $T(\mathbf{p})$, in the plane of the polygon. We show that T is a congruence of the plane.

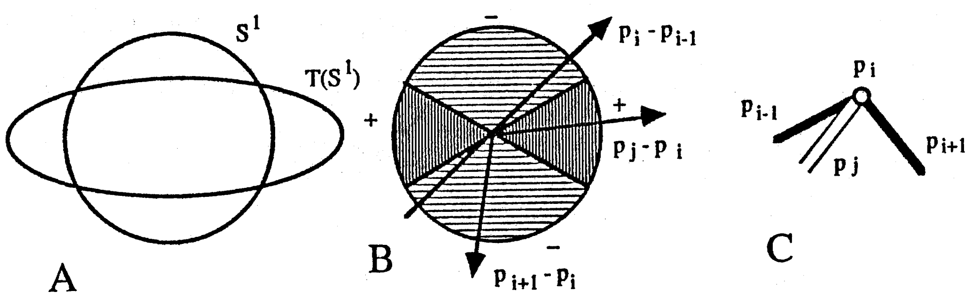


FIGURE 16.7.

If T is a nontrivial affine motion, we know by Theorem 10.22 that there is a flex of the framework, whose derivative A is a nontrivial affine infinitesimal motion. Now this affine infinitesimal motion must preserve, instantaneously, the lengths of all cables and struts which are self–stressed in the framework. At any vertex, there is at least two struts and one cable with a non-zero coefficient - so there are at least three constant directions for the affine motion. A nontrivial affine motion of the plane moves the unit circle to an ellipse (Figure 16.7). The only affine motions of the plane with three distinct constant directions are the trivial congruences. This contradiction completes the proof. ▯

The next corollary shows one effect of reversing from cables on the boundary of the polygon to struts on the boundary and cables in the interior, as conjectured by Grünbaum&Shephard (1975).

COROLLARY 16.13. Connelly (1982) Let $G^*(\mathbf{p})$ be the tensegrity framework on a convex plane polygon with rods on the boundary and cables on the interior. If $G^*(\mathbf{p})$ is rigid then the reversal $G(\mathbf{p})$, with cables on the outside and struts on the interior is uniquely realizable in E^d for $d \geq 2$. globally rigid

Proof. The framework $G^*(\mathbf{p})$ must have a non–zero self–stress ω. By its form at the convex vertices, there must be compression in some bar and tension in some cable. The compression at one vertex passes around the polygon - requiring compression in all bars. Thus $-\omega$ is a proper self–stress on $G(\mathbf{p})$, so $G(\mathbf{p})$ is uniquely realized, and rigid in E^d. □

We close by stating a stronger result which was conjectured by Roth (Roth&Whiteley (1981)). One step of the proof wil be given in the section 16.6. This is not stronger, only different

THEOREM 16.14. Connelly&Whiteley (1987). If a tensegrity framework $G(\mathbf{p})$ on the vertices of a convex polygon, with bars on the exterior and cables on the interior, is rigid then it is infinitesimally rigid.

EXERCISES.

16.13. Given the vertices of a regular plane n-gon as joints ($n \geq 4$), prove that the smallest rigid tensegrity framework has $3n/2$ members for n even, $(3n+1)/2$ members for n odd.

16.14. Give an example of a non-rigid tensegrity framework on the vertices of a convex polygon with one interior vertex, with cables on the boundaries, bars in the inside, and a strict self–stress.

16.15. Consider a simple, strictly convex polyhedron with each face replaced by a self-stressed polygon with cables on the boundary and struts on the interior.

(i) Show that there entire framework has a strict self-stress with a positive semidefinite stress matrix of nullity 4.

(ii) Show that the tensegrity framework is globally rigid.

(iii) Show that if a simple, strictly convex d–polytope is realized with each 2-face as such a self-stressed cable polygon with interior struts then the tensegrity framework is globally rigid in d–space.

16.5. Gale Transforms and Global Rigidity.

How can we generate examples of frameworks in d–space which have positive semidefinite stress matrices of nullity d+1? We offer one technique which is adapted from the study of convex polytopes.

Consider any positive semidefinite stress matrix Ω of nullity $d+1$. We can make this into a diagonal matix D by an invertible orthogonal transformation $D=O\Omega O^{tr}$, where $O^{tr}=O^{-1}$. Since D is still positive semidefinite of nullity $d+1$, we can take the square root of the $v-(d+1)$ positive diagonal entries. This gives $D=(\sqrt{D})^{tr}(\sqrt{D})$ and:

$$\Omega= O^{tr}DO = O^{tr}(\sqrt{D})^{tr}(\sqrt{D})O= [O^{tr}\sqrt{D}O]^{tr}[O^{tr}\sqrt{D}O]=(\sqrt{\Omega})^{tr}\sqrt{\Omega}$$

It is a simple exercise to check that $[\mathcal{O}^{tr}\sqrt{D}\mathcal{O}]=\sqrt{\Omega}$ also has rank $v-(d+1)$.

Conversely, consider any (n-d-1) by n matrix A, with independent rows. The matrix $A^{tr}A$ is automatically positive semidefinite of rank (n–d-1), since $\mathbf{q}^{tr}A^{tr}A\mathbf{q}= (A\mathbf{q}).(A\mathbf{q})\geq 0$. To read these entries as the coeficients of a self-stress on a tensegrity framework, we solve the equations $A^{tr}A\underline{\mathbf{P}}^{tr}=0$ for positions of the points in d–space, as above. If we now choose the members corresponding to positive entries in $A^{tr}A$ to be struts, those with negative entries to be cables, and omit those with zero coeficients, we have the induced tensegrity graph G_A such that $G_A(\mathbf{p})$ has the stress–stress with stress matrix $A^{tr}A=\Omega$.

Since $A^{tr}A$ is positive semidefinite of nullity d+1, we know that the solution set to $A^{tr}A\underline{\mathbf{P}}^{tr}=\mathbf{0}$ is a vector space of dimension d+1. Therefore this must equal the subspace of solutions to $A\underline{\mathbf{P}}^{tr}=\mathbf{0}$, which also has dimension d+1. Given a set of points $\mathbf{p}$ spanning d–space, the rows of a matrix satisfying $A\underline{\mathbf{P}}^{tr}=\mathbf{0}$ must be affine dependencies of the points. Since A has rank n-d-1, the rows of A must generate the space of all affine dependencies. In the theory of convex polytopes the columns of such a matrix A are treated as points in (n-d-1)–space, forming the **Gale transform** of the original points. For this reason we call this matrix A a **Gale matrix** for the points $\mathbf{p}$.

We summerize the argument so far.

PROPOSITION 16.15. Given a set of points **p** in d–space with affine matrix P, the induced tensegrity framework $G_A(\mathbf{p})$ for any Gale matrix A is positive definite of nullity d+1.

Consider the following three examples.

EXAMPLE 16.16. Consider the set of points (1,0), (-1,0), (0,1), (0,-1) in the plane (Figure 16.8 A). This creates the matrix <u>P</u>

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

This matrix has rank 3, so we have an orthogonal space of dimension 1 spanned by the Gale matrix $A = [\ 1 \quad 1 \quad -1 \quad -1\ ]$. This gives the stress matrix $\Omega = A^{tr}A$.

$$\Omega = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

This matrix corresponds to the tensegrity framework of Figure 16.8A. Since the matrix A is unique up to a scalar, this is, up to a positive scalar, the only positive definite stress matrix for these points.

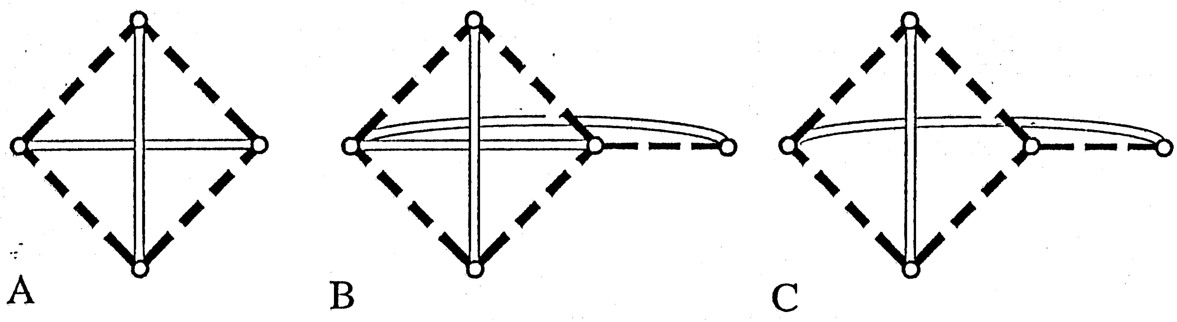


FIGURE 16.8.

