M. Terrell

May 1987

Chapter 3: The Basic Concepts of Static Rigidity

Robert Connelly

1. Introduction

Yet another form of rigidity is the notion of static rigidity, a concept fimiliar to structural engineers. If vectors are assigned to the joints of a framework in such a way that their sum is zero and such that there is no net angular velocity, then we say that this set of vectors, now thought of as forces, is in equilibrium. Such a set of equilibrium forces is said to be resolved if there are scalars that can be assigned to each bar such that at each joint the vector sum of the scalars times the bars is equal to the given force at that joint. The framework is called staticly rigid if every set of equilibrium forces can be resolved. The fundamental result of this chapter is that static and infinitesimal rigidity are duals, and thus they are equivalent.

We also investigate which transformations of the ambient space $\mathbb{R}^d$ preserve static rigidity. It turns out that projective transformations are the natural objects for this property.

2. Equilibrium Forces

Dual to the notion of infinitesimal rigidity is static rigidity. But as with infinitesimal rigidity we must take into account the motions of the ambient space $\mathbb{R}^d$.

Recall that the trivial infinitesimal flexes $\mathbf{p}'$ for a configuration $\mathbf{p}$ are

$$T_{\mathbf{p}} = \{ \mathbf{p}' \in \mathbb{R}^{vd} \mid \mathbf{p}'_i = S\mathbf{p}_i + \mathbf{p}_0,\ i = 1,\ldots, v,\ S^T = -S,\ \mathbf{p}_0 \in \mathbb{R}^{vd} \}.$$

We define the <u>equilibrium</u> <u>forces</u> at $\mathbf{p}$ as

$$E_{\mathbf{p}} = \{ \mathbf{F} \in \mathbb{R}^{vd} \mid \mathbf{F}\cdot\mathbf{p}' = 0,\ \mathbf{p}' \in T_{\mathbf{p}} \} = T_{\mathbf{p}}^{\perp}.$$

To understand these better we consider some alternate descriptions of $E_{\mathbf{p}}$. Note that $T_{\mathbf{p}}$ is the internal direct sum $R_{\mathbf{p}} \oplus D_{\mathbf{p}}$, where

$$R_{\mathbf{p}} = \{ \mathbf{p}' \in \mathbb{R}^{vd} \mid \mathbf{p}'_i = S\mathbf{p}_i,\ i = 1,\ldots, v,\ S = -S^T \},$$

$$D_{\mathbf{p}} = \{ \mathbf{p}' \in \mathbb{R}^{vd} \mid \mathbf{p}' = (\mathbf{p}'_0,\ldots, \mathbf{p}'_0),\ \mathbf{p}'_0 \in \mathbb{R}^d \}.$$

$R_{\mathbf{p}}$ is the set of <u>infinitesimal rotations</u> of $\mathbf{p}$, and $D_{\mathbf{p}}$ is the set of <u>infinitesimal translations</u> of $\mathbf{p}$. Thus $E_{\mathbf{p}} = R_{\mathbf{p}}^{\perp} \cap D_{\mathbf{p}}^{\perp}$.

Suppose $\mathbf{F} = (\mathbf{F}_1,\ldots, \mathbf{F}_v) \in D_{\mathbf{p}}^{\perp}$. Then

(3.1) $$\sum_{i=1}^{v} \mathbf{F}_i\cdot\mathbf{p}'_0 = 0, \text{ for all } \mathbf{p}'_0 \in \mathbb{R}^{vd}.$$

Thus

$$D_{\mathbf{p}}^{\perp} = \{ \mathbf{F} \in \mathbb{R}^{vd} \mid \sum_{i=1}^{v} \mathbf{F}_i = 0 \}.$$

Thinking of $\mathbf{F}$ as velocities as in physics, we see that $D_{\mathbf{p}}^{\perp}$ is the set of those velocities that preserve the <u>center of gravity</u>

$$\bar{\mathbf{p}} = \frac{1}{v}\sum_{i=1}^{v} \mathbf{p}_i \in \mathbb{R}^d.$$

Suppose $\mathbf{F} \in R_{\mathbf{p}}^{\perp}$. Then

$$\sum_{i=1}^{v} \mathbf{F}_i\cdot S\mathbf{p}_i = 0, \text{ for all } S = -S^T.$$

One way to interpret this formula is with exterior calculus. Regard S as an exterior $n - 2$ form so that $Sp_i \equiv S \wedge p_i$ is an $n - 1$ form. So $F_i \cdot Sp_i = F_i \wedge S \wedge p_i = - F_i \wedge p_i \wedge S$. Thus $F \in R_p^{\perp}$ if and only if

$$\sum_{i=1}^{v} F_i \wedge p_i = 0. \tag{3.2}$$

For $n = 3$ this is the same thing as

$$\sum_{i=1}^{v} F_i \times p_i = 0,$$

where $\times$ is the usual cross product.

We also give an explicit description of equation (3.2) without the use of wedge or cross products, but using the coordinates of p instead. For $i = 1, \ldots, v$, let

$$p_i = (x_{1i}, x_{2i}, \ldots, x_{di})^T$$

and

$$F_i = (y_{1i}, y_{2i}, \ldots, y_{di})^T.$$

For $j,k = 1, \ldots, d$ let S_{jk} be the skew symmetric matrix with entry j,k equal to 1, entry k,j equal to -1, and all the other entries equal to 0. Then

$$F_i \cdot S_{jk} p_i = y_{ji} x_{ki} - y_{ki} x_{ji}.$$

Since skew symmetric matrices S_{jk} generate all skew symmetric matrices, we see that the following relations are equivalent to the equations (3.2).

$$\sum_{i=1}^{v} F_i \cdot S_{jk} p_i$$

$$= \sum_{i=1}^{v} y_{ji} x_{ki} - y_{ki} x_{ji} = 0, \text{ for all } j,k = 1, \ldots, d. \tag{3.3}$$

Thus these equations (3.3) and the following (3.4) are equivalent to $\mathbf{F}$ being an equilibrium force at $\mathbf{p}$.

(3.4)
$$\sum_{i=1}^{v} \mathbf{F}_i = 0.$$

We now can see that the following Proposition is an immediate consequence of equations (3.3) and (3.4). This was the result that was delayed from the second proof of Proposition 2.47 that was used for the second proof of Theorem 2.54.

Proposition 3.1: The following set

$$\{ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{vd} \times \mathbb{R}^{vd} \mid \mathbf{q} \text{ is an equilibrium force at } \mathbf{p} \}$$

is a closed subset of $\mathbb{R}^{vd} \times \mathbb{R}^{vd}$.

Remark 3.2: Note that the pair of conditions (3.3) and (3.4) are independant of the coordinate system used, since the notion of trivial infinitesimal flex is independant. Also it is easy to check that (3.4) alone does not depend on the coordinate system either. However, condition (3.3) alone, if condition (3.4) does not hold, does depend on the coordinate system.

In physics if a system of particles moves in $\mathbb{R}^d$ in such a way that (3.4) holds, then it is said that linear momentum is preserved. This is the same as saying that the center of gravity $\overline{\mathbf{p}}$ is preserved, as mentioned before. If in addition (3.3) holds as well, then it is said that angular momentum is preserved. In this case there is no quantity which is a function of the coordinates of the configuration space alone, which is preserved. There is no "center of angular momentum".

