Draft #2
April, 1987

CHAPTER 4

THE RIGIDITY OF FRAMEWORKS GIVEN BY CONVEX SURFACES

Ben Roth
Department of Mathematics
University of Wyoming
Laramie, Wyoming 82071

4.1. Introduction.

In this chapter, we deal almost exclusively with frameworks which are given, in one way or another, by the surfaces of convex polyhedra in $\mathbb{R}^3$. In Section 4.2, we examine the simplest and perhaps most natural case – namely, that in which the vertices and edges of the framework coincide precisely with the vertices and edges of a convex polyhedron in $\mathbb{R}^3$. Then, in Section 4.3 and 4.4, we complicate matters by allowing other points and line segments on the surface of the convex polyhedron to serve as vertices and edges of the framework.

Understanding the role that convexity plays is the major theme of the chapter. Two other themes are central to Section 4.3 and at least peripheral to most of the remainder of the chapter. One deals with the connections between self-stresses of the whole framework and self-stresses of its "facial" frameworks (which consist of the vertices and edges that lie in a single face of the convex polyhedron). Incidentally, a "self-stress" will usually be referred to simply as a "stress" in this chapter. The other major theme explores the relationships between the spatial rigidity of a framework and the planar rigidity of its facial frameworks. Three separate kinds of conditions are pertinent to the development of these themes:

1. the location of the vertices of the framework (are there vertices of the framework

in the interior of edges or even faces of the convex polyhedron?);

2. the nature and location of any crossings of edges of the framework (are there edges of the framework that intersect in a point that is not an endpoint of both edges and, if so, where do these crossings occur?);

3. the presence of each edge of the convex polyhedron in the framework (are there edges or portions of edges of the convex polyhedron that are absent from the framework?).

The results in this chapter are often of a delicately balanced and somewhat technical nature, perhaps primarily because of the subtle interplay between these three independent conditions. However, with some experience and a small number of examples in mind, one rather quickly is led to the appropriate conditions for the kind of result one seeks.

Of all the topics in this book, the rigidity of convex surfaces surely has the richest historical tradition. Euclid's definition of "equal and similar solid figures" (Definition 10, Book XI, of the **Elements**) has met with much criticism, precisely because of a question regarding the rigidity of surfaces that the definition raises. For a complete discussion of both the definition and the issues it raises, see Heath (1956). However, it is illuminating for our purposes in this chapter to discuss the problem in at least an informal way. Roughly speaking, Euclid's definition says that equal and similar solid figures are those with corresponding faces congruent and arranged in the same way. And the question is: does this definition really guarantee that the solid figures are congruent in the sense that one of the solid figures can be made to coincide with the other by applying a congruence of $\mathbb{R}^3$? (This is surely the notion that Euclid was attempting to capture by his definition.) We can frame the question in the following very concrete, down-to-earth way. Suppose one cuts out pieces of cardboard and tapes them together in such a way as to form the surface of a solid figure. Might someone else, taping the same faces together in the same arrangement, arrive at a solid figure which is not congruent to the first? The example shown in Figure 4.1 (which goes back at least to Legendre (1794)) shows that this can actually occur if the solid figures are not required to be convex! Or worse still, might the surface one obtains be "flexible"? This would provide an infinite family of noncongruent solid figures with corresponding faces congruent and arranged in the same way.

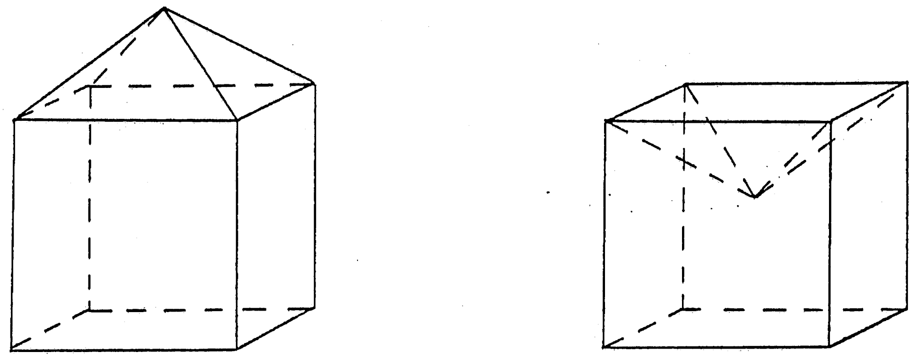
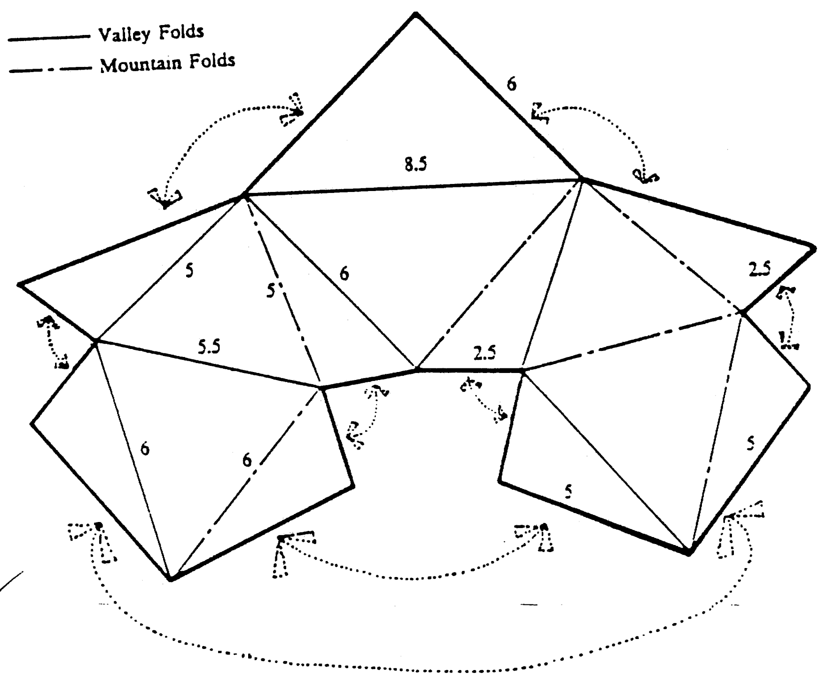