EXAMPLE 16.17. What happens if we extend this set of points by a point (2, 0)? This extends the matrix $\underline{P}$ to $\underline{Q}$

$$\underline{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Since this has rank 3, we have a choice for the Gale transform, which has rank 2. We extend the previous matrix by the obvious affine dependence of the points (1,0), (-1,0) and (2,0).

Has rank 2!

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 3 & -1 & 0 & 0 & -2 \end{bmatrix}$$

This leads to the stress matrix

$$\Omega = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 \\ 3 & -1 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -1 & -1 & -6 \\ -2 & 2 & -1 & -1 & 2 \\ -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ -6 & 2 & 0 & 0 & 4 \end{bmatrix}$$

and the tensegrity framework of Figure 16.8B.

However we can simplify this tensegrity framework by a more delicate choice for the Gale matrix, which gives more zeros in the

the stress matrix by making another pair of columns orthogonal. Consider the Gale matrix B:

$$B = \begin{bmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} & -\sqrt{3} & 0 \\ 3 & -1 & 0 & 0 & -2 \end{bmatrix}$$

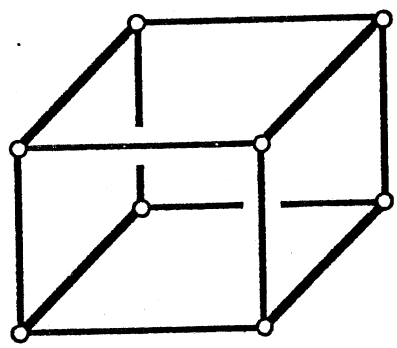
This gives the stress matrix:

$$\Omega = \begin{bmatrix} 12 & 0 & -3 & -3 & -6 \\ 0 & 4 & -3 & -3 & 2 \\ -3 & -3 & 3 & 3 & 0 \\ -3 & -3 & 3 & 3 & 0 \\ -6 & 2 & 0 & 0 & 4 \end{bmatrix}$$

and the corresponding tensegrity framework of Figure 16.8C.

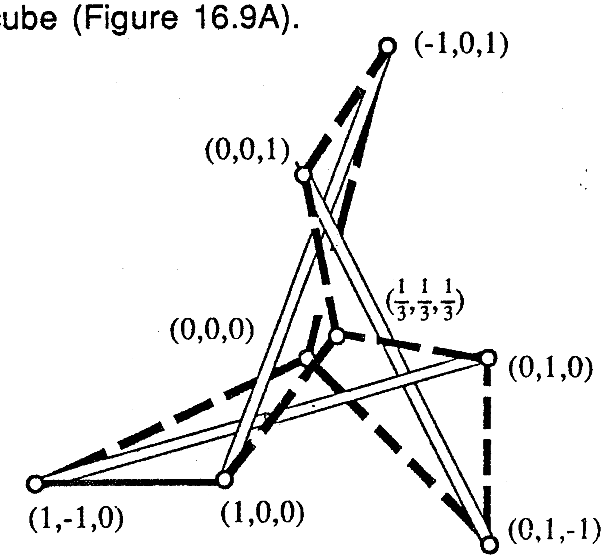
Note that all these examples have no affine flexes, so they are all globally rigid.

EXAMPLE 16.18. Consider the graph of a cube (Figure 16.9A).



A

FIGURE 16.9.



B

Can we realize this in 3-space with a positive semidefinite self–stress matrix of nullity 4? This will require a stress matrix with a pattern of zeros:

What is the labeling?

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 & x & x & x \\ 0 & x & 0 & 0 & x & 0 & x & x \\ 0 & 0 & x & 0 & x & x & 0 & x \\ 0 & 0 & 0 & x & x & x & x & 0 \\ 0 & x & x & x & x & 0 & 0 & 0 \\ x & 0 & x & x & 0 & x & 0 & 0 \\ x & x & 0 & x & 0 & 0 & x & 0 \\ x & x & x & 0 & 0 & 0 & 0 & x \end{bmatrix}$$

The matrix $\underline{P}$ will have rank 4, and the Gale matrix will also have rank 4. This stress matrix requires that the Gale matrix A has A_1, A_2, A_3, A_4 as a set of mutually orthogonal vectors in R^4. Similarly A_5, A_6, A_7, A_8, are mutually orthogonal, with the additional constraint that corresponding vectors in the two sets, such as A_1 and A_5, are also orthogonal. The following matrix has these properties:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We solve the equations $A\,\underline{P}^{tr} = \mathbf{0}$ for the positions of the points:

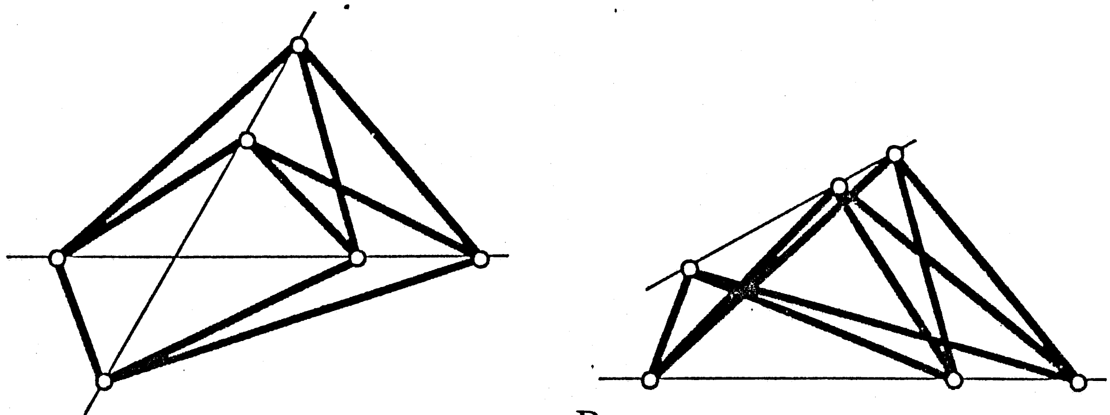
$$\underline{P} = \begin{bmatrix} -1 & 0 & 1 & \frac{1}{3} & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{3} & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The reader can now check that this gives the tensegrity framework in Figure 16.9B. Since each joint is only 3 valent, we know that the joint and its 3 neighbours are coplanar. Such a pattern of two "tetrahedron" $\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 \mathbf{p}_4$ and $\mathbf{p}_5 \mathbf{p}_6 \mathbf{p}_7 \mathbf{p}_8$ with the vertices of one, e.g. $\mathbf{p}_1$ on the faces of the other, e.g. its neighbours $\mathbf{p}_6 \mathbf{p}_7 \mathbf{p}_8$, is called a **Möbius pair** of tetrahedra.

How do we know that this is globally rigid? By Proposition 16.7 we know that all deformations are affine transformations. By Proposition 10.22, any affine flex begins with an affine infinitesimal flex. Since there is a self-stress in all members the affine flex must preserve the length of all the members. A nontrivial affine infinitesimal flex in 3–space preserves at most a conic of directions in the plane at infinity. At any joint the 3 members define three collinear directions at infinity which must be on this conic - so the conic contains this line. We conclude that the required conic at infinity contains the 8 lines at infinity for the eight joints. This contradiction shows that the framework is globally rigid.

Note that any n by n stress matrix Ω which is positive semidefinite of nullity d+1, can be written as $A^{tr}A$ where A is a n-d-1 by n matrix, with row sums zero. This matrix is the Gale transform of an appropriate set of points, so every positive semidefinite stress matrix comes by this construction.

EXAMPLE 16.19 Consider the example of $K_{3,3}$ with the vertices of the two sides on two distinct lines in the plane (Figure 16.10). From chapter 5 we know that this has a self-stress given by the product of the two affine dependencies on the two lines. A direct analysis of the corresponding stress matrix shows that it has rank 2 or nullity 4 (see example 16.27). This reminds us that the same self-stress can be realized for two skew lines in 3–space (Figure 16.10B).



A B

FIGURE 16.10.

Can this self-stress ever be positive semidefinite? We show that answer is no, by showing there is no appropriate Gale transform. The matrix P in 3-space has rank 4, so any Gale transform of these points has rank 2, spanned by the two affine dependencies on the two lines. However to get the stress matrix of $K_{3,3}$ we must have two sets of three mutually orthogonal non–zero columns in this transform matrix. This contradicts the assumption that the rank is 2. We conclude that no such Gale transform exists and none of these realizations is positive semidefinite.