Problems:

Problem 3.1: Show that if the configurations p and q are congruent by a congruence $h: \mathbb{R}^d \to \mathbb{R}^d$, and $F = (F_1, \ldots, F_v)$ is an equilibrium force at p, then $hF = (hF_1, \ldots, hF_v)$ is an equilibrium force at $q = h(p) = (hp_1, \ldots, hp_v)$. If F only satisfies (3.4), show that hF satisfies (3.4) at $h(p) = q$.

Problem 3.2: Show that if only (3.3) holds, then by changing the coordinate system, (3.3) may not hold.

Problem 3.3: Find a motion for a configuration $p(t)$ of particles in $\mathbb{R}^d$, $0 \leq t \leq 1$, such that linear and angular momentum is preserved, $p(0)$ is congruent to $p(1)$, but $p(0) \neq p(1)$. For $p(0)$ fixed, for any $\epsilon > 0$, show that it can be arranged that $|p(t) - p(0)| < \epsilon$. Use this to show that there is no smooth function $f: \mathbb{R}^{vd} \to \mathbb{R}^1$ such that for all motions $p(t)$ linear and angular momentum is preserved if and ond only if $f(p(t))$ is constant.

3. Examples and Calculations of Equilibrium Forces

We present some methods which can be helpful for determining when a given force F is in equilibrium at p.

We observe that $p_i \wedge F_i = (p_i + tF_i) \wedge F_i$ when p_i is replaced by $p_i + tF_i$, where t is any scalar since $F_i \wedge F_i = 0$. Geometrically this means that the angular momentum (3.2) remains the same if the point p_i at which the force F_i is applied is moved along the line in the direction of the force. The same force is applied at the new position. See Figure 3.1.

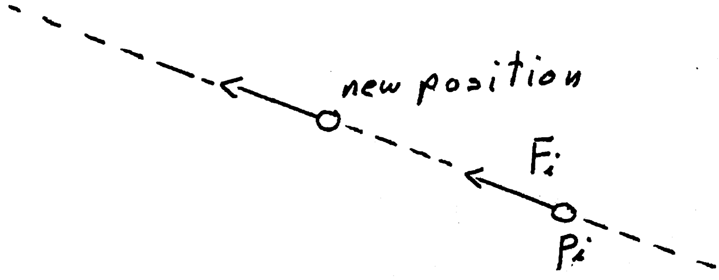

Figure 3.1

In the plane this observation can help calculate when $\mathbf{F}$ is in equilibrium.

Recall that in $\mathbb{R}^2$ $(x_1, x_2) \wedge (y_1, y_2) = x_1 y_2 - x_2 y_1$.

<u>Example 3.2</u>: Suppose that $v = 3$ and $d = 2$. In other words, we have three points in the plane $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, $\mathbf{p}_i \in \mathbb{R}^2$. Suppose also that the points are not colinear. Consider $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$, $\mathbf{F}_i \in \mathbb{R}^2$ as three forces at the three points. Suppose that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$.

<u>Case 1</u>: The lines through $\mathbf{p}_i$ in the direction of $\mathbf{F}_i$ meet at the point O.

We take O as the origin and observe that $\mathbf{F}_i \wedge \mathbf{p}_i = 0$ for each i. Thus $\mathbf{F}$ is clearly an equilibrium force at $\mathbf{p}$.

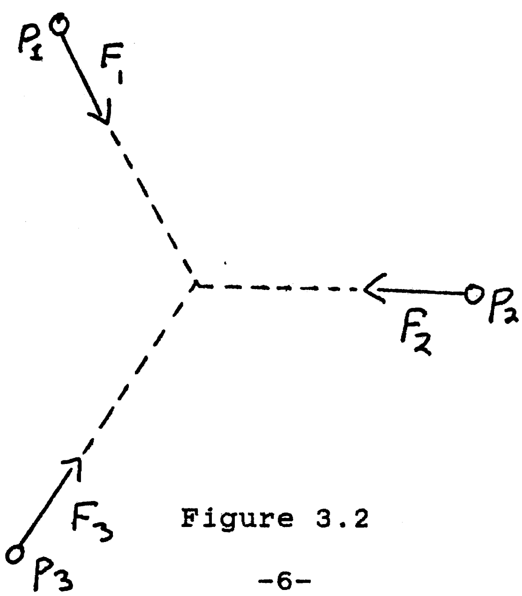

Figure 3.2

Note that the same conclusion holds if F_1, F_2, and F_3 are parallel.

<u>Case 2</u>: The lines through the p_i do not meet at a point and are not parallel.

We take the intersection of one pair of lines which are not parallel as the origin. Say the lines through p_1 and p_2 intersect at 0, the origin. Then $F_1 \wedge p_1 = F_2 \wedge p_2 = 0$, but $F_3 \wedge p_3 \neq 0$. So F is never in equilibrium in this case.

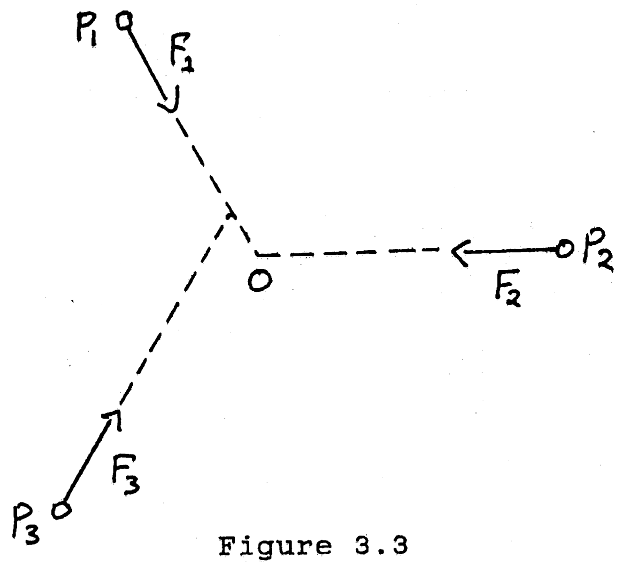

Figure 3.3

<u>Example 3.3</u>: Suppose we have four non-colinear points $p = (p_1, p_2, p_3, p_4)$ in the plane (so $v = 4$, $d = 2$), and four forces $F = (F_1, F_2, F_3, F_4)$ such that $F_1 + F_2 + F_3 + F_4 = 0$. Then we can move F_1 and F_2, say, to the intersection of the lines through p_1 and p_2. We can then do the same thing for F_3 and F_4. Then we see that F is in equilibrium at p if and only if $F_1 + F_2 = -(F_3 + F_4)$ is parallel to the line through the two points of intersection. See Figure 3.4.

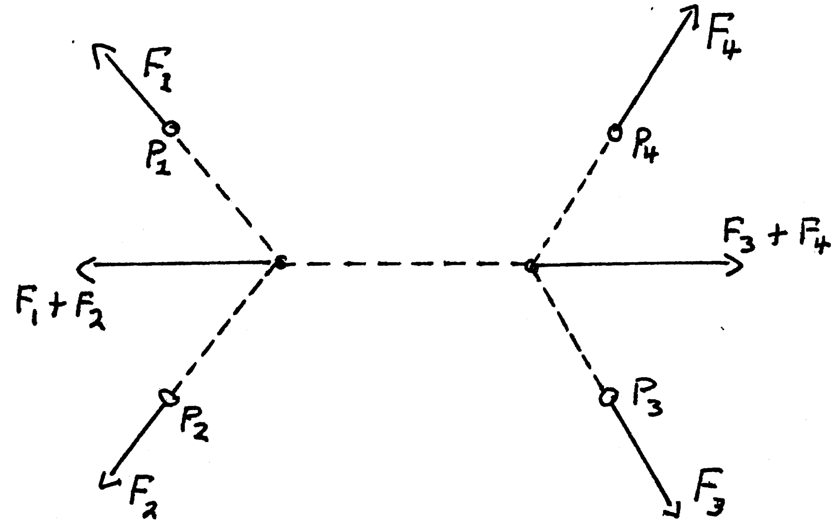

Figure 3.4

A similar analysis can be done with any configuration and force in the plane.