Figure 4.1

The question raised by Euclid's definition was resolved early in the nineteenth century by Cauchy (1813) who (almost) proved that two convex polyhedra with corresponding faces congruent and arranged in the same way are themselves congruent. However, like Euclid's definition, Cauchy's proof was eventually to become the subject of a good deal of criticism. The proof consists basically of two parts, one topological and the other geometric in nature. Imperfections in the topological part of the argument were detected by Hadamard (1907) and corrected by Lebesgue (1909). However, Steinitz later found defects in the geometric part of Cauchy's argument as well, and the first complete proof of Cauchy's theorem appears finally in Steinitz & Rademacher (1934). Basically, this is the proof presented in Lyusternik (1963). Stoker (1968) both proves and extends Cauchy's theorem. In any case, history supports the contention that Definition 10 of Book XI of Euclid's **Elements** is not properly a definition at all – it is, in fact, a rather substantial theorem, involving ideas that are somewhat subtle and technically difficult.

Understanding the part played by convexity when Cauchy's theorem is interpreted as a result about the rigidity of convex polyhedral surfaces is a great deal more difficult.

Perhaps all closed polyhedral surfaces, even nonconvex ones, are rigid. The conjecture, a version of which Euler proposed in 1766, that all embedded polyhedral surfaces are rigid has recently been settled by an ingenious counterexample of Connelly (1978). His flexible polyhedral surface is a collection of triangles joined together along common edges in such a way as to form a closed (but not convex) polyhedron without self-intersections in $\mathbb{R}^3$. Connelly (1979) describes several such flexible polyhedral surfaces, including the one shown in Figure 4.2 due to Klaus Steffen. Note that this figure is symmetric about a vertical line. Each unlabeled edge has the same length as its corresponding edge under the symmetry. Building this surface by cutting out pieces of cardboard and taping them together as shown by the arrows in Figure 4.2 is well worth the effort.



careful about valley and mountain folds

Figure 4.2

Although it is not difficult to formulate the appropriate notions of rigidity and flexibility for polyhedral surfaces, we instead focus once again on frameworks and relate Cauchy's theorem to the rigidity of frameworks. Since as a framework any triangle is rigid, it is not difficult to convince oneself that for a convex polyhedron with all triangular faces, Cauchy's theorem implies that the framework consisting of the vertices and edges (the "one-skeleton") of such a polyhedron is rigid in $\mathbb{R}^3$. However, rather than relying on Cauchy's theorem for convex polyhedra in our work, we will instead formulate and prove a version of Cauchy's theorem directly for frameworks. Section 4.2 does precisely this.

Finally, what can be said about the rigidity of smooth surfaces? Results analogous to Cauchy's can be found in Liebmann (1900) for analytic surfaces and in Cohn-Vossen (1936) for smooth surfaces. More recent work includes that of Efimov (1957), Pogorelov (1973), and others. Chapter 12 of Volume V of Spivak (1979) is perhaps the most accessible reference. In general, one finds that although important ideas seem to transfer between the smooth category and the piecewise-linear category with considerable success, specific techniques may not and it has not yet been possible to establish results in one category by simply appealing to analogous results in the other. Nonetheless, the importance of each as a source of inspiration for the other is substantial.

4.2. Frameworks Given by Convex Polyhedra in $\mathbb{R}^3$.

Let C be a convex polyhedron in $\mathbb{R}^3$, i.e., the convex hull of a finite set of noncoplanar points in $\mathbb{R}^3$. A *vertex* of C is a point which is the intersection of C with a support plane of C, while an *edge* of C is a closed line segment which is the intersection of C with a support plane of C. Suppose C has v vertices located at the points $\mathbf{p}_1, \cdots, \mathbf{p}_v \in \mathbb{R}^3$. Let $V = \{1, \cdots, v\}$ and

$$E = \{\{i,j\} : [\mathbf{p}_i, \mathbf{p}_j] \text{ is an edge of } C\} \ .$$

Definition 4.1. The framework $G(\mathbf{p})$ where $G = (V, E)$ and $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_v)$ is the framework in $\mathbb{R}^3$ *given by* the convex polyhedron C.

Our goal in this section is to characterize the frameworks given by convex polyhedra in $\mathbb{R}^3$ that are rigid in $\mathbb{R}^3$.

Suppose $G(\mathbf{p})$ is the framework given by a convex polyhedron C in $\mathbb{R}^3$. We will show that $G(\mathbf{p})$ admits only the trivial self-stress. That is, we show that rank $df_{\mathbf{p}} = e$ where $f : \mathbb{R}^{3v} \to \mathbb{R}^e$ is the rigidity map of G and e is the number of edges of G. This implies that $\max\{\text{rank } df_{\mathbf{x}} : \mathbf{x} \in \mathbb{R}^{3v}\} = e$ and thus $\mathbf{p}$ is a regular point of f. Consequently, the Implicit Function Theorem implies that $f^{-1}(f(\mathbf{p}))$ is a $(3v - e)$-dimensional manifold near $\mathbf{p}$. By Proposition 2.32, $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$ if and only if $f^{-1}(f(\mathbf{p}))$ and the six-dimensional manifold $H_{\mathbf{p}}$ of points congruent to $\mathbf{p}$ coincide near $\mathbf{p}$. But this is merely a matter of dimension since both are manifolds. Therefore one can determine the rigidity or flexibility of $G(\mathbf{p})$ by the simplest imaginable procedure – just count the number e of edges of C and compare the result to $3v - 6$. Furthermore, the results can even be interpreted "infinitesimally" if desired, since rigidity (flexibility) and infinitesimal rigidity (infinitesimal flexibility) amount to the same thing at regular points.