Why can't same points overlap?

EXERCISES.

16.16. Consider the six points in the plane (1,1), (2,0), (1,-1), (-1,-1), (–2,0), (-1,1). Find Gale transforms which give positive semidefinite stress matrices for the three tensegrity frameworks shown in Figure 16.11.

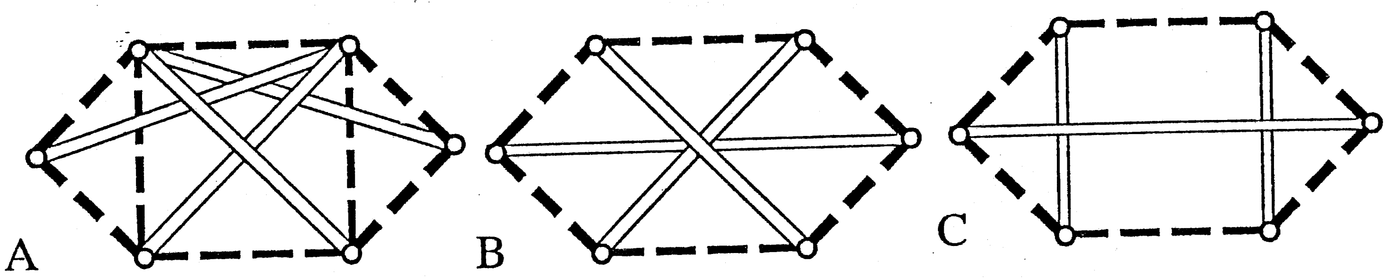


FIGURE 16.11.

16.17. Show that the tensegrity framework in Figure 16.12A has a positive semidefinite stress matrix, by giving an appropriate Gale matrix.

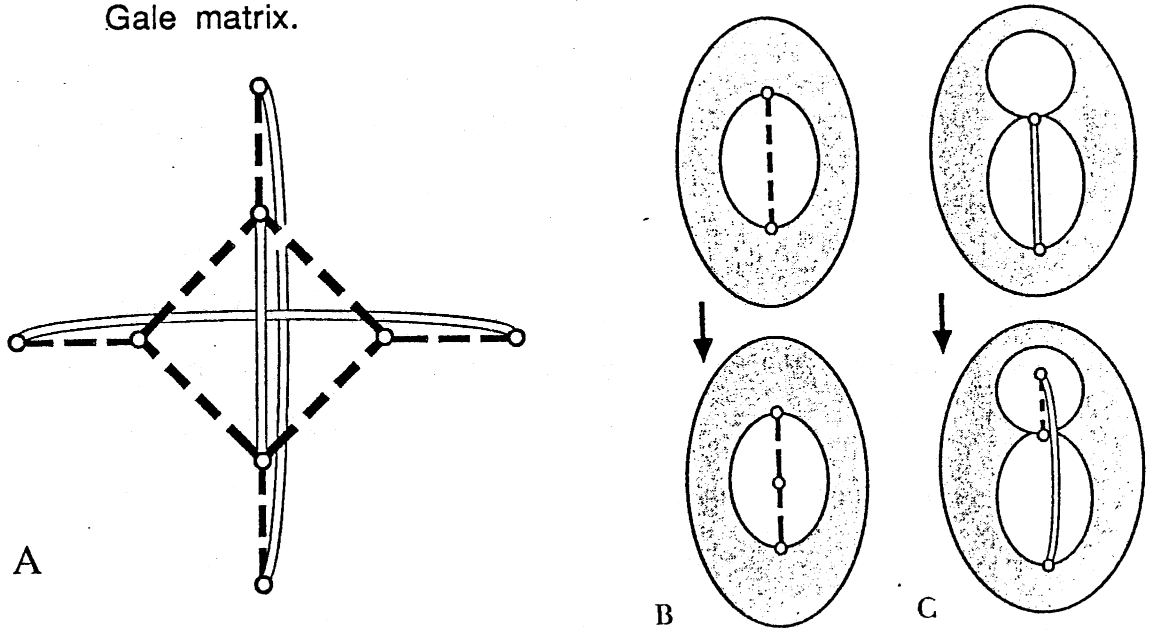


FIGURE 16.12.

16.18. Assume that a tensegrity framework G(**p**) has a positive semidefinite stress matrix of nullity $d+1$.

(i) Show that a stressed cable (a,b) can be split in the interior by a point c (Figure 16.12B) to give a new framework with a positive semidefinite stress matrix of nullity $d+1$.

(ii) Show that a stressed strut (a,b), can be split by a point c exterior to b with longer collinear strut (a,c), and cable (c,b) (Figure 16.12 C) to give a new tensegrity framework with a positive semidefinite stress matrix of nullity $d+1$.

(iii) Show that both of these operations also preserve global rigidity.

16.19. Recall from chapter 5 that any bipartite framework $K(A,B)$ realized with A spanning an affine subspace $\langle A\rangle$ of affine dimension $|A|-1$, B spanning an affine subspace $\langle B\rangle$ of affine dimension $|B|-1$, and with $\langle A\rangle \cap B = \emptyset = \langle B\rangle \cap A$, has a single self–stress of which is the product of the affine dependencies.

(i) Show that this self–stress has nullity $|A|+|B|-2$;

(ii) Show that the stress matrix for this self-stress is indefinite.

16.6. Second–Order Rigidity of Tensegrity Frameworks.

In chapter 4, second–order rigidity was introduced to show the rigidity of certain non-infinitesimally rigid bar frameworks. We recall that a bar framework is **second–order rigid** if for each nontrivial infinitesimal motion $\mathbf{p}'$ (infinitesimal flex), with $R(\mathbf{p}).\mathbf{p}'=0$, no second-order deformation $\mathbf{p}''$ satisfies:

$$R(\mathbf{p}').\mathbf{p}' + R(\mathbf{p}).\mathbf{p}''=0.$$

One critical feature of this rigidity was given in Theorem 4.xx:

If a bar framework is second–order rigid, then it is rigid.

We complete our summary on second–order rigidity for bar frameworks with a basic **second–order stress test** for the second–order rigidity of the framework.

PROPOSITION 16.20. Connelly&Whiteley (1987) An infinitesimal flex **p'** of a bar framework G(**p**) extends to a second-order flex **p"** if and only if for every self-stress ω,

$$\omega R(\mathbf{p}').\mathbf{p}' = \mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' = 0 .$$

Proof. An infinitesimal motion **p'** has a second-order extension **p"** if and only if $R(\mathbf{p}).\mathbf{p}'=0$ and $R(\mathbf{p}').\mathbf{p}' + R(\mathbf{p}).\mathbf{p}''=0$ are consistent if and only if every row dependence ω of R(**p**), which is a self-stress of G(**p**), is a row dependence of $-R(\mathbf{p}').\mathbf{p}'$ if and only if for each self-stress ω of R(**p**): $\omega R(\mathbf{p}').\mathbf{p}'=0$ □

We now extend the definitions and these results to tensegrity frameworks. The analogue of Proposition 16.20 will show the connection between our previous results on global rigidity and this local second-order rigidity.

DEFINITION 16.21. A **second–order flex** for a tensegrity framework is a solution **p', p"** to the equations:

(i) for bars: $(\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}'_i-\mathbf{p}'_j)=0$ and
$(\mathbf{p}'_i-\mathbf{p}'_j)(\mathbf{p}'_i-\mathbf{p}'_j) + (\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}''_i-\mathbf{p}''_j)= 0$;

bracket (ii) for cables: either $[(\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}'_i-\mathbf{p}'_j)<0$ or $[(\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}'_i-\mathbf{p}'_j) = 0$ and
$(\mathbf{p}'_i-\mathbf{p}'_j)(\mathbf{p}'_i-\mathbf{p}'_j) + (\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}''_i-\mathbf{p}''_j)\leq 0;]$

(iii) for struts: either $[(\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}'_i-\mathbf{p}'_j)>0$ or $[(\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}'_i-\mathbf{p}'_j) = 0$
and $[(\mathbf{p}'_i-\mathbf{p}'_j)(\mathbf{p}'_i-\mathbf{p}'_j) + (\mathbf{p}_i-\mathbf{p}_j).(\mathbf{p}''_i-\mathbf{p}''_j)\geq 0.]$

A tensegrity framework is **second–order rigid** if all second-order flexes have a trivial infinitesimal motion as **p'**. Otherwise G(**p**) is **second–order flexible.**

Figure 16.13 shows first and second–order flexes of some tensegrity frameworks, (light arrows for **p"**, heavy for **p'**). The flex in Figure 16.13A is trivial for both **p'**, and **p"**, that in Figure 16.13B is trivial for **p'**, but not **p"**, while that in C is nontrivial for **p'**, and **p"**. This means that only the flex in Figure 16.13C is a nontrivial second–order flex.