<u>Example 3.4</u> Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_v)$ be any configuration in $\mathbb{R}^d$, and let $i \neq j$. Then define a force $\mathbf{F} = \mathbf{F}(i,j)$ at $\mathbf{p}$ by

$$\mathbf{F}_k = \begin{cases} \mathbf{p}_i - \mathbf{p}_j & k = i \\ \mathbf{p}_j - \mathbf{p}_i & k = j \\ 0 & \text{otherwise} \end{cases} .$$

Thus

$$\mathbf{F} = (0, \ldots, 0, \mathbf{p}_i - \mathbf{p}_j, 0, \ldots, 0, \mathbf{p}_j - \mathbf{p}_i, 0, \ldots, 0).$$

It is easy to see that $\mathbf{F}$ is an equilibrium force, which we call the <u>elementary equilibrium force between</u> $\mathbf{p}_i$ and $\mathbf{p}_j$. See Figure 3.5

Figure 3.5

Note that any other linear combination of such elementary equilibrium forces is an equilibrium force. We will see later that if the affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$, then any equilibrium force is such a linear combination of elementary forces.

Note that $\mathbf{F}(i, j)$ is the $\{i,j\}$-th row of the rigidity matrix $R(\mathbf{p})$ defined in section 15 of Chapter 2.

4. Stresses

Let $G(\mathbf{p})$ be any tensegrity framework in $\mathbb{R}^d$ with e members. A <u>stress</u> $\omega = (\ldots, \omega_{ij}, \ldots) \in \mathbb{R}^e$ is an assignment of a scalar $\omega_{ij} = \omega_{ji}$ for each member of $\{i,j\}$ of G. We say that ω is <u>proper</u> if $\omega_{ij} \geq 0$ for $\{i,j\}$ a cable of G, and $\omega_{ij} \leq 0$ for $\{i,j\}$ a strut of G. If $\{i,j\}$ is a bar of G, then ω_{ij} can have any value.

Let $\mathbf{F}$ be any equilibrium force at $\mathbf{p}$. We say that a stress ω for $G(\mathbf{p})$ resolves $\mathbf{F}$ if for each (variable) vertex i of G,

$$\mathbf{F}_i + \sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0, \tag{3.5}$$

where the sum is taken over all j such that $\{i,j\}$ is a member of G. In case G has fixed vertices, there is no condition as in (3.5).

The language here is taken from structurial engineering, where $\mathbf{F}$ as above is called an <u>equilibrium</u> load on $G(\mathbf{p})$. For example, imagine a truck in the middle of a bridge as in Figure 3.6. The weight of the truck acts a "load" on the bridge, which is regarded as a bar framework. Of course, the points where the bridge is attched to the ground have a corresponding force vector as well, and these forces must be such that together they form an equilibrium load.

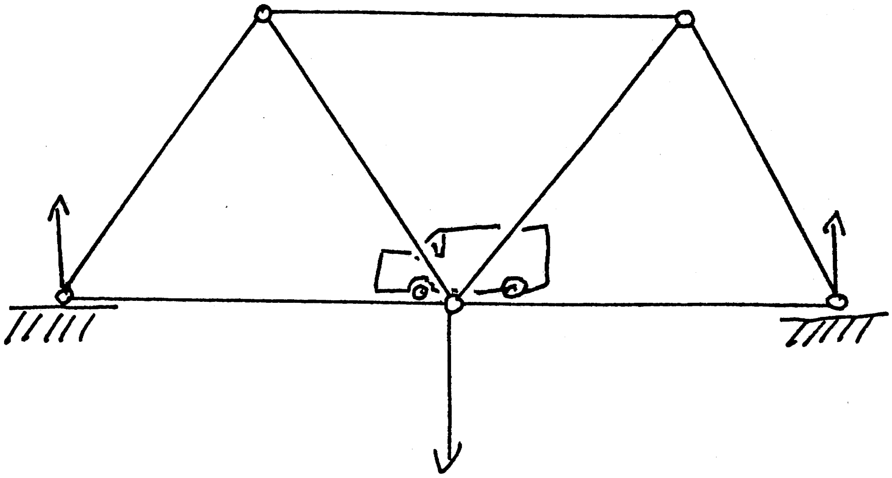

Figure 3.6

Naturally one would hope that the bridge will resolve all such "reasonable" loads. The sign of the stress ω_{ij} on the member $\{i,j\}$ indicates whether the member is in tension or compression, so we would hope that the resolving stress would be proper as well.

We must be careful, however, since the definition of a stress here and the way we use the concept are different from what the word means in the engineering literature. The engineering definition of stress in a member is the magnitude of the force per unit area of a cross section of the member. Thus if t_{ij} represents the "engineering stress" in member $\{i,j\}$, and A_{ij} represents the cross sectional area of member $\{i,j\}$, then

$$\omega_{ij}(p_i - p_j) = t_{ij}A_{ij}\,\frac{(p_i - p_j)}{|p_i - p_j|}.$$

Note that the "engineering stress" t_{ij} is in units of force/area, Newtons per square meter, say. Our stress ω_{ij} is in units of force/length, Newtons per meter, say.

Of course t_{ij} is more useful for deciding, say, the appropriate materials and member size for resolving any given set of loads, but such a definition is inconvenient mathematically since it does not have agreeable linear properties for our analysis.

Remark 3.5: Note that in our definition of when a stress ω for a framework $G(p)$ resolves a load $\mathbf{F}$, it must turn out that $\mathbf{F}$ is in equilibrium. To see this observe that if we define

$$\tilde{\mathbf{F}}_i = \sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i), \quad i = 1, 2, \ldots, v,$$

then

$$\tilde{\mathbf{F}} = (\tilde{\mathbf{F}}_1, \ldots, \tilde{\mathbf{F}}_v) = -\sum_{i,j} \omega_{ij}\mathbf{F}(i,j).$$

Thus any stress ω for $G(p)$ defines a force in equilibrium. Hence in the definition we need only consider equilibrium forces, since if a force $\mathbf{F}$ is resolved by ω, then $\mathbf{F} = \tilde{\mathbf{F}}$ as above, and thus $\mathbf{F}$ is an equilibrium force.

On the other hand, if there are pinned joints whose affine span is all of $\mathbb{R}^d$, then we do not include the equilibrium equation at those points, and we do not require that the force on the other (variable) points be in equilibrium. We can always apply forces to the pinned points in such a way that the whole force is put in equilibrium.

5. Static Rigidity

Our basic definition is the following. A tensegrity framework $G(p)$ in $\mathbb{R}^d$ is called staticly rigid if every equilibrium force $\mathbf{F}$ at $\mathbf{p}$ can be resolved by a proper stress ω.