Before proving that frameworks given by convex polyhedra in $\mathbb{R}^3$ are stress free, it seems useful to provide an example of a nonconvex polyhedron which admits a nontrivial stress.

Example 4.1. Consider the nonconvex octahedron $G(\mathbf{p})$ shown in Figure 4.3 with vertices located at the points $\mathbf{p}_1 = (0,1,0)$, $\mathbf{p}_2 = (-2,3,0)$, $\mathbf{p}_3 = (0,-3,0)$, $\mathbf{p}_4 = (2,3,0)$, $\mathbf{p}_5 = (0,0,3)$, $\mathbf{p}_6 = (0,0,-3)$. Using the definition of a self-stress (see Equation (4.1)), it is easy to verify that $\{\omega_{ij}\}$ is a self-stress when $\omega_{12} = \omega_{14} = 3$, $\omega_{23} = \omega_{34} = 1$, $\omega_{15} = \omega_{16} = 6$, $\omega_{25} = \omega_{26} = \omega_{35} = \omega_{36} = \omega_{45} = \omega_{46} = -2$.

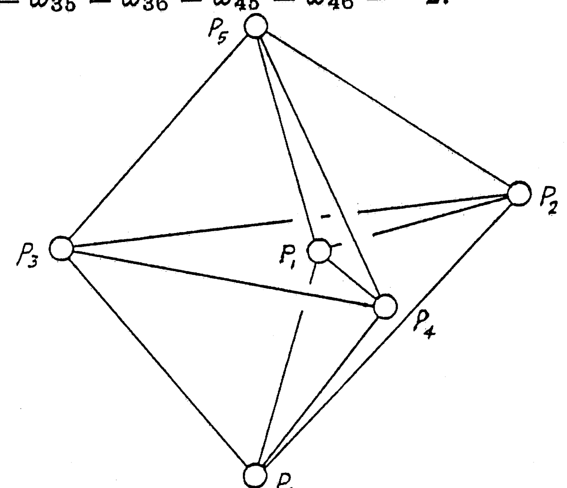


Figure 4.3

The proof that rank $df_{\mathbf{p}} = e$ has two parts, both of which originate in the proof by Cauchy (1813) of the fact that two convex polyhedra in $\mathbb{R}^3$ with corresponding faces congruent and "arranged in the same way" are themselves congruent. (More formally, the hypothesis of Cauchy's theorem says that there exists a one-to-one correspondence ψ between the sets of vertices of the two polyhedra such that S is the set of vertices of a face of one polyhedron if and only if $\psi(S)$ is the set of vertices of a face of the other and, furthermore, the map ψ preserves distances between vertices on corresponding faces. Thus in Cauchy's theorem one considers the faces of the convex polyhedra to be rigid in $\mathbb{R}^3$.) One part is of a global topological nature and deals with graphs on a polyhedron, while the other is of a local geometrical nature and relies on the convexity of the polyhedron. The particular arrangement of ideas used here is due to Alexandrov (1958) and Gluck (1975), and related results appear in Dehn (1916) and Weyl (1917).

Consider a framework $G(\mathbf{p})$ in $\mathbb{R}^3$ given by a convex polyhedron C and suppose there exists a nontrivial stress $\{\omega_{ij}\}$ of $G(\mathbf{p})$. This means that for $1 \leq i \leq v$

$$\sum_{j \in a(i)} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{4.1}$$

where $a(i) = \{j : [\mathbf{p}_i, \mathbf{p}_j] \text{ is an edge of } G(\mathbf{p})\}$, the set of vertices adjacent to vertex i.

The signs of the coefficients ω_{ij} are now used to attach the symbols $+$ and $-$ to some of the edges of C. If $\omega_{ij} > 0$, then the $\{i,j\}$ edge of C is marked $+$ while if $\omega_{ij} < 0$, then the $\{i,j\}$ edge of C is marked $-$. The edge $\{i,j\}$ is left unmarked if $\omega_{ij} = 0$. Consider the graph G' on the surface ∂C of C induced by the marked edges of C, which means that the edges of G' are the edges of C marked $+$ or $-$ and the vertices of G' are the vertices of C incident with at least one edge marked $+$ or $-$. For each vertex $\mathbf{p}_i$ of G', the edges of G' incident with $\mathbf{p}_i$ can by cyclically ordered according to their occurrence on the surface of C as the vertex $\mathbf{p}_i$ is circled once. The *index of* $\mathbf{p}_i$ is the number of changes of sign encountered in this cycle of edges around the vertex $\mathbf{p}_i$ and the *index* I is the sum of the indices of the vertices of G'. Actually these notions of index make sense even for nonconvex polyhedra. For instance, in Example 4.1 the index of $\mathbf{p}_1$ is zero, the index of $\mathbf{p}_5$ and $\mathbf{p}_6$ is two, and the index of $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ is four. Thus $I = 16$ for this nonconvex octahedron. The topological part of the proof deals with graphs induced by some subset of the edges of a polyhedron where each edge in the subset is marked $+$ or $-$. Convexity is irrelevant for this lemma; what is crucial is that the polyhedron is topologically a sphere so one can use Euler's formula.