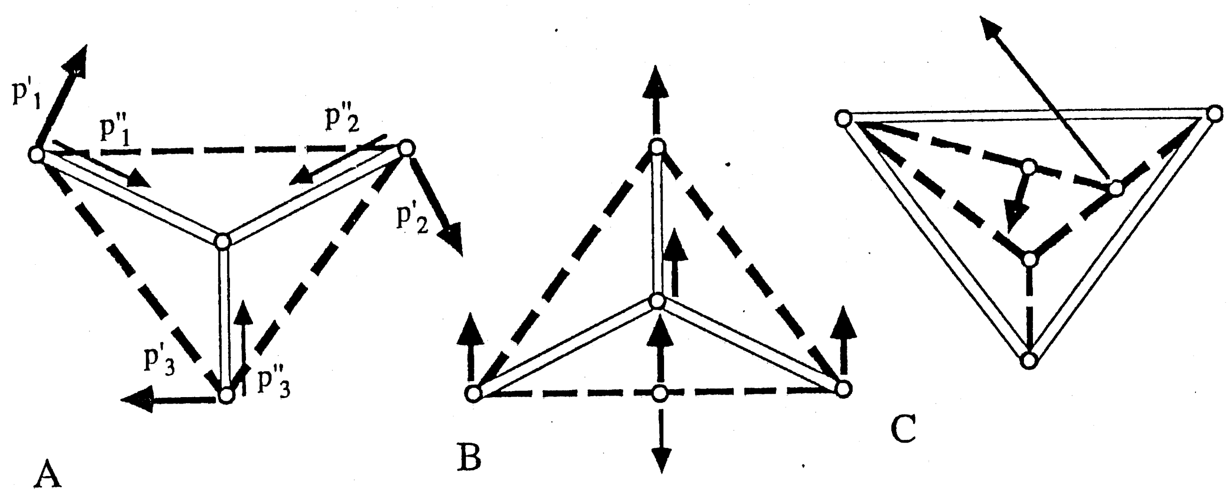


FIGURE 16.13

Since the second-order extensions are solutions to an inhomogeneous system of equations and inequalities, we can add on any solution **q'** to the homogeneous system to the second–order flex **p"**.

PROPOSITION 16.22. If $(\mathbf{p}',\mathbf{p}'')$ is a second–order flex of a tensegrity framework $G(\mathbf{p})$ and $\mathbf{q}'$ is any infinitesimal flex of $G(\mathbf{p})$ then $(\mathbf{p}',\mathbf{p}''+\mathbf{q}')$ is a second–order flex of $G(\mathbf{p})$.

Proof. Assume that for each cable (resp. bar, strut) with $(p_i-p_j).(p'_i-p'_j)=0$,

$$(p'_i-p'_j).(p'_i-p'_j)+(p_i-p_j).(p''_i-p''_j)\leq 0 \quad (\text{resp. } =0, \geq 0)$$

$$\text{and } (p_i-p_j).(q'_i-q'_j)\leq 0 \quad (\text{resp. } =0, \geq 0).$$

Therefore:

$$(p'_i-p'_j).(p'_i-p'_j)+(p_i-p_j).[(p''_i+q'_i)-(p''_j+q'_j)]\leq 0 \quad (\text{resp. } =0, \geq 0). \qquad []$$

If we add a multiple of $\mathbf{p}'$ itself to any extension $\mathbf{p}''$ we can make the second–order extension satisfy the second–order inequalities for all members with $(p_i-p_j).(p'_i-p'_j)\neq 0$. Since this observation will simplify some of our arguments, we summerize in the following corollary.

COROLLARY 16.23. An infinitesimal motion $\mathbf{p}'$ has a second–order extension $\mathbf{p}''$ if and only if there is another second–order extension $\mathbf{q}''$ with $(p'_i-p'_j)(p'_i-p'_j)+(p_i-p_j).(p''_i-p''_j)\leq 0$ (resp. $=0, \geq 0$) for all cables (resp. bars, struts).

We now confirm that second–order rigidity remains an appropriate tool for checking the rigidity of a tensegrity framework.

THEOREM 16.24. Connelly&Whiteley (1987) If a tensegrity framework $G(\mathbf{p})$ is second–order rigid, then it is rigid.

Proof. Assume that $G(\mathbf{p})$ has a nontrivial analytic flex $\mathbf{p}(t)$. Let the first nontrivial derivative at $t=0$ be $\mathbf{p}^k(t)$. (After adding a trivial motion of the space, we can assume that $\mathbf{p}^m(t)=\mathbf{0}$ for $0<m<k$.) Differentiating $(\mathbf{p}_i-\mathbf{p}_j)^2\leq c_{ij}$ for each cable we have:

$$D_t^{k-1}(\mathbf{p}_i(0)-\mathbf{p}_j(0))^2=0 \text{ and}$$

$$D_t^k(\mathbf{p}_i(t)-\mathbf{p}_j(t))^2=\Sigma\binom{k}{m}[\mathbf{p}^{(k-m)}{}_i(t)-\mathbf{p}^{(k-m)}{}_j(t)].[\mathbf{p}^m{}_i(t)-\mathbf{p}^m{}_j(t)]$$

However $\mathbf{p}^m{}_i(0)=0$ for $0<m<k$, and at $t=0$ $D_t^k(\mathbf{p}_i(t)-\mathbf{p}_j(t))^2\leq 0$, so we have:

$$[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{p}^k{}_i(0)-\mathbf{p}^k{}_j(0)] \leq 0.$$

A similar equation ($=0$), or inequality (≥ 0), holds for each bar and for each strut, respectively. This means that $\mathbf{p}^k(0)$ is a nontrivial infinitesimal motion of the tensegrity framework $G(\mathbf{p})$.

We will show that $\mathbf{p}^k(0)$ has a second-order extension, built from $\mathbf{p}^{2k}(0)$. Consider the derivative

$$D_t^{2k}(\mathbf{p}_i(t)-\mathbf{p}_j(t))^2 = \Sigma\binom{2k}{m}[\mathbf{p}^{(2k-m)}{}_i(t)-\mathbf{p}^{(2k-m)}{}_j(t)].[\mathbf{p}^m{}_i(t)-\mathbf{p}^m{}_j(t)]$$

At $t=0$, this is

$$[\mathbf{p}^k{}_i(0)-\mathbf{p}^k{}_j(0)].[\mathbf{p}^k{}_i(0)-\mathbf{p}^k{}_j(0)]+[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{p}^{2k}{}_i(0)-\mathbf{p}^{2k}{}_j(0)].$$

For a cable $\{i,j\}$ either $D_t^{2k}(\mathbf{p}_i(t)-\mathbf{p}_j(t))^2\leq 0$ or there was a previous derivative: $D_t^n(\mathbf{p}_i(0)-\mathbf{p}_j(0))^2=[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{p}^n{}_i(0)-\mathbf{p}^n{}_j(0)]<0$ with $D_t^m(\mathbf{p}_i(0)-\mathbf{p}_j(0))^2=0$ $0<m<n$.