Regard ω as a row vector in $\mathbb{R}^{e}$. Recalling the rigidity matrix $R(\mathbf{p})$ of section 15, chapter 2, we see that

$$\omega\, R(\mathbf{p}) = (\ldots, \omega_{ij}, \ldots) \left[\ \ldots, (\mathbf{p}_i - \mathbf{p}_j)^T, \ldots, (\mathbf{p}_j - \mathbf{p}_i)^T, \ldots \right]$$

$$= (\ldots, \sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j), \ldots) \in \mathbb{R}^{vd}.$$

Thus regarding a force $\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_v)$ as a row vector, ω resolves $\mathbf{F}$ if and only if

$$\mathbf{F} + \omega\, R(\mathbf{p}) = \mathbf{0}.$$

By taking the transpose of this equation and regarding the equilibrium forces as column vectors now, we can rephrase the defintion of static rigidity. Recall that $E_{\mathbf{p}}$ is the set of equilibrium forces at $\mathbf{p}$.

Proposition 3.6: A bar framework $G(\mathbf{p})$ is staticly rigid if and only if

$$\text{Image } R(\mathbf{p})^T = E_{\mathbf{p}}.$$

We now have the following basic equivalence.

Theorem 3.7: A bar framework $G(\mathbf{p})$ in $\mathbb{R}^d$ is infinitesimally rigid if and only if $G(\mathbf{p})$ is staticly rigid in $\mathbb{R}^d$.

Proof: From Proposition 3.6 $G(\mathbf{p})$ is staticly rigid if and only if

$$\text{Ker } R(\mathbf{p}) = [\text{Image } R(\mathbf{p})^T]^{\perp} = E_{\mathbf{p}}^{\perp} = T_{\mathbf{p}},$$

where $T_{\mathbf{p}}$ is the set of trivial infinitesimal flexes at $\mathbf{p}$, and $(\ )^{\perp}$ denotes the orthogonal conplement. By Proposition 2.33 of Chapter 2, this is precisely the condition for infinitesimal rigidity.

Remark 3.8: The same result is true for tensegrity frameworks and will be shown later in Chapter xx. The proof is very similar to the proof above, except that the notion of dual cones replaces the notion of perpendicular subspaces.

Note also that we needed the configuration space $\mathbb{R}^{vd}$ to be finite dimensional only in the proof that infinitesimal rigidity implies static rigidity. If $G(p)$ is staticly rigid and p' is an infinitesimal flex of $G(p)$, then by subtracting a trivial flex we may assume that p' satisfies the equilibrium conditions (3.3), (3.4). Then there is a stress ω such that $\omega R(p) = p'$, and

$$p' \cdot p' = \omega R(p) p' = \omega 0 = 0.$$

Thus p' is trivial and $G(p)$ is infinitesimally rigid.

6. Other Formulations of Static Rigidity

Instead of considering every possible equilibrium force in the definition of static rigidity, under most circumstances, it is enough to consider only the elementary equilibrium forces. The following follows from Theorem 3.7.

Corollary 3.9: Let p be any configuration of v points in $\mathbb{R}^d$ whose affine span is $\mathbb{R}^d$ or which are affine independent. Then

$$\{ \mathbf{F}(i, j) \mid 1 \le i < j \le v \}$$

generates E_p.

Proof: Let G be the bar graph with bars between all pairs of distinct vertices. Then by Corollary 2.24 and Proposition 2.20 $G(p)$ is infinitesimally rigid in $\mathbb{R}^d$, and thus $G(p)$ is staticly rigid. The span of the $\mathbf{F}(i, j)$ is clearly the image of $R(p)^T$ and by Proposition 3.6, this image is all of E_p.

We now wish to get even more precise information about the relation between infinitesimal flexes and equilibrium forces.

Proposition 3.10: Let $G(\mathbf{p})$ be any bar framework in $\mathbb{R}^d$, and let i, j be any pair of vertices of G. Then $\mathbf{F}(i, j)$ cannot be resolved if and only if there is an infinitesimal flex $\mathbf{p}'$ of $G(\mathbf{p})$ such that $(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \neq 0$.

Proof: Suppose that $\mathbf{p}'$ is any infinitesimal flex of $G(\mathbf{p})$ such that $(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \neq 0$. Recall that $R(\mathbf{p})\mathbf{p}' = 0$. Let $\omega \in \mathbb{R}^e$ be any stress. Then

$$
\begin{aligned}
[\omega\, R(\mathbf{p}) + \mathbf{F}(i, j)]\, \mathbf{p}' &= \omega\, R(\mathbf{p})\mathbf{p}' + \mathbf{F}(i, j)\mathbf{p}' \\
&= \mathbf{F}(i, j)\mathbf{p}' \\
&= [\ldots\ (\mathbf{p}_i - \mathbf{p}_j)^T, \ldots, (\mathbf{p}_j - \mathbf{p}_i)^T \ldots] \begin{bmatrix} \mathbf{p}'_1 \\ \vdots \\ \mathbf{p}'_i \\ \vdots \\ \mathbf{p}'_j \\ \vdots \end{bmatrix} \\
&= (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \\
&\neq 0.
\end{aligned}
$$

Thus $\omega\, R(\mathbf{p}) + \mathbf{F}(i, j) \neq \mathbf{0}$, and no ω can resolve $\mathbf{F}(i, j)$.

Conversly suppose that $\mathbf{F}(i, j)$ cannot be resolved. The image of $R(\mathbf{p})^T$ and the kernel of $R(\mathbf{p})$ are complementary orthogonal subspaces in $\mathbb{R}^{vd}$, and $\mathbf{F}(i, j)$ is not in the image. Thus there is a vector $\mathbf{p}'$ in the kernel of $R(\mathbf{p})$, hence an infinitesimal flex of $G(\mathbf{p})$, such that

$$\mathbf{F}(i, j)\cdot\mathbf{p}' = (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \neq 0.$$

Remark 3.11: Note that Proposition 3.10 implies Theorem 3.7 in the case when the affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$, which of course includes all the cases of infinitesimally rigid frameworks except bar simplices. The equivalence of infinitesimal and static rigidity is so fundamental that we have included both points of view.

It will be shown later in Chapter xx that the equivalence also holds for tensegrity frameworks as well. Orthogonal complements are replaced by dual cones.

7. Self Stresses

We can use our new notion of static rigidity to provide other methods of calculating infinitesimal rigidity. Dual to the notion of an infinitesimal flex is the notion of a self stress. Let $G(\mathbf{p})$ be any framework, and let $\omega \in \mathbb{R}^d$. We say that ω is a <u>self stress</u> for $G(\mathbf{p})$ if $\omega R(\mathbf{p}) = 0$, where $R(\mathbf{p})$ is the rigidity matrix for $G(\mathbf{p})$. In other words ω resolves the 0 load. At the level of vectors we have

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0,$$

for vertices i of G, and the sum is taken over all vertices j such that $\{i,j\}$ is a member of G.

Note that the set of all self stresses for a fixed framework $G(\mathbf{p})$ in $\mathbb{R}^d$ is a linear subspace of $\mathbb{R}^e$. In fact this set of self stresses can be regarded as the orthogonal complement of the column space of $R(\mathbf{p})$. Thus the dimension of the space of self stresses can be used to determine the static (or infinitesimal) rigidity of $G(\mathbf{p})$ as follows.