Lemma 4.1. *The index satisfies*

$$I \leq 4v' - 8$$

where v' is the number of vertices of G'.

Proof. Let e' be the number of edges of G' and f' the number of regions (or topological components) of $\partial C - G'$. Each such region has a boundary consisting of edges of G'. Let f'_n be the number of regions with exactly n boundary edges where an edge is counted twice for a region if the region lies on both sides of the edge. Clearly $f'_1 = 0$ and f'_2 is nonzero only when G' has just one edge. Since $I = 0 = 4v' - 8$ in this case, we assume $f'_2 = 0$. Then

$$2e' = \sum_{n\geq 3} nf'_n \quad \text{and} \quad f' = \sum_{n\geq 3} f'_n \ .$$

We now compute the index I by circling regions rather than vertices. Since the number of sign-changes as one traverses the boundary of a region with n edges is an even number less than or equal to n, we have

$$\begin{aligned} I &\leq 2f'_3 + 4f'_4 + 4f'_5 + 6f'_6 + 6f'_7 + \cdots \leq \sum_{n\geq 3}(2n-4)f'_n \\ &= 2\sum_{n\geq 3} nf'_n - 4\sum_{n\geq 3} f'_n = 4e' - 4f' \ . \end{aligned}$$

By Euler's formula, $v' - e' + f' = 1 + N \geq 2$ where N is the number of components of the graph G'. Therefore

$$I \leq 4e' - 4f' \leq 4v' - 8 \ . \qquad \square$$

Next, we present the geometrical part of the argument which relies on the convexity of C together with Equation (4.1). Note that the nonconvex octahedron in Example 4.1 has vertices of index zero and two.

Lemma 4.2. *The index of every vertex of G' is greater than or equal to four.*

Proof. Consider any vertex $\mathbf{p}_i$ of G' and let $a'(i) = \{j : [\mathbf{p}_i, \mathbf{p}_j] \text{ is an edge of } G'\}$. By Equation (4.1), we have

$$\sum_{j\in a'(i)} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{4.2}$$

since the coefficients ω_{ij} of edges of G but not G' are zero. First, the index of $\mathbf{p}_i$ cannot be zero since the scalars ω_{ij} for $j \in a'(i)$ are either all positive or all negative in this case. By the convexity of C, there exists a plane in $\mathbb{R}^3$ which intersects C only at $\mathbf{p}_i$; say an equation of the plane is $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{x}) = 0$, where $\mathbf{n} \in \mathbb{R}^3$ is a normal to the plane. Since all vertices of G' except $\mathbf{p}_i$ lie on one side of the plane, $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j)$ is either positive for all $j \in a'(i)$ or negative for all $j \in a'(i)$. Therefore,

$$\sum_{j \in a'(i)} \omega_{ij} \left[\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j)\right] \neq 0 ,$$

which is impossible since Equation (4.2) gives

$$\sum_{j \in a'(i)} \omega_{ij} \left[\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j)\right] = \mathbf{n} \cdot \left[\sum_{j \in a'(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j)\right] = 0 .$$

Moreover, a similar argument shows that the index of $\mathbf{p}_i$ cannot be two, since in this case there is a set of edges of G' marked $+$ followed by a set of edges marked $-$ in the cycle of edges around $\mathbf{p}_i$. By the convexity of C, there exists a plane through $\mathbf{p}_i$ with the edges of G' incident with $\mathbf{p}_i$ marked $+$ on one side of the plane and those marked $-$ on the other side of the plane. If an equation of this plane is $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{x}) = 0$, we have $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j)$ of one sign for all the edges marked $+$ and of the opposite sign for those marked $-$. Thus by Equation (4.2)

$$0 = \mathbf{n} \cdot \left[\sum_{j \in a'(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j)\right] = \sum_{j \in a'(i)} \omega_{ij} \left[\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j)\right] \neq 0 .$$

This contradiction completes the proof of Lemma 4.2. □

Theorem 4.3. *Let $G(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^{3v}$, be the framework in $\mathbb{R}^3$ given by a convex polyhedron C and suppose f is the rigidity map of G. Then*

$$\text{rank } df_{\mathbf{p}} = e ,$$

the number of edges of C.

Proof. We suppose that $G(\mathbf{p})$ admits a nontrivial stress and arrive at a contradiction. We attach the symbols $+$ and $-$ to some of the edges of C according to the signs in a nontrivial stress and let G' be the graph induced by the marked edges of C. By Lemmas 4.1 and 4.2 we have

$$I \leq 4v' - 8 < 4v' \leq I$$

where I is the index and v' the number of vertices of G'. This contradiction shows that $G(\mathrm{p})$ is stress free and thus rank $df_{\mathrm{p}} = e$. □

Therefore, as was observed in the comment following Definition 4.1, the rigidity or flexibility of a framework given by a convex polyhedron is determined by a simple comparison of e and $3v - 6$. However, this same comparison arises in another quite different way. Consider a convex polyhedron C in $\mathbb{R}^3$ with v vertices, e edges, and f faces of which f_n have exactly n edges. By Euler's formula,

$$3v - 6 = 3(v - 2) = 3(e - f) = e + (2e - 3f) \ .$$

But

$$3f = 3\sum_{n\geq 3} f_n \leq \sum_{n\geq 3} n f_n = 2e$$

with equality if and only if $f = f_3$, i.e., every face of C is a triangle. Therefore $e \leq 3v - 6$ with equality if and only if every face of C is a triangle. This leads to the following corollary, one implication of which was previously observed in Section 4.1 to be a consequence of Cauchy's theorem for convex polyhedra. The other implication is due to Asimow & Roth (1978).