In this second case, for each cable (resp. strut) we will create an infinitesimal flex $\mathbf{q}^{ij}$ with $[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{q}^{ij}{}_i-\mathbf{q}^{ij}{}_j]<0$, (resp. >0). This is done by induction on n. If $n=k$, then $\mathbf{q}^{ij}=\mathbf{p}^k(0)$. Assume the construction works for all cables (resp. struts) $\{h,k\}$ with $D_t^r(\mathbf{p}_h(0)-\mathbf{p}_k(0))^2<0$ (resp. >0) $r<n<2k$, and that $D_t^n(\mathbf{p}_i(0)-\mathbf{p}_j(0))^2<0$ for a cable (resp. >0 for a strut). For each edge $\{g,h\}$ either $D_t^n(\mathbf{p}_g(0)-\mathbf{p}_h(0))^2=[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{p}^n{}_i(0)-\mathbf{p}^n{}_j(0)]<0$ for a cable (resp. >0 for a strut) or we have previously constructed an infinitesimal motion $\mathbf{q}^{gh}$. For a suitable scalar α_{gh}, $\mathbf{p}^n(0)+\alpha_{gh}\mathbf{q}^{gh}$ satisfies

$[\mathbf{p}_g-\mathbf{p}_h].[\mathbf{p}^n{}_g(0)-\mathbf{p}^n{}_h(0)]+\alpha_{gh}[\mathbf{p}_g-\mathbf{p}_h].[\mathbf{q}^{gh}{}_g-\mathbf{q}^{gh}{}_h] <0$ (resp. >0 for a strut) or $[\mathbf{p}_g-\mathbf{p}_h].[(\mathbf{p}^n{}_g(0)+\alpha_{gh}\mathbf{q}^{gh}{}_g)-(\mathbf{p}^n{}_h(0)+\alpha_{gh}\mathbf{q}^{gh}{}_h)] <0$ (resp. >0 for a strut) and for all other cables (resp. struts, bars) $\{i,j\}$

$[\mathbf{p}_i-\mathbf{p}_j].[(\mathbf{p}^n{}_i(0)+\alpha_{gh}\mathbf{q}^{gh}{}_j)-(\mathbf{p}^n{}_i(0)+\alpha_{gh}\mathbf{q}^{gh}{}_j)] \leq [\mathbf{p}_i-\mathbf{p}_j].[\mathbf{p}^n{}_i(0)-\mathbf{p}^n{}_i(0)]$

(resp. $\geq, =$). Thus we can add all these previous $\alpha_{gh}\mathbf{q}^{gh}$ to make $\mathbf{q}^{ij}=\mathbf{p}^n(0)+\Sigma\alpha_{gh}\mathbf{q}^{gh}$ into an infinitesimal motion with $[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{q}^{ij}{}_i-\mathbf{q}^{ij}{}_j]<0$, (resp. >0).

For the equations of $D_t^{2k}(\mathbf{p}_i(t)-\mathbf{p}_j(t))^2$ we similarly create $\mathbf{q}^{2k}=\mathbf{p}^{2k}(0)+\Sigma\beta_{gh}\mathbf{q}^{gh}$ so that all inequalities are satisfied. For each cable (resp. bar, strut):

$$[\mathbf{p}^k{}_i(0)-\mathbf{p}^k{}_j(0)].[\mathbf{p}^k{}_i(0)-\mathbf{p}^k{}_j(0)]+[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{q}^{2k}{}_i-\mathbf{q}^{2k}{}_j]\leq 0 \quad \text{(resp. } =, \geq \text{)}$$

since $[\mathbf{p}_i-\mathbf{p}_j].[\mathbf{q}^{gh}{}_i-\mathbf{q}^{gh}{}_j]\leq 0$ (resp. $=, \geq$). This shows that $\mathbf{p}^k(0),\mathbf{q}^{2k}$ is the required second–order extension of $\mathbf{p}^k(0)$. []

How can we check the second-order rigidity of a tensegrity framework? We offer an extension of Proposition 16.20 which connectes the proper self-stresses with the second-order rigidity of the framework. The following result is a variant of the standard duality of linear programming (Rockafellar (1970)). We will prove it directly from the hyperplane separation theorem for convex cones.

For convenience in this proof we use some modified notation. For a tensegrity framework $G(\mathbf{p})$, the rigidity matrix is rewritten in two parts: $R^+(\mathbf{p})$ is the matrix with rows:

for each cable $\{i,j\}$, $i<j$: $[0 \ \dots \ 0 \ \ \mathbf{p}_i\text{-}\mathbf{p}_j \ \ 0 \dots \ 0 \ \ \mathbf{p}_j\text{-}\mathbf{p}_i \ \ 0 \ \dots \ 0]$

for each strut $\{i,j\}$, $i<j$: $[0 \ \dots \ 0 \ \ \mathbf{p}_j\text{-}\mathbf{p}_i \ \ 0 \dots \ 0 \ \ \mathbf{p}_i\text{-}\mathbf{p}_j \ \ 0 \ \dots \ 0]$

while the $R^\circ(\mathbf{p})$ is the matrix with rows:

for each bar $\{i,j\}$, $i<j$: $[0 \ \dots \ 0 \ \ \mathbf{p}_i\text{-}\mathbf{p}_j \ \ 0 \dots \ 0 \ \ \mathbf{p}_j\text{-}\mathbf{p}_i \ \ 0 \ \dots \ 0]$

In this notation, an infinitesimal flex is a solution to the equations:

$$R^+(\mathbf{p})\mathbf{p}' \le \mathbf{0} \text{ and } R^\circ(\mathbf{p})\mathbf{p}' = \mathbf{0}.$$

A proper stress is now an assignment of scalars ω^+, ω° such that:

$$\omega^+_{ij} \ge 0 \text{ for all } \{i,j\} \in E_- \cup E_+ \text{ and } \omega^+ R^+(\mathbf{p}) + \omega^\circ R^+(\mathbf{p}) = \mathbf{0}.$$

The second-order equations become:

$$R^+(\mathbf{p}')\mathbf{p}' + R^+(\mathbf{p})\mathbf{p}'' \le \mathbf{0} \text{ for cables and struts,}$$

$$R^\circ(\mathbf{p}')\mathbf{p}' + R^\circ(\mathbf{p})\mathbf{p}'' = \mathbf{0} \text{ for bars.}$$

THEOREM 16.25. Connelly&Whiteley (1987) An infinitesimal motion $\mathbf{p}'$ on a tensegrity framework $G(\mathbf{p})$ extends to a second-order flex $\mathbf{p}', \mathbf{p}''$ if and only if $\mathbf{p}'^{tr} \underline{\Omega}\, \mathbf{p}' \le 0$ for all proper self-stresses ω.

Proof: In terms of the vocabulary of R^+, R° etc. this statement translates the following more general fact:

For any deformation $\mathbf{p}'$ there is an extension $\mathbf{p}''$ satisfying:

$$R^\circ(\mathbf{p}')\mathbf{p}' + R^\circ(\mathbf{p})\mathbf{p}'' = \mathbf{0} \text{ and } R^+(\mathbf{p}')\mathbf{p}' + R^+(\mathbf{p})\mathbf{p}'' \le \mathbf{0},$$

if and only if for all $\omega^+ \ge \mathbf{0}$:

$$\omega^+ R^+(\mathbf{p}) + \omega^\circ R^\circ(\mathbf{p}) = 0 \text{ implies } (\omega^+) R^+(\mathbf{p}')\mathbf{p}' \le 0.$$

(i) Assume that $R^\circ(\mathbf{p}')\mathbf{p}' + R^\circ(\mathbf{p})\mathbf{p}'' = \mathbf{0}$ and $R^+(\mathbf{p}')\mathbf{p}' + R^+(\mathbf{p})\mathbf{p}'' \le \mathbf{0}$ has a solution. Multiplying by any proper self-stress $\omega^+ \ge \mathbf{0}$, ω°, with $\omega^+ R^+(\mathbf{p}) + \omega^\circ R^\circ(\mathbf{p}) = \mathbf{0}$, we have:

$$\omega^+ R^+(\mathbf{p}')\mathbf{p}' + \omega^+ R^+(\mathbf{p})\mathbf{p}'' + \omega^\circ R^\circ(\mathbf{p}')\mathbf{p}' + \omega^\circ R^\circ(\mathbf{p})\mathbf{p}''$$

$$= \omega^+ R^+(\mathbf{p}')\mathbf{p}' + \omega^\circ R^\circ(\mathbf{p}')\mathbf{p}' + 0 = \omega^+ R^+(\mathbf{p}')\mathbf{p}' + \omega^\circ R^\circ(\mathbf{p}')\mathbf{p}' \le 0.$$

(ii) Assume that for all proper self-stresses $\omega^+ \ge \mathbf{0}$, with $\omega^+ R^+(\mathbf{p}) + \omega^\circ R^\circ(\mathbf{p}) = \mathbf{0}$, we have $\omega^+ R^+(\mathbf{p}')\mathbf{p}' + \omega^\circ R^\circ(\mathbf{p}')\mathbf{p}' \le \mathbf{0}$.