<u>Proposition 3.12</u>: Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$ with v vertices and e bars such that the affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$. Let s be the dimension of self stresses of $G(\mathbf{p})$. Then

$$s \geq e - [vd - \frac{d(d+1)}{2}],$$

and equality holds if and only if $G(\mathbf{p})$ is staticly rigid (or equivalently infinitesimally rigid).

Proof: Since the dimension of the column space of $R(p)$ is the rank of $R(p)$, the discussion before the Proposition implies that

$$s + \mathrm{rank}(R(p)) = e.$$

But Remark 2.34 of Chapter 2 implies that

$$\mathrm{rank}(R(p)) \le vd - \frac{d(d+1)}{2},$$

with equality if and only if $G(p)$ is infinitesimally rigid. Thus

$$s = e - \mathrm{rank}(R(p)) \ge e - [vd - \frac{d(d+1)}{2}],$$

with equality if and only if $G(p)$ is infinitesimally rigid.

Remark 3.13: A very useful consequence of the above Proposition is that if G has $e = vd - \frac{d(d+1)}{2}$, then $G(p)$ is staticly rigid if and only if $G(p)$ has only the 0 self stress (assuming the affine span of p is $\mathbb{R}^d$ of course).

It is also useful to observe that if we have the equilibrium equation for a self stress for a vertex i

$$\sum_j \omega_{ij}(p_j - p_i) = 0,$$

then it follows that for any vector v,

$$\sum_j \omega_{ij}(p_i - p_j)\cdot v = 0.$$

Often this can be used to show certain (and then all) of the ω_{ij}'s are 0. We illustrate this with an example below.

Example 3.14: Consider the framework $K_{3,3}(p)$ as in Figure 3.7.

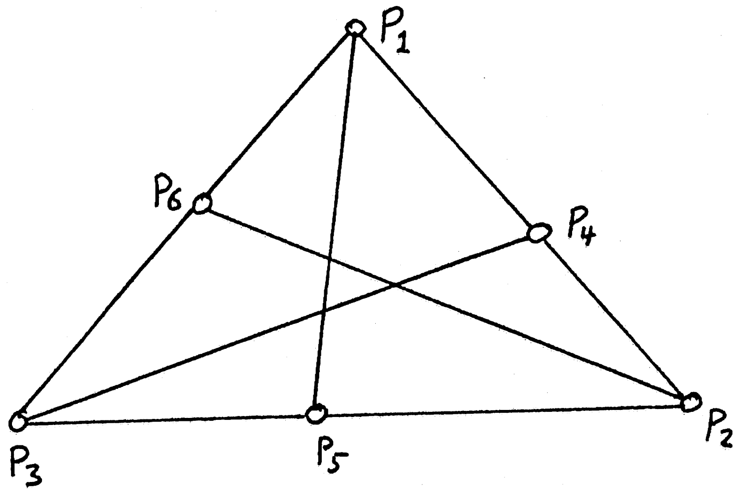

Figure 3.7

We make sure that p_4, p_5, p_6 are on the (open) line segments $< p_1, p_2 >$, $< p_2, p_3 >$, $< p_3, p_1 >$ respectively.

Note that $e = 9$, $v = 6$ so $e = 2v - 3$. So $K_{3,3}(p)$ is staticly rigid if and only if it has no self stress.

Suppose that $K_{3,3}(p)$ has a self stress ω. At p_4 we take the inner product of both sides of the equilibrium equation with a vector v perpendicular to $< p_1, p_2 >$. Then

$$\omega_{34}(p_3 - p_4)\cdot v = 0,$$

and thus $\omega_{34} = 0$. Similarly $\omega_{15} = \omega_{26} = 0$. Next we apply the same idea to p_1, p_2, and p_3 to get all the other ω_{ij}'s equal to 0. Thus 0 is the only self stress, and $K_{3,3}(p)$ is staticly rigid and hence infinitesimally rigid as well.

Note also by section 19 of Chapter 2, the above shows that $K_{3,3}$ is generically 2-rigid.

Problems:

Problem 3.4: For a triangulated two dimensional sphere in $\mathbb{R}^3$, show that it is statically rigid if and only if it has no self stress. would be easier if one recalled Euler characteristic

Problem 3.5: Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$ with v vertices and e bars such that the affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$ as in the Proposition. Let s be the dimension of the space of self stresses of $G(\mathbf{p})$, and let m be the dimension of the infinitesimal flexes of $G(\mathbf{p})$. Show that

$$s - m = e - vd.$$

Problem 3.6: Let G be a bar graph, and let H be any subgraph of G. Suppose that ω is a self stress for a realization $G(\mathbf{p})$ of G. Define a (force) vector $\mathbf{F}_i$ for each vertex i of H as follows.

$$\mathbf{F}_i = \sum_{\{i,j\} \in G \setminus H} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i),$$

where the sum for each fixed i is taken over those bars in G but not H which have i as a vertex. If all the members of G having i as a vertex are members of H, then define $\mathbf{F}_i = \mathbf{0}$. Show that this defines an equilibrium force at the realization of the vertices of H.

Problem 3.7: Consider any realization $G(\mathbf{p})$ in $\mathbb{R}^2$ of the "Desargues configuration" as in Figure 3.8, such that no three of the joints lie on a line.

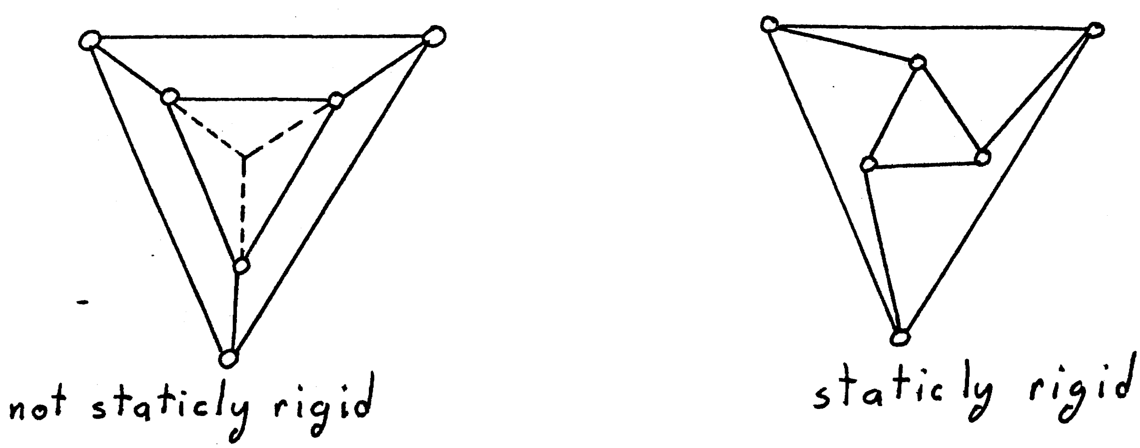

Figure 3.8

Show that $G(p)$ is not staticly rigid if and only if the line through p_1, p_4, the line through p_2, p_5, and the line through p_3, p_6 are either parallel or intersect in a point. To do this use Problem 3.6, where H is the triangle determined by the vertices 1, 2, 3.

Problem:

Problem 3.8: By only using the notion of self stresses and not infinitesimal rigidity, show that any bar simplex is staticly rigid in any $\mathbb{R}^d$.

8. A Matrix Reformulation

We present a compact alternate description of the equilibrium equations used for self stresses.