Corollary 4.4. *The framework $G(\mathrm{p})$ given by a convex polyhedron C is rigid in $\mathbb{R}^3$ if and only if every face of C is a triangle.*

Proof. By Theorem 4.3, rank $df_{\mathrm{p}} = e$ where e is the number of edges of C and f is the rigidity map of G. Thus $\mathrm{p} = (\mathrm{p}_1, \cdots, \mathrm{p}_v)$ is a regular point of f and therefore $G(\mathrm{p})$ is rigid in $\mathbb{R}^3$ if and only if $e =$ rank $df_{\mathrm{p}} = 3v - 6$. But, as we just observed, $e = 3v - 6$ if and only if every face of C is a triangle. □

A similar argument using Theorem 4.3 establishes that the removal of any bar from a framework given by a convex polyhedron in $\mathbb{R}^3$ with all triangular faces gives a flexible framework.

This $e = 3v-6$ "test for rigidity" arising from Theorem 4.3 has certainly not escaped the attention of engineers. In fact, the inaccuracies that mar many accounts of rigidity stem from attempts to apply this simple formula to all frameworks in $\mathbb{R}^3$. It is not difficult to find examples showing this is inappropriate.

For instance, consider the framework $G(\mathbf{p})$ in $\mathbb{R}^3$ shown in Figure 4.4, which is a tetrahedron with a triangle in the plane of its base. A simple geometrical argument using the very special location of the three vertices in the interior of the base of the tetrahedron shows that $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$ even though $e < 3v - 6$. (Note Theorem 4.3 is not applicable since $G(\mathbf{p})$ is not the framework given by a convex polyhedron in $\mathbb{R}^3$ in the sense of Definition 4.1.) However, Proposition 2.36 says that the framework G (in fact, any framework with $e < 3v - 6$) is always infinitesimally flexible in $\mathbb{R}^3$ which is equivalent to saying that G is generically flexible in $\mathbb{R}^3$.

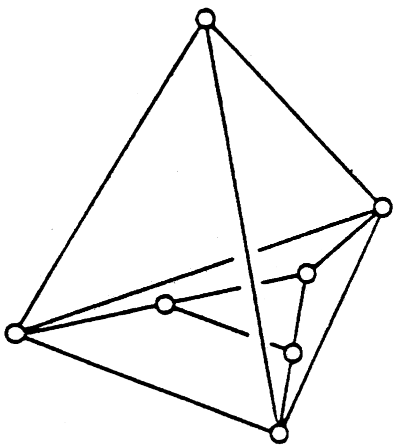


Figure 4.4

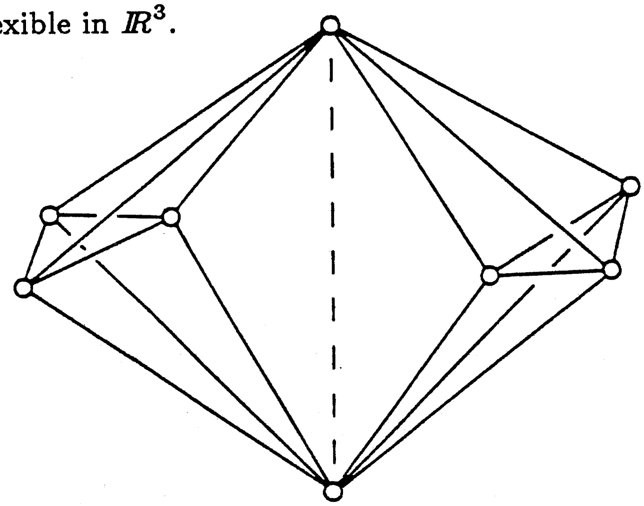


Figure 4.5

On the other hand, there exist frameworks with $e = 3v - 6$ which are not only flexible but even generically flexible in $\mathbb{R}^3$. For example, the framework $G(\mathbf{p})$ shown in Figure 4.5 is flexible in $\mathbb{R}^3$ since one half of the framework rotates relative to the other around the dotted line shown, and this is clearly the typical behavior of G in $\mathbb{R}^3$.

Theorem 4.3 is also the key to Gluck's elegant proof that triangulated spheres are generically rigid in $\mathbb{R}^3$. The main result of Gluck (1975) is that "almost all" frameworks given by any triangulation of the two-sphere are rigid in $\mathbb{R}^3$. To see this, suppose G is the graph corresponding to a triangulation of the two-sphere. By Steinitz' theorem (see Lyusternik (1963)), there exists $\mathbf{p} \in \mathbb{R}^{3v}$ such that $G(\mathbf{p})$ is the framework given by a convex polyhedron in $\mathbb{R}^3$ with all triangular faces. Hence $G(\mathbf{p})$ is rigid (even infinitesimally rigid) in $\mathbb{R}^3$ by Corollary 4.4. Moreover, since each $G(\mathbf{q})$ for $\mathbf{q}$ sufficiently close to $\mathbf{p}$ is also a framework given by a convex polyhedron with all triangular faces, each such $G(\mathbf{q})$ is infinitesimally rigid in $\mathbb{R}^3$. Since $G(\mathbf{q})$ is infinitesimally rigid for all $\mathbf{q}$ in a neighborhood of $\mathbf{p}$, G is generically rigid in $\mathbb{R}^3$ (see Chapter 3).

While Corollary 4.4 tells us that the framework given by a cube is flexible in $\mathbb{R}^3$, it gives no information about the "braced" cube obtained by adding new diagonal edges across some (or perhaps all) of the six faces of the cube. Since such frameworks are surely of interest, much of the remainder of this chapter deals with frameworks which arise in various more general ways from convex polyhedra in $\mathbb{R}^3$. Incidentally, this is not meant to imply that frameworks arising in one way or another from convex polyhedra are the only interesting or important ones from either a mathematical or a structural point of view – these just form one of the broad classes of frameworks about which a great deal is known. However, our understanding of nonconvex frameworks is growing. For example, the fact that infinitesimal rigidity is projectively invariant (see Chapter 5) enables one to create nonconvex infinitesimally rigid frameworks from convex ones. The generic rigidity of triangulated tori is also of current interest.