Consider the convex cone, with apex at the origin, generated by positive multiples of the rows $[R^+(\mathbf{p}')\mathbf{p}', R^+(\mathbf{p})]$ and arbitrary multiples of $[R^\circ(\mathbf{p}')\mathbf{p}', R^\circ(\mathbf{p})]$. This cone is the set of vectors

$$(\omega^+)[R^+(\mathbf{p}')\mathbf{p}', R^+(\mathbf{p})] + \omega^\circ[R^\circ(\mathbf{p}')\mathbf{p}', R^\circ(\mathbf{p})].$$

We claim that $[1,0,\ldots,0]$ is not in this cone. If it is, then there is a $\omega^+ \geq 0$, and a ω° such that:

$$[1,0,\ldots,0] = (\omega^+)[R^+(\mathbf{p}')\mathbf{p}', R^+(\mathbf{p})] + \omega^\circ[R^\circ(\mathbf{p}')\mathbf{p}', R^\circ(\mathbf{p})]$$

or $\omega^+ R^+(\mathbf{p}')\mathbf{p}' + \omega^\circ R^\circ(\mathbf{p}')\mathbf{p}' = 1 > 0$ and $\omega^+ R^+(\mathbf{p}) + \omega^\circ R^\circ(\mathbf{p}) = \mathbf{0}$.

This is a contradiction of our assumption.

Since $[1,\mathbf{0}]$ is not in the cone, there is a hyperplane through $[t, \mathbf{u}]$ through the origin with $[R^+(\mathbf{p}')\mathbf{p}', R^+(\mathbf{p})][t, \mathbf{u}]^\dagger \leq \mathbf{0}$, $[R^\circ(\mathbf{p}')\mathbf{p}', R^\circ(\mathbf{p})][t, \mathbf{u}]^\dagger = \mathbf{0}$ and $t = [1,\mathbf{0}][t, \mathbf{u}] > 0$. Dividing by $t > 0$ and setting $\mathbf{p}'' = [\mathbf{u}/t]^\dagger$, we have

$$R^+(\mathbf{p}')\mathbf{p}' + R^+(\mathbf{p})\mathbf{p}'' \leq \mathbf{0} \quad \text{and} \quad R^\circ(\mathbf{p}')\mathbf{p}' + R^\circ(\mathbf{p})\mathbf{p}'' = \mathbf{0}. \qquad \square$$

We call this the **second-order stress test for tensegrity frameworks.** To illustrate this test we examine the second-order rigidity of the convex polygons of Section 16.4.

PROPOSITION 16.26. Connelly&Whiteley (1987)

(i) If a tensegrity framework $G(\mathbf{p})$ on the vertices of a convex polygon, with cables on the exterior and struts on the interior, is rigid then it is second–order rigid. (In fact prestress rigid & globally rigid)

(ii) If a tensegrity framework $G(\mathbf{p})$ on the vertices of a convex polygon, with bars on the exterior and cables on the interior, is second–order rigid then it is infinitesimally rigid.

Proof. (i) If the tensegrity polygon G(**p**) is rigid, then it has a self–stress. By Theorem 16.11, the stress matrix is positive semidefinite, so $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' \geq 0$ for all infinitesimal motions. We also know that Ω has nullity 3, so if $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' = 0$ for some nontrivial infinitesimal flex, this is an affine flex. Such an affine flex implies G(**p**) is not rigid, so we conclude that $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' > 0$ for all nontrivial infinitesimal motions. Theorem 16.25 now gives the second-order rigidity.

(ii) We know that all self–stresses on G(**p**) have a negative semidefinite stress matrix. Therefore every infinitesimal flex **p'** will extend to a second–order flex. The second-order rigidity now implies that the only infinitesimal flexes are trivial infinitesimal flexes. ▯

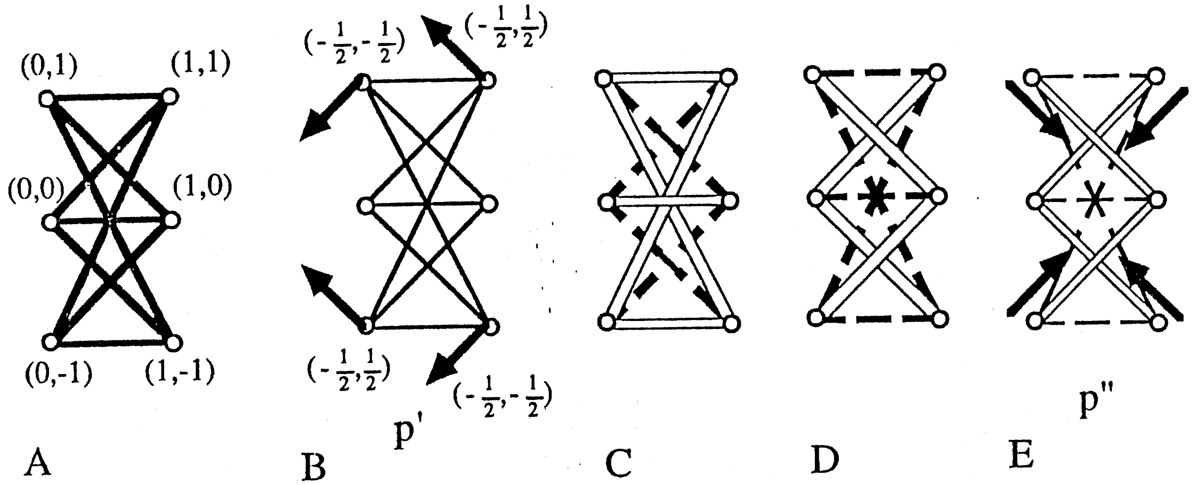


FIGURE 16.14.

EXAMPLE 16.27. Consider the bar framework of Figure 16.14A. Since this is $K_{3,3}$ with the joints on two lines, we know that it has a single self–stress and a nontrivial infinitesimal flex. It is a simple exercise to check that the vectors of Figure 16.14B are a nontrivial infinitesimal flex **p'**, and all nontrivial infinitesimal flexes are a linear combination of this **p'** and the trivial infinitesimal motions. From chapter 5, we know that the self–stress is a product of the affine dependencies of the two sets of collinear points, so the stress matrix has the form:

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

The reader can now check that for this stress matrix and the given infinitesimal motion: $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' = 2 \neq 0$. By Theorem 16.20, this **p'** does not have a second-order extension. Clearly **p'** has a second–order extension if and only if some (all) infinitesimal flexes **p'+p#** which differ by a trivial infinitesimal motion **p#** also have a second-order extension (see Exercise 16.23.) This also holds if we allow non-zero multiples α**p'**. We conclude that no nontrivial infinitesimal motions extend, and the bar framework is infinitesimally rigid. Note that this bar framework is not globally rigid.

If we follow the signs of the stress matrix to create a tensegrity framework (Figure 16.14C), the calculation with the proper stress shows that $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' = 2 > 0$. By Theorem 16.25, this infinitesimal flex still does not have a second–order extension. By the same argument given above, only trivial infinitesimal flexes have a second-order extension, and the tensegrity framework is second–order rigid.

If we reverse the signs in the stress matrix, and follow these new signs for a tensegrity framework (Figure 16.14 D), the calculation gives $\mathbf{p}'^{tr}(-\underline{\Omega})\mathbf{p}'=-2\leq 0$. Since this now holds for all proper self-stresses, we conclude that the tensegrity framework is second-order flexible. Figure 16.14E shows the vectors of a second-order extension **p"** for **p'** on this new framework.

REMARK 16.28. As an intermediate stage between the global rigidity induced by a positive semidefinite stress matrix, and second–order rigidity, we have frameworks for which a single proper self-stress ω gives $\mathbf{p}'^{tr}(+\underline{\Omega})\mathbf{p}'>0$ for all nontrivial infinitesimal flexes. Such a tensegrity framework is called **prestress stable** (Connelly&Whiteley (1987)). Such prestress stable frameworks are the basic kind which are built with tension in the cables or compression in the struts.

This concept can be restated in energy terms. A tensegrity framework $G(\mathbf{p})$ is **prestress stable** if and only if there is a self-stress ω such that the energy function:

$$H(\mathbf{p}', \omega) = \sum \omega_{ij}(\mathbf{p}'_j-\mathbf{p}'_i)^2 + \sum((\mathbf{p}_j-\mathbf{p}_i)\cdot(\mathbf{p}'_j-\mathbf{p}'_i))^2$$
$$= \mathbf{p}'^{tr}[\underline{\Omega} + R(\mathbf{p})^{tr}R(\mathbf{p})]\ \mathbf{p}'$$

is positive semidefinite with only trivial infinitesimal flexes in the kernal. The matrix $R(\mathbf{p})^{tr}R(\mathbf{p})$ is clearly positive semidefinite with only the infinitesimal flexes in the kernal. If a single self-stress ω gives $\mathbf{p}'^{tr}(+\underline{\Omega})\mathbf{p}'>0$ for all nontrivial infinitesimal flexes, then some small multiple $t\omega$ will make $H(\mathbf{p}', t\omega)$ positive semidefinite on all but the trivial infinitesimal motions. For more properties of this stability, see Exercise 16.24, the next remark and Connelly&Whiteley (1987).