First we change the notation for the configuration $p \in \mathbb{R}^{vd}$ itself. Recall that p was regarged as a column vector grouped into v sets of d coordinates, where each set of d coordinates is regarded as a column vector $p_i \in \mathbb{R}^d$, $i = 1, \ldots, v$. We now instead define

$$P = [p_1, \ldots, p_v],$$

which is a d by v matrix, where the i-th column represents the i-th point in $\mathbb{R}^d$.

Next we represent a stress differently as well. Let $G(p)$ be any framework with v vertices and e members in $\mathbb{R}^d$. Recall that a stress $\omega \in \mathbb{R}^e$ is simply an assignment of a scalar ω_{ij} for each member of G, where ω is regarded as a row vector. Define a v by v matrix Ω as follows. Define the i,j-th entry Ω_{ij} of Ω by

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{for } i \neq j \\ \sum_k \omega_{ki} & \text{for } i = j. \end{cases}$$

Thus Ω is a symmetric matrix whose diagonal entries are equal to the negative of the sum of the off diagonal entries in that row or column. We call Ω the <u>stress matrix</u> for ω.

We compute

$$P\Omega = [\ldots, \sum_{k \neq i} \omega_{ki} p_i - \sum_{j \neq i} \omega_{ij} p_j, \ldots]$$

$$= [\ldots, \sum_j \omega_{ij}(p_i - p_j), \ldots].$$

Thus ω is a self stress for $G(p)$ if and only if $P\Omega = 0$, the 0 matrix.

Note also that if Ω is any symmetric matrix such that $P\Omega = 0$ and the row and column sums of Ω are zero, then Ω corresponds to a self stress ω for $G(p)$. In fact we can record this information as follows. For any vector $p_i \in \mathbb{R}^d$, regarded as a column vector of course, define $\hat{p}_i \in \mathbb{R}^{d+1}$ by

$$\hat{p}_i = \begin{bmatrix} p_i \\ 1 \end{bmatrix}.$$

In other words $\hat{p}_i$ is obtained by adding an extra coordinate 1 to the bottom of p_i. Similarly we define a $d + 1$ by v matrix $\hat{P}$ by

$$\hat{P} = [\hat{p}_1, \ldots, \hat{p}_v],$$

obtained by adding an extra row of 1's to the bottom of P. Since column sums of Ω are 0, we see that $\hat{P}\Omega = 0$ if and only if ω is a self stress for $G(p)$. Furthermore, any symmetric matrix Ω corresponds to a self stress for $G(p)$ if and only if $\hat{P}\Omega = 0$.

This reformulation is helpful for the discussion of the projective invariance of static rigidity discussed in section 10.

9. The Affine Invariance of Static Rigidity

If a given framework $G(\mathbf{p})$ in $\mathbb{R}^d$ is staticly rigid, it is natural to ask for what other configurations $\mathbf{q}$ will $G(\mathbf{q})$ be staticly rigid. In particular, what transformations of the ambient space $\mathbb{R}^d$ preserve static rigidity? Obviously if $\mathbf{q}$ is congruent to $\mathbf{p}$, then $G(\mathbf{q})$ will be staticly rigid if and only if $G(\mathbf{p})$ is staticly rigid. Can we find transformations more general than congruences of $\mathbb{R}^d$ which preserve static rigidity?

From Proposition 3.12 we see that if the dimension of the space of self stresses of a bar framework $G(\mathbf{q})$ is the same as the dimension of the space of self stresses of $G(\mathbf{p})$, then $G(\mathbf{q})$ is staticly rigid if and only if $G(\mathbf{p})$ is staticly rigid. With this in mind we have the following.

<u>Proposition 3.15</u>: Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$, and let $L: \mathbb{R}^d \to \mathbb{R}^d$ be any affine map. If $\omega \in \mathbb{R}^e$ is a self stress for $G(\mathbf{p})$, then ω is a self stress for $G(L\mathbf{p})$. Hence when L is invertible, $G(L\mathbf{p})$ is staticly rigid if and only if $G(\mathbf{p})$ is staticly rigid.

<u>Proof</u>: Let ω be any self stress for $G(\mathbf{p})$. Write $L\mathbf{x} = A\mathbf{x} + \mathbf{x}_0$ for all $\mathbf{x} \in \mathbb{R}^d$, where $\mathbf{x}_0 \in \mathbb{R}^d$ is a fixed vector and A is a d by d matrix as in section 2 of Chapter 2. Then the equilibrium equation for ω being a self stress is

$$\sum_j \omega_{ij}(Lp_i - Lp_j) = \sum_j \omega_{ij}A(p_i - p_j)$$

$$= A\left[\sum_j \omega_{ij}(p_i - p_j)\right]$$

$$= 0.$$

Remark 3.16: It follows that affine maps also preserve infinitesimal rigidity, but the correspondence between the corresponding spaces of infinitesimal flexes is not always obvious. Namely when L is an affine map as above it seems natural to guess that if $\mathbf{p}'$ is an infinitesimal flex of $G(\mathbf{p})$, then $A\mathbf{p}' = (A\mathbf{p}_1', \ldots, A\mathbf{p}_v')$ is an infinitesimal flex of $G(L\mathbf{p})$, where A is the linear part of L. In the Figure 3.9 we see two realizations of $K_{3,3}$ on two lines in $\mathbb{R}^2$, which differ by an affine equivalence. However the infinitesimal flexes do not correspond in the obvious way as indicated above. In both cases the infinitesimal flex on the non-horizontal line is vertical.

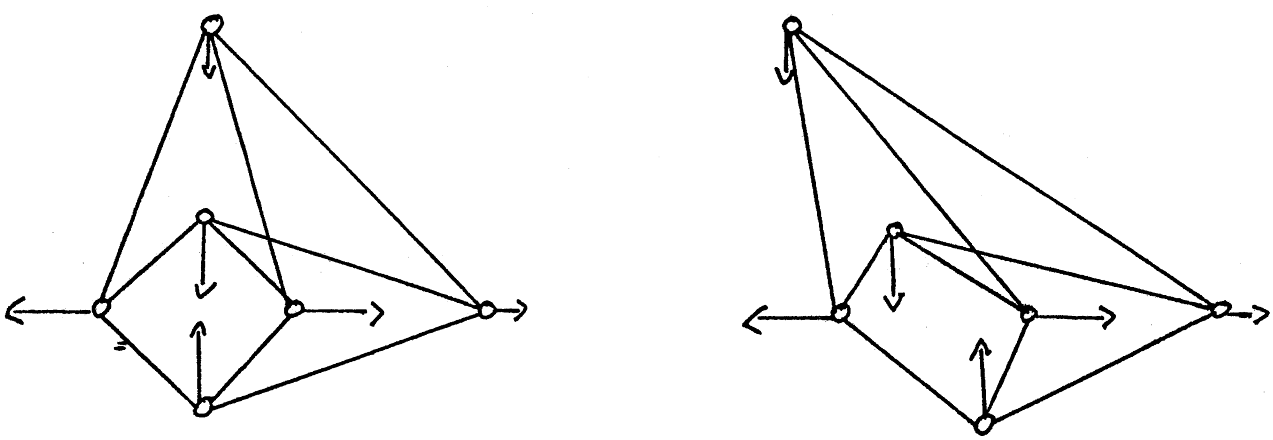

Figure 3.9

Remark 3.17: For tensegrity frameworks it is also true that affine equivalences preserve static (and infinitesimal) rigidity. This will be shown in Chapter xx.