For a framework $G(\mathbf{p})$ obtained from a convex polyhedron C by first adding new vertices in the interior of edges of C (so each edge of C is now subdivided by edges of $G(\mathbf{p})$) and then adding new noncrossing diagonal edges across each face of C, Alexandrov (1958) shows that $G(\mathbf{p})$ is stress free, *i.e.*, rank $df_{\mathbf{p}}$ equals the number of edges of G. Therefore, for such frameworks, $G(\mathbf{p})$ is rigid (or, equivalently, infinitesimally rigid) in $\mathbb{R}^3$ if and only if $G(\mathbf{p})$ forms a triangulation of the surface of C. If new vertices are also allowed in the interior of faces of C before the addition of the new noncrossing edges in the faces of C, then the resulting framework may admit nontrivial stresses. A stress $\{\omega_{ij} : \{i,j\} \text{ an edge of } G\}$ of a framework $G(\mathbf{p})$ is called a *facial stress* if there exists a face of C such that $\omega_{ij} = 0$ for all edges $[\mathbf{p}_i, \mathbf{p}_j]$ which do not lie in the face.

Whiteley (1984) proves a significant generalization of Alexandrov's result which states that, for appropriate frameworks, every stress is a sum of facial stresses. Related results based on the study of infinitesimal flexes rather than stresses appear in Connelly (1980). In the next section of this chapter, a version of Whiteley's theorem is proved.

Finally, what happens if new vertices are present in the interior of faces of C before the faces of C are triangulated by noncrossing edges? In this case, the framework $G(\mathbf{p})$ is not infinitesimally rigid in $\mathbb{R}^3$. (To the vertices in the interior of a face, one can assign vectors which are perpendicular to the face and assign zero vectors to the remaining vertices in order to obtain a nontrivial infinitesimal flex of $G(\mathbf{p})$.) Moreover, $\mathbf{p}$ is not a regular point of the rigidity map and $G(\mathbf{p})$ admits nontrivial stresses. Nevertheless, Connelly (1980) shows that $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$ by examining the second derivative of the preservation of edge length conditions. In other words, every triangulation of the surface of a convex polyhedron in $\mathbb{R}^3$ gives a rigid framework in $\mathbb{R}^3$. Connelly's theorem is proved in the last section of this chapter.

4.3. Whiteley's Rigidity Theorem.

Theorem 4.3, the version of Cauchy's rigidity theorem for frameworks which we proved in Section 4.2, applies only to frameworks given by the edges and vertices of a convex polyhedron C in $\mathbb{R}^3$ in the sense of Definition 4.1. As mentioned at the end of the previous section, we now consider frameworks arising from the surfaces of convex polyhedra in $\mathbb{R}^3$ in other ways. Recall that Lemma 4.2, the geometric part of the proof of Theorem 4.3, consisted of showing that every vertex of G' has index at least four (where G' is the graph induced by the marked or nonzero edges of a framework $G(\mathrm{p})$). In particular, we showed that no vertex of G' has index two by using a plane through the vertex which separates the positive edges from the negative ones. Once we allow edges in the interior of faces of C, this separating plane may no longer exist and vertices of index two must be dealt with in another way.

One can further complicate matters by allowing new vertices, either ***edge vertices*** (which lie in the interior of edges of the polyhedron C) or ***face vertices*** (which lie in the interior of faces of the polyhedron C), before introducing the new diagonal edges in the faces of the polyhedron. Once these new vertices appear, the possibility of vertices of index zero also arises since there may no longer exist a plane through each vertex with the remaining vertices in one open half-space.

Alexandrov (1958) proves that if $G(\mathrm{p})$ is a framework obtained from a convex polyhedron C in $\mathbb{R}^3$ by first adding new edge vertices (no face vertices are allowed) and then triangulating each face of C by adding noncrossing diagonal edges across the face, then rank $df_{\mathrm{p}} = e$, the number of edges of $G(\mathrm{p})$. Note that, in contrast to Cauchy's result which requires the faces of the polyhedron to be rigid in $\mathbb{R}^3$, Alexandrov is only requiring the faces to be planarly rigid. Alexandrov's proof, a version of which appears in Asimow & Roth (1979), involves a very detailed analysis of the occurrence of vertices of index two. Whiteley (1984) introduces some important new ideas in his proof of a more general and more illuminating theorem. Whiteley's result concerns frameworks $G(\mathrm{p})$

arising from a convex polyhedron by allowing new edge vertices and new face vertices before the addition of new noncrossing edges to the natural faces of the polyhedron; roughly speaking, it states that if the set of edges of $G(\mathbf{p})$ in each face admits only the trivial stress, then the entire framework $G(\mathbf{p})$ admits only the trivial stress. However, it is important to note that this and many related results require that every edge of the convex polyhedron be an edge (or a union of edges) of $G(\mathbf{p})$. The following example, which will be pertinent to many results in this section, establishes the necessity of the condition that the edges of the convex polyhedron be present in the framework.

Example 4.2. Consider a tetrahedron with one face vertex in each of two faces. After triangulating these two faces, delete the common edge of the two faces (as shown by the dotted line in Figure 4.6). The resulting framework $G(\mathbf{p})$ has stress free faces but there exists a nontrivial stress of $G(\mathbf{p})$ involving the edges in the two faces which share the deleted edge.