REMARK 16.29. If we take gradients of this energy function, we find a static interpretation of prestress stability. The forces that are resolved, at first-order, by a displacement **p'** of the joints are:

$$\mathbf{F} = -\nabla H(\mathbf{p}',\omega) = 2\Sigma\omega_{ij}(\mathbf{p}'_j-\mathbf{p}'_i) + 2\Sigma((\mathbf{p}_j-\mathbf{p}_i).(\mathbf{p}'_j-\mathbf{p}'_i))(\mathbf{p}_j-\mathbf{p}_i)$$
$$= [\Omega+R(\mathbf{p})^T R(\mathbf{p})]\,\mathbf{p}'$$

All equilibrium loads are resolved if and only if the matrix $[\Omega+R(\mathbf{p})^T R(\mathbf{p})]$ is invertible when restricted to the orthogonal complement of the trivial motions. In this case, the displacement **p'** resolves the load $[\Omega+R(\mathbf{p})^T R(\mathbf{p})]\mathbf{p}'$. This is a feasible physical response of the structure, corresponding to positive work by the force, if and only if **p'** is in the same direction as **F** or $\mathbf{p}'.\mathbf{F} \geq 0$, with equality only if $\mathbf{F}=\mathbf{0}$ or **p'** is a trivial infinitesimal flex. This is a restatement of the fact that $H(\mathbf{p}',\omega)$ is positive definite on the complement of the trivial motions.

If $H(\mathbf{p}',\omega)$ is only positive semidefinite on the complement of the trivial infinitesimal motions, then there is a direction $\mathbf{p}^*$ for which there is no change in energy. In the real world there may still be third, or higher order effects of a real energy which produce rigidity. However if $H(\mathbf{p}',\omega)$ is indefinite there is a direction **p'** in which the energy is strictly lower, and the framework is **unstable** for ω in the direction **p'**. Such an instability means that the framework cannot be physically built with ω as a prestress in the cables and struts.

EXERCISES.

16.21. Prove that if a tensegrity framework $G(\mathbf{p})$ is second–order rigid, then removing all struts and cables which have $\omega_{ij}=0$ in all self–stresses leaves a second–order rigid tensegrity framework on the same joints.

16.22. (i) Prove that if a tensegrity framework $G(\mathbf{p})$ is second–order rigid, then the adding a joint in the interior of any cable with $\omega_{ij}>0$ in some self-stress, gives a second–order rigid tensegrity framework.

(ii) Prove that if a tensegrity framework $G(\mathbf{p})$ is second–order rigid, then the adding a joint in the exterior of any strut with $\omega_{ij}<0$ in some self-stress gives a second–order rigid tensegrity framework.

(iii) If a tensegrity framework contains a cable which is zero in all self–stresses, show that the framework formed by splitting this cable in the interior is second-order flexible.

(iv) Prove that if a tensegrity framework $G(\mathbf{p})$ is second–order rigid and there is a strict self-stress, then the **equivalent bar framework**, with all struts and cables replaced by appropriately split bars gives a second-order rigid bar framework.

16.23. (i) Show that for any proper stress ω and any trivial infinitesimal flex $\mathbf{p}^*$ of a tensegrity framework $G(\mathbf{p})$, $\underline{\Omega}\mathbf{p}^*=\mathbf{0}$.

(ii) Use part (i) to show that for any infinitesimal flex $\mathbf{p}'$, any proper stress ω and any trivial infinitesimal flex $\mathbf{p}^*$ of a tensegrity framework $G(\mathbf{p})$, $\mathbf{p}'^{tr}\underline{\Omega}\,\mathbf{p}' \leq 0$ if and only if $(\mathbf{p}'+\mathbf{p}^*)^{tr}\underline{\Omega}\,(\mathbf{p}'+\mathbf{p}^*) \leq 0$.

16.24. Extend Example 16.27 to the following general results (Connelly&Whiteley (1987)):

(i) If a bar framework $G(\mathbf{p})$ is second-order rigid, with a 1–dimensional space of equilibrium motions then for some proper self–stress ω the tensegrity framework following this self-stress is prestress stable.

(ii) If a bar framework $G(\mathbf{p})$ is second-order rigid with a 1–dimension space of self-stresses generated by ω, then for either ω or $-\omega$ the corresponding tensegrity framework is prestress stable.

16.25. (i) Assume that a tensegrity framework $G(\mathbf{p})$ has an independent set of bars, and a one dimensional cone of self–stresses. Show that if $G(\mathbf{p})$ and $G^{-}(\mathbf{p})$ are both second–order rigid, then they are infinitesimally rigid.

(ii) Give an example of a plane tensegrity framework $G(\mathbf{p})$ which is second-order rigid, and its reverse framework $G^{-}(\mathbf{p})$ is second–order rigid, but neither is infinitesimally rigid.

How about a bar framework?

16.26. For a tensegrity framework $G(\mathbf{p})$ an **extreme** set of self–stresses is a finite set $\{ \omega_i \}$ such that every self–stress is a non-negative combination of the ω_i. Similarly, an extreme set of infinitesimal equilibrium flexes is a set which generates the cone of equilibrium infinitesimal flexes as non–negative linear combinations.

(i) Show that an infinitesimal motion $\mathbf{p}'$ of a tensegrity framework has a second–order extension if and only if

$p'^{tr}\underline{\Omega}_i p' \leq 0$ for all self-stresses in a fixed extreme set of self-stresses $\{ \omega_i \}$.

(ii) Show, by an example, that a second–order flexible framework may satisfy $p'^{tr}_i \underline{\Omega}_j p'_i > 0$ for an extreme set of equilibrium infinitesimal motions p'_i and a set of proper self–stresses ω_j.

16.27. (i) If $G(\mathbf{p})$ is second–order rigid, show there is a $\delta>0$ such that $G(T(\mathbf{p}))$ is second–order rigid for any projective transformation T with $|T(\mathbf{p})-\mathbf{p}|<\delta$.

(ii) Give an example of a framework $G(\mathbf{p})$ which is second–order rigid, but $G(T(\mathbf{p}))$ is flexible for some projective transformation T.

16.7. Structures in 3-space.

The rigid spatial frameworks in Fuller (1975), Snelson (1975), Pugh (1976), Motro (1983) etc. continue to pose a problem. The known patterns from the previous sections and previous chapters: the infinitesimal rigidity, the generic rigidity, the spider webs, the convex polygons, do not apply directly.

These experimental tensegrity frameworks in space are special position, rigid structures which fit the pattern: the vertices of a convex polyhedron are joined by cables for the edges of the polyhedron and struts are inserted in the interior to force an equilibrium self–stress. They are not, in general, infinitesimally rigid. Motro (1983) shows the prestress stability of a few of these spatial patterns. Hindrich (1984) shows that some simple patterns, based on symmetric prisms, are rigid, and Connelly (1982) applies the methods of Sections 3 and 4 to show the global rigidity of a few examples.

It is probable that all of these experimental examples are prestress rigid, and many are globally rigid with positive semidefinite stress matrices. However no general qualitative arguments, like those of the Section 16.4, have worked for the larger class.