Notice also that orthogonal projection $\pi: \mathbb{R}^d \to \mathbb{R}^{d-1}$, for instance, takes a framework $G(p)$ in $\mathbb{R}^d$ with self stress ω to a framework $G(\pi p)$ in $\mathbb{R}^{d-1}$ with the same self stress ω, but $G(\pi p)$ may have more self stresses. So, in particular, π may take a staticly rigid framework in $\mathbb{R}^3$ to one that is not staticly rigid in $\mathbb{R}^2$, as in Figure 3.10.

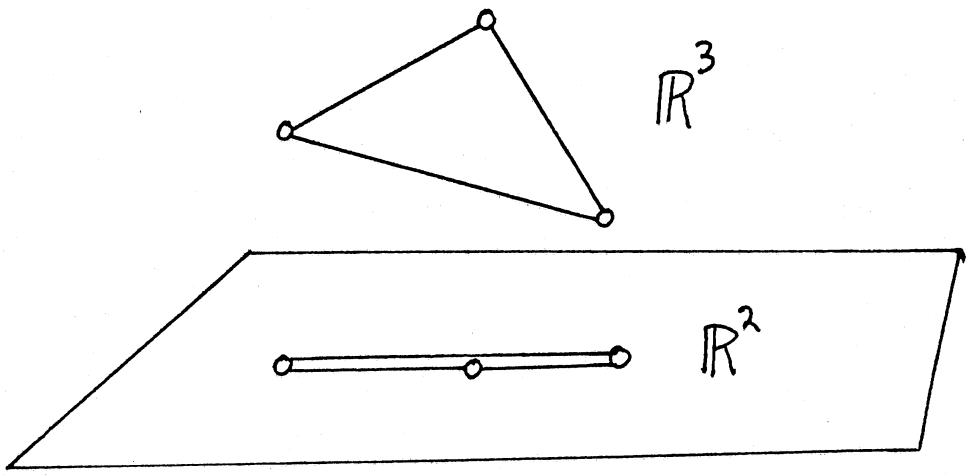

Figure 3.10

Also π may take a framework which is not staticly rigid in $\mathbb{R}^3$ to one that is staticly rigid in $\mathbb{R}^2$. For instance, any staticly rigid bar framework in $\mathbb{R}^2$ with $v \geq 4$ is not staticly rigid in $\mathbb{R}^3$, but its "projection" into $\mathbb{R}^2$ is staticly rigid by construction.

Problem:

Problem 3.9: Suppose a bar graph G is generically rigid in $\mathbb{R}^d$. Show that G is genericllly rigid in $\mathbb{R}^{d-1}$. See Section 19 of Chapter 1 for the definition of generically rigid.

10. Projective Transformations

There is more to the story of which transformations preserve static rigidity. In the study of Projective Geometry a natural function that arrises is a projective map. One way to describe this map is as follows. Let $\mathbf{x} \in \mathbb{R}^d$, and let $\hat{\mathbf{x}} \in \mathbb{R}^{d+1}$ be as in section 8. Let $\hat{A}$ be any invertible $d + 1$ by $d + 1$ matrix. Then

$$\hat{A}\hat{\mathbf{x}} = \begin{bmatrix} A\mathbf{x} + \mathbf{x}_0 \\ \lambda \end{bmatrix},$$

where A is a d by d matrix and $\mathbf{x}_0 \in \mathbb{R}^d$ is fixed. Note that λ depends on $\mathbf{x}$. Then the function that takes $\mathbf{x}$ to $(A\mathbf{x} + \mathbf{x}_0)/\lambda$ is called a <u>projective transformation</u>. It turns out that this transformation may not be defined on a $(d - 1)$-dimensional hyperplane in $\mathbb{R}^d$, those points whose image under $\hat{A}$ have last coordinate 0, but it has the pleasant property that it takes any set of points that lie on a line (or hyperplane) to a set of points that lie on a line (or hyperplane). In fact, roughly speaking, for $\mathbb{R}^d$ this line preserving property characterizes what we are calling a projective transformation. See Artin, Geometric Algebra (19xx), for a careful but somewhat detailed description of this characterization. See Coxeter (19xx), or Greenberg (19xx), for a discusion of the virtues of Projective Geometry itself.

Theorem 3.18: Let $G(p)$ be any bar framework in $\mathbb{R}^d$, and let T be any projective transformation such that Tp_i is defined for every joint p_i of $G(p)$. Then $G(p)$ is staticly rigid if and only if $G(Tp)$ is staticly rigid.

Proof: Let $\hat{A}$ be the $d+1$ by $d+1$ matrix used in the definition T as above. Then for each vertex i of G,

$$\hat{A}\hat{p}_i = \begin{bmatrix} Ap_i + x_0 \\ \lambda_i \end{bmatrix},$$

where $\lambda_i \neq 0$ for $i = 1, \ldots, v$.

Let ω be any self stress for $G(p)$, and let Ω be the corresponding stress matrix as in section 8. Thus $\hat{P}\Omega = 0$. Let D be the v by v diagonal matrix, whose i-th diagonal entry is λ_i and whose off diagonal entries are 0.

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & \\ 0 & & \lambda_v \end{bmatrix}.$$

Then

$$\hat{A}\hat{P}D^{-1} = \begin{bmatrix} (Ap_1 + x_0)/\lambda_1 & \cdots & (Ap_v + x_0)/\lambda_v \\ 1 & & 1 \end{bmatrix}.$$

We observe also that $D\Omega D$ is a symmetric matrix and

$$\hat{A}\hat{P}D^{-1}D\Omega D = \hat{A}\hat{P}\Omega D = \hat{A}OD = 0.$$

Thus $D\Omega D$ corresponds to a self stress for $G(Tp)$, where the self stress on member $\{i,j\}$ is $\lambda_i\lambda_j\omega_{ij}$. This is clearly a vector space isomorphism between the space of self stresses of $G(p)$ and the space of self stresses of $G(Tp)$. Thus by Proposition 3.12 when the affine span of p is all of $\mathbb{R}^d$, then $G(p)$ is staticly rigid if and only if $G(Tp)$ is staticly rigid.

In case the affine span of p is not all of $\mathbb{R}^d$, then $G(p)$ is staticly rigid if and only if $G(p)$ is a bar simplex. But by the hyperplane preserving nature of T, $G(p)$ is a bar simplex if and only if $G(Tp)$ is a bar simplex. Hence $G(p)$ is staticly rigid if and only if $G(Tp)$ is staticly rigid in this case as well. This finishes the Theorem.

Remark 3.19: It is not true for tensegrity frameworks that projective transformations preserve static rigidity. In Figure 3.11 we have two tensegrity frameworks in $\mathbb{R}^2$ with bars and cables which have the same tensegrity graph.

Figure 3.11

In $\mathbb{R}^2$ for any two sets of four points, no three on a line for each set, there is a projective transformation that takes one set onto the the other. Thus the two frameworks in Figure 3.11 are projectively equivalent, but one is staticly rigid while the other is not.

Despite the counterexample above there is something that can said about tensegrity frameworks. In the description of the projective transformation T each point p_i of $G(p)$ is associated to a non-zero scalar λ_i. It turns out that if we interchange cables and struts on those members $\{i,j\}$ where

$\lambda_i\lambda_j < 0$ to get a new tensegrity graph G^*, then $G(p)$ is staticly rigid if and only if $G^*(Tp)$ is staticly rigid.