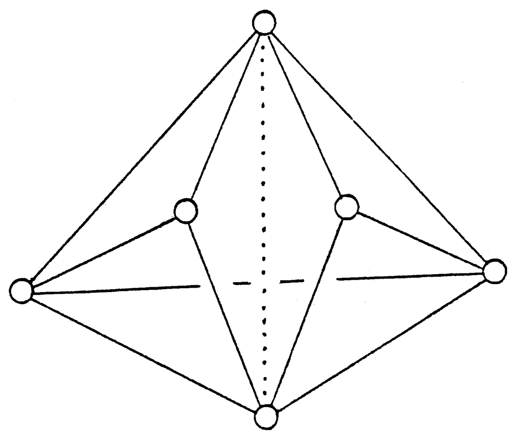


Figure 4.6

Until now, we have "built up" the frameworks under discussion by adding new vertices and edges to convex polyhedra. There is a convenient alternative description of the frameworks in question.

Definition 4.2. Consider a framework $G(\mathbf{p})$ in $\mathbb{R}^3$ and let C be the convex hull of its vertices $\mathbf{p}_1, \cdots, \mathbf{p}_v$. $G(\mathbf{p})$ is a *convex framework* in $\mathbb{R}^3$ if

(i) every edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $G(\mathbf{p})$ is contained in ∂C,

(ii) $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$, and

(iii) the affine span of $\mathbf{p}_1, \cdots, \mathbf{p}_v$ is $\mathbb{R}^3$.

Condition (i), which says that the edges lie on the boundary of the convex hull of the vertices, is the essential one. Condition (ii) says that distinct vertices don't have the same location while condition (iii) merely says the framework really is three-dimensional.

We say the edges of a framework are *noncrossing* if the intersection of any two edges of the framework is either empty or an endpoint of both edges. Incidentally, later in this section we will allow edges to cross in restricted (and occasionally unrestricted) ways. The following simple fact about convex polygons in the plane will be needed on several occasions.

Exercise 4.1. Consider a finite set $\{\mathbf{p}_1, \cdots, \mathbf{p}_v\}$ of points in $\mathbb{R}^2$ whose affine span is $\mathbb{R}^2$ such that each $\mathbf{p}_i$ belongs to the boundary of the convex hull C of $\{\mathbf{p}_1, \cdots, \mathbf{p}_v\}$. A framework $G(\mathbf{p})$, $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_v)$, consisting of the boundary edges $[\mathbf{p}_1, \mathbf{p}_2]$, $[\mathbf{p}_2, \mathbf{p}_3], \cdots, [\mathbf{p}_v, \mathbf{p}_1]$ of C and any collection of noncrossing edges in the interior of C admits only the trivial stress. Furthermore, if C is triangulated by these noncrossing edges, then $G(\mathbf{p})$ is also infinitesimally rigid in $\mathbb{R}^2$. Figure 4.7 shows such a triangulated framework.

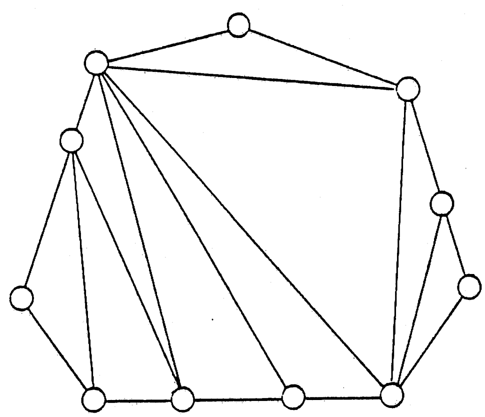


Figure 4.7

Finally, it is useful to make an observation about vertices of index zero or two in a convex framework.

Remark 4.1. Suppose $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$ with convex hull C which has a nontrivial stress $\{\omega_{ij}\}$. For each i, let $a'(i) = \{j : \omega_{ij} \neq 0\}$, the set of vertices adjacent to vertex i in the graph G' induced by the nonzero edges of $G(\mathbf{p})$. If a vertex $\mathbf{p}_i$ of G' has index zero, then

$$\sum_{j \in a'(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j) = 0$$

where every ω_{ij} for $j \in a'(i)$ has the same sign. Choose a plane through $\mathbf{p}_i$ supporting C, say an equation of the plane is $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{x}) = 0$. Then

$$0 = \mathbf{n} \cdot \sum_{j \in a'(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j) = \sum_{j \in a'(i)} \omega_{ij} \, \mathbf{n} \cdot (\mathbf{p}_i - \mathbf{p}_j) \tag{4.3}$$

and hence every $\mathbf{p}_j$ for $j \in a'(i)$ lies in the supporting plane. Thus vertices of index zero have all incident nonzero edges in a single face of C. We now observe that the same conclusion holds for vertices of index two. If the index of a vertex $\mathbf{p}_i$ of G' is two, then there is a set of positive edges followed by a set of negative edges in the cycle of edges around $\mathbf{p}_i$. There cannot exist a plane strictly separating the positive and negative edges (as is seen by looking at the proof of the index two case in Lemma 4.2). Hence the two sign changes must occur between edges in a single face of C. If $\mathbf{n} \cdot (\mathbf{p}_i - \mathbf{x}) = 0$ is an equation of the plane containing this face, then Equation (4.3) shows that all nonzero edges incident with $\mathbf{p}_i$ lie in the face.

We are now in a position to prove Whiteley's theorem for convex frameworks $G(\mathbf{p})$. It has three hypotheses; the first requires that the faces of the convex hull C be stress free, the second that the edges of $G(\mathbf{p})$ be noncrossing, and the third that each edge of C be present in the framework $G(\mathbf{p})$. If there are no vertices of $G(\mathbf{p})$ in the interior of faces of C, then the second and third hypotheses together with Exercise 4.1 guarantee that the faces of C are stress free. Later we will find that the requirement that the edges be noncrossing can be relaxed.