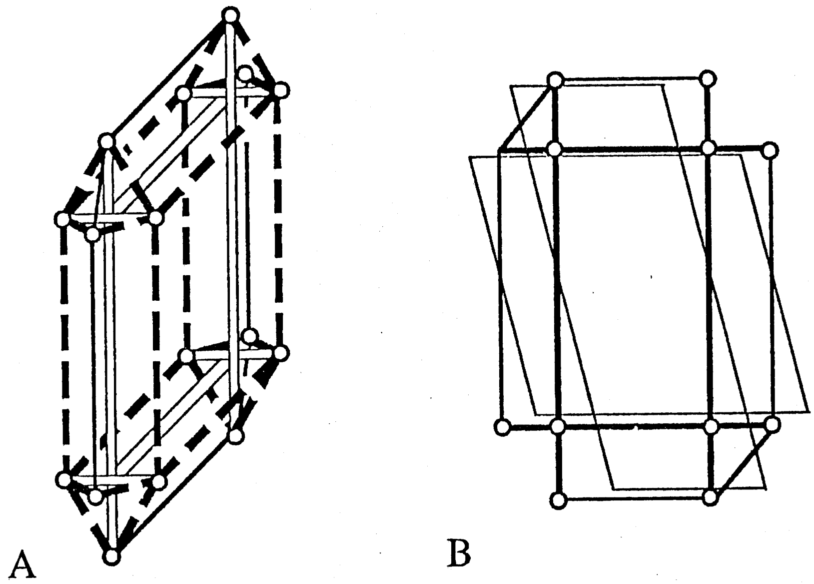
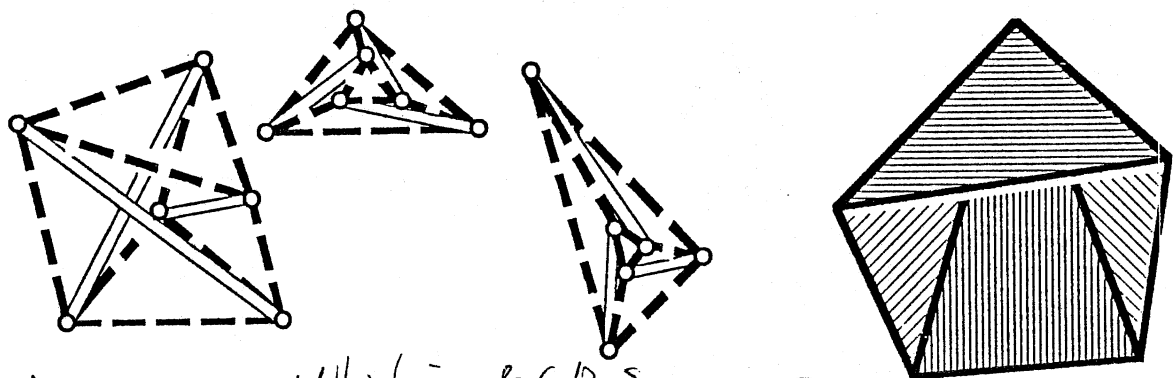


FIGURE 16.15.

Proposition 16.7 still guarantees that a spatial tensegrity framework with a positive semidefinite stress matrix of nullity 4 has at most affine flexes. Figure 16.15, adapted from Grünbaum (private communication), shows a basic counter–example to the use of this result for the rigidity (global rigidity) of the entire class. Figure 16.15A has a nontrivial motion based on an affine motion (Figure B). Since the self–stress is the sum of the four self–stresses in the four planes, it is positive semidefinite. (This self–stress is not strict, and the cables not in the self–stress are drawn with thin lines.) However, the interior struts are not enough to block all affine flexes.

Moreover, a local projective transformation will bring the point of intersection of the 4 "hinges" in from infinity, leaving a flex which is not an affine motion (Figure C). This confirms that the underlying stress–matrices of both examples have nullity >4.



A Which is BCD? B

FIGURE 16.16.

Figure 16.16 illustrates a second counterexample, with a strict self–stress with cables on the graph of a convex polyhedron. This is formed by using four versions of the basic "simplex" framework of Figures A,B,C on the four triangular faces of the pattern of Figure D. Each piece is itself rigid, with a positive semidefinite reduced stress matrix of nullity 4 (see Exercise 16.27). However the combined framework has the finite motion of the underlying cycle of 4 triangular panels. Although the stress matrix is positive semidefinite, the stress matrix has nullity >4.

These examples prevent any simple extension of Corollary 16.12. However, they suggest that the stress matrix is positive semidefinite.

CONJECTURE 16.30. A proper self–stress on a tensegrity framework, built on the vertices of a convex set with cables on the boundary of the set and struts in the interior, has a positive semidefinite stress matrix. in faces?

With the techniques of section 6 and Connelly&Whiteley (1987), this conjecture would imply:

CONJECTURE 16.31. If a tensegrity framework, built on the vertices of a convex set, with struts on the boundary of the set and cables on the interior, is rigid then it is infinitesimally rigid and the reverse framework with cables on the boundary and struts on the interior is infinitesimally rigid. in $\mathbb{R}^3$??

We know of two classes of spatial frameworks which have a positive semidefinite stress matrix of nullity 4 (see Exercises 16.15, 28). However we do not have a class of frameworks for which all self-stresses have reduced stress–matrices of nullity 4.

We close with a variant of a conjecture first offered by Grünbaum&Shephard (1975).

CONJECTURE 16.32. If a convex polyhedron in 3-space is built with a plane-rigid tensegrity framework on the natural vertices of each face, the spatial framework is rigid.

EXERCISES.

16.28. (i) Show that the stress matrix of the example in Figure 16.16A is positive semidefinite of nullity 4.

16.29. If a convex polyhedron, with a 3-valent vertex v_0 is built with cables on all edges except the triangle around the 3–valent vertex, and struts on this triangle as well as from v_0 to all vertices off this triangle, show that any proper self–stress is positive semidefinite of nullity 4. (In chapter 12 we saw that this framework has a strict self–stress.)

16.30. (i) Show that if a convex polyhedron is realized as the cables of a tensegrity framework, and one interior vertex is added, joined to each vertex by a strut, the resulting tensegrity framework is not globally rigid.

(ii) CONJECTURE: If a convex polyhedron is realized as the cables of a tensegrity framework, and one interior vertex is added, joined to each vertex by a strut, the resulting tensegrity framework is prestress rigid. (Compare to Example 10.16 for triangulated convex polyhedra.)

BIBLIOGRAPHY

Calladine (1978) C.R. Calladine, Buckminster Fuller's "tensegrity structures" and Clerk Maxwell's rules for the construction of stiff frames, *Int. J. Solids and Structures* 14, 161-172.

Cauchy (1831) A. Cauchy, Deuxiéme memoire sur les polygons et les polyédres, *J. École Polytechnique* XVIe Cahier, 87-98.

Connelly (1980) R. Connelly, The rigidity of certain cabled frameworks and the second order rigidity of arbitrarily triangulated convex surfaces, *Adv. in Math* 37, 272-298.

Connelly (1982) R. Connelly, Rigidity and energy, *Inventiones mathematicae* 66 , 11-33.

Connelly&Whiteley (1987) R. Connelly and W. Whiteley, Prestress and second-order rigidity for tensegrity frameworks, preprint, Champlain Regional College, 900 Riverside Drive, St. Lambert, Québec, J4P-3B8.

Crapo&Whiteley (1986) H. Crapo and W. Whiteley, Plane stresses and projected polyhedra, preprint Champlain Regional College, 900 Riverside Drive, St. Lambert, Québec, J4P-3B8.

Fuller (1975) R.B. Fuller, Synergetics: Explorations in the geometry of thinking, Macmillan, New York.

Grünbaum (1967) B. Grünbaum, *Convex Polytopes* , Interscience, New York.

Grünbaum&Shephard (1975) B. Grünbaum and G. Shephard, Lectures in lost mathematics, Mimeographed notes, Univ. of Washington.

Hindrichs (1984) L. Hindrich, Prismic tensegrids, *Structural Topology* 9, 3-14.

Motro (1983) Formes et Forces dans les Systèmes Constructifs. Cas des Systèmes Réticulés Autocontraints, Thèse d'Etat Université des Sciences et Technologique du Langedoc, Montpellier, France.

Pellegrino&Calledine (1986) S. Pellegino and C.R. Calledine, Matrix analysis of statically and kinematically indeterminate frameworks ,*I. Int J. Solids and Structures* 22, 409–428.

Pugh (1976) A. Pugh, *Introduction to Tensegrity*, University of California Press.

Rockafellar (1970) R.T. Rockafellar, *Convex analysis*, Princton University Press, Princeton N.J..

Roth (1981) B. Roth, Rigid and flexible frameworks, *Amer. Math. Monthly* 88, 6-21.

Roth&Whiteley (1981) B. Roth and W. Whiteley, Tensegrity frameworks, *Trans. Amer. Math. Soc.* 265, 419-446.

Tutte (1964) W.T. Tutte, How to draw a graph, *Proc. London Math. Soc.* (3) 13, 743-68.

White&Whiteley (1983) N. White and W. Whiteley, The algebraic geometry of stresses in frameworks, *SIAM J. Alg. & Disc. Methods* 4, 481-511.

Whiteley (1984) W. Whiteley, Infinitesimally rigid polyhedra I: statics of frameworks, *Trans. Amer. Math. Soc.* 285, 431-465.

Whiteley (1985) W. Whiteley, Infinitesimally rigid polyhedra II: modified spherical frameworks, *Trans. Amer. Math. Soc.* to appear.

Author's Address

Champlain Regional College
900 Riverside Drive
St. Lambert, Québec
J4P–3P2 CANADA