We can even see geometrically which of these members to change. The projective transformation T is not defined on a certain hyperspace H. The interchange is performed on those members $\{i,j\}$ such p_i and p_j are on opposite sides of H. For instance in Figure 3.12 starting with the same frameworks as in Figure 3.11, we have indicated the line H and which cable to change to a strut to obtain $G^*(Tp)$.

Figure 3.12

See Chapter xx for a proof and more discussion.

Problems:

Problem 3.10: Use the example of Figure 2.27 of Chapter 2 to show that if the points of $K_{3,3}(p)$ in $\mathbb{R}^2$ lie on any ellipse, parabola, or hyperbola, then $K_{3,3}(p)$ is not staticly rigid.

Problem 3.11: Let $F: \mathbb{R}^d \to \mathbb{R}^d$ be any function such that $G(p)$ is staticly rigid if and only if $G(Fp)$ is staticly rigid, when p is in some open subset of $\mathbb{R}^d$. Show that F is a projective transformation.

11. Some Combinatorics of Rigidity

Since we now know about the concepts of stresses, self stresses, infinitesimal flexes etc., we can use them in a combinatorial way to investigate related properties of frameworks.

Suppose that G(**p**) is a bar framework and i, j are two of the vertices of G. Regardless of whether G(**p**) is staticly rigid or not, we can ask whether the elementary equilibrium force **F**(i, j) can be resolved. If **F**(i, j) can be resolved by some stress for G(**p**), then we say that {i,j} is an <u>implicit bar</u> of G(**p**). Note that this is the same thing as saying that **F**(i, j) is in the linear span of the rows of the rigidity matrix R(**p**).

Similarly we say that a bar {i,j} of G(**p**) is <u>redundant</u> if {i,j} is an implicit bar in the framework obtained by removing {i,j}. Clearly adding implied bars or removing redundant ones does not change the static rigidity of a framework G(**p**).

<u>Example 3.20</u>: In the bar framework of Figure 3.13 in $\mathbb{R}^2$, {1,2} is an implicit bar.

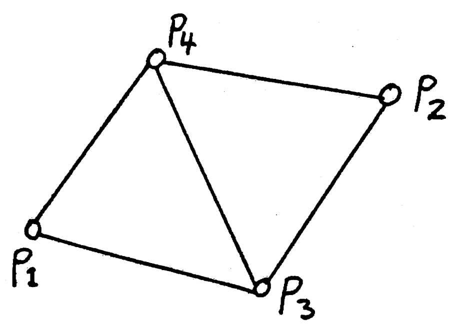

Figure 3.13

Furthermore when {1,2} is added as in Figure 3.14, every bar becomes redundant.

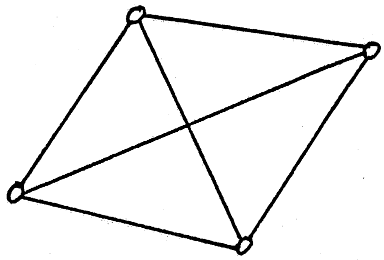

Figure 3.14

Note that whether $\{i,j\}$ is an implicit bar or a redundant bar may depend on the position of the joints.

Example 3.21: Suppose that G is a bar triangle. If all the joints of $G(\mathbf{p})$ are distinct, then each bar is redundant if and only if the joints lie on a line.

Note that a bar $\{i,j\}$ of G is redundant if and only if there is a self stress ω for $G(\mathbf{p})$ such that $\omega_{ij} \neq 0$. With this in mind we say that a bar framework is dependent if there are redundant bars, and $G(\mathbf{p})$ is independent otherwise. We see that this definition coincides with the usual definition of linear dependence and linear independence of the rows of the rigidity matrix $R(\mathbf{p})$. If $G(\mathbf{p})$ is statically rigid and independent, we say that $G(\mathbf{p})$ is isostatic. If $G(\mathbf{p})$ is a dependent framework such that the removal of any bar creates an independent framework, then we say that $G(\mathbf{p})$ is a circuit. For example the framework of Figure 3.13 is isostatic and the framework of Figure 3.14 is a circuit. Isostatic frameworks are minimally (in the sense containing bars) statically rigid frameworks, and circuits are minimally dependent frameworks.

Of course we know from Proposition 2.30 of Chapter 2 that if the affine span of **p** is all of $\mathbb{R}^d$, and if all possible $\{i,j\}$ are implied bars, then $G(\mathbf{p})$ is statically rigid.

Lastly we observe that being independent, isostatic, and statically rigid are all generic properties for bar frameworks. If $G(\mathbf{p})$ has one of these properties for one configuration **p**, then $G(\mathbf{q})$ has the property for an open dense set of of configurations in $\mathbb{R}^{vd}$. This is because all of the above properties are equivalent to certain determinants not vanishing. The set of configurations where the property does not hold is a closed algebraic set in $\mathbb{R}^{vd}$. Thus the complement is open and dense in $\mathbb{R}^{vd}$.

Problems:

Problem 3.12: Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$ with the affine span of **p** all of $\mathbb{R}^d$. Define the *degree of freedom* f for $G(\mathbf{p})$ as the minimum number of bars needed to add to $G(\mathbf{p})$ (adding no new vertices) to make $G(\mathbf{p})$ statically rigid. Let e be the number of bars of G. If $G(\mathbf{p})$ is independent, show that

$$f = vd - d(d + 1)/2 - e.$$

Problem 3.13: Let $G(\mathbf{p})$ be as in Problem 3.12. Define the *degree of redundancy* r of $G(\mathbf{p})$ as the minimum number of bars one needs to remove to make $G(\mathbf{p})$ independent.

a. If $G(\mathbf{p})$ is statically rigid, show that

$$r = e - vd + d(d + 1)/2.$$

b. If $G(\mathbf{p})$ is not statically rigid, show that

$$f - r = vd - d(d + 1)/2 - e.$$

Problem 3.14: Consider a configuration p of six distinct points in $\mathbb{R}^2$, where no three are on a line. Consider any bar framework $G(p)$ which is not statically rigid, but such that every member is redundant. What is the most number of bars which G can have, and what configurations p allow this maximum number of bars?

For the next problem, we say a graph G is n vertex connected if G remains connected upon the removal of any $n - 1$ or fewer vertices, but some n vertices do disconnect G. Similarly, G is n edge connected if G remains connected upon the removal of $n - 1$ or fewer edges, but some n edges do disconnect G.

Problem 3.15(G. Kilai): Let $G(p)$ be a bar framework in $\mathbb{R}^d$ with e bars and v vertices such that $v > d$ and

$$e > vd - d(d + 1)/2.$$

a. Show that there is a subgraph of G with more than d vertices which is $d + 1$ edge connected.

b. Show that there is an edge $\{i,j\}$ of G such that there are $d + 2$ paths from i to j with only i and j in common.

Suppose instead that

$$e = vd - d(d + 1)/2.$$

c. Show that a. is true with $d + 1$ edge connected replaced by d edge connected.

d. Show that b. is true with $d + 2$ replaced by $d + 1$.

Problem 3.16: Let p be any configuration of points in $\mathbb{R}^d$ such no $d + 1$ of the points lie on a $d - 1$ dimensional hyperplane. Let $G(p)$ be obtained from a complete bar graph by successively deleting bars only if they are part of a remaining complete bar graph with $d + 2$ vertices. Show that $G(p)$ is statically rigid.