Theorem 4.5. *(Whiteley) Consider a convex framework $G(\mathbf{p})$ in $\mathbb{R}^3$ and let C be the convex hull of its vertices $\mathbf{p}_1, \cdots, \mathbf{p}_v$. If*

(a) *the set of edges of* $G(\mathbf{p})$ *in each face of* C *forms a stress free framework,*

(b) *the edges of* $G(\mathbf{p})$ *are noncrossing, and*

(c) *each edge of* C *is a union of edges of* $G(\mathbf{p})$*,*

then $G(\mathbf{p})$ *admits only the trivial stress.*

Proof. Suppose there exists a convex framework in $\mathbb{R}^3$ satisfying properties (a), (b), and (c) with a nontrivial stress. Then there exists such a convex framework $G(\mathbf{p})$ in $\mathbb{R}^3$ with a nontrivial stress such that the set of edges of this framework with nonzero coefficients induces a subgraph G' with a minimum number of vertices. It cannot be the case that every vertex of G' has index at least four since we then have

$$I \leq 4v' - 8 < 4v' \leq I$$

where v' is the number of vertices of G' and I is the index. (Note that $I \leq 4v' - 8$ follows from Lemma 4.1.) Therefore, G' has a vertex of index zero or two and, by Remark 4.1, all edges of G' incident with this vertex lie in a single face F of C.

We now create a new convex framework with a nonzero stress in which the nonzero edges induce a graph with a smaller number of vertices than G'. This involves the removal of a vertex of index zero or two from the face F. The notions of equilibrium and resolvable forces (Chapter 2.??) play a crucial role in the creation of this new framework. Among other things, it is useful to observe that if a system of forces can be resolved, then the system of forces is an equilibrium one.

Our assumption that $G(\mathbf{p})$ has a nontrivial stress means that there exists a set $\{\omega_{ij} : \{i,j\} \text{ an edge of } G\}$, the nonzero elements of which induce the graph G', such that for $1 \leq i \leq v$

$$\sum_{j \in a(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j) = 0$$

where, as usual, $a(i) = \{j : \{i,j\} \text{ an edge of } G\}$. Suppose the vertices of $G(\mathbf{p})$ which belong to the face F are $\mathbf{p}_1, \cdots, \mathbf{p}_m$ where the first n of these, namely $\mathbf{p}_1, \cdots, \mathbf{p}_n$, are the vertices of $G(\mathbf{p})$ on F which are incident with an edge of G' off the face F. Note that the set $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ is nonempty by hypothesis (a), contains no vertices of index zero or two by Remark 4.1, and contains no face vertices. Also $n < m$ since F contains a vertex of index zero or two. Note that $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ is a subset of the boundary of the convex polygon F. For $1 \leq i \leq m$, let $b(i) = \{j : \text{the edge } \{i,j\} \text{ of } G \text{ lies on } F\}$ and let

$$\mathbf{F}_i = \sum_{j \in b(i)} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j) \ . \tag{4.4}$$

Note that the vectors $\mathbf{F}_i$, $1 \leq i \leq m$, all lie in the plane of F. Then, by definition, $(\mathbf{F}_1, \cdots, \mathbf{F}_m)$ is a resolvable force for the set of edges of $G(\mathbf{p})$ on the face F and hence an equilibrium force for $(\mathbf{p}_1, \cdots, \mathbf{p}_m)$. However, $\mathbf{F}_i = 0$ for $n < i \leq m$ and thus it is easy to verify that $(\mathbf{F}_1, \cdots, \mathbf{F}_n)$ is an equilibrium force for $(\mathbf{p}_1, \cdots, \mathbf{p}_n)$.

Assume now that the affine span of $\mathbf{p}_1, \cdots, \mathbf{p}_n$ is two-dimensional. Next delete everything (all vertices and edges) on face F except for the vertices $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$. We then rebuild an appropriate framework on face F. The appropriate framework on F is given by any triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ by noncrossing edges. By Exercise 4.1, the resulting framework with vertices $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ is both infinitesimally rigid and stress free. Since the forces $\mathbf{F}_i$, $1 \leq i \leq n$, all lie in the plane of F, the equilibrium force $(\mathbf{F}_1, \cdots, \mathbf{F}_n)$ can be resolved by our triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$. Thus there exist scalars $\{\omega'_{ij}\}$, one for each edge of this triangulation, such that for $1 \leq i \leq n$

$$\mathbf{F}_i = \sum \omega'_{ij} (\mathbf{p}_i - \mathbf{p}_j) \tag{4.5}$$

where for each i the sum is over those j with $\{i,j\}$ an edge of the triangulation.

At this point, the basic idea is to simply combine the triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ and its accompanying scalars $\{\omega'_{ij}\}$ with the remainder of the original framework $G(\mathbf{p})$ and its scalars $\{\omega_{ij}\}$ in such a way that we obtain a new framework with a nontrivial stress for which the nonzero edges induce a graph with a smaller number of vertices than G'. Of course, one needs to be sure that conditions (a), (b), and (c) are satisfied for the new framework. More formally, the edges of the new framework are:

(i) the edges of the triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$;

(ii) the edges of the original framework which are both off F and nonzero (that is, original edges $\{i, j\}$ not on F for which $\omega_{ij} \neq 0$); and

(iii) whatever additional edges or portions of edges of C, both on and off face F, that are needed to satisfy condition (c). The dotted lines in Figure 4.8 illustrate two portions of edges of C on F and an edge of C off F that have been added in (iii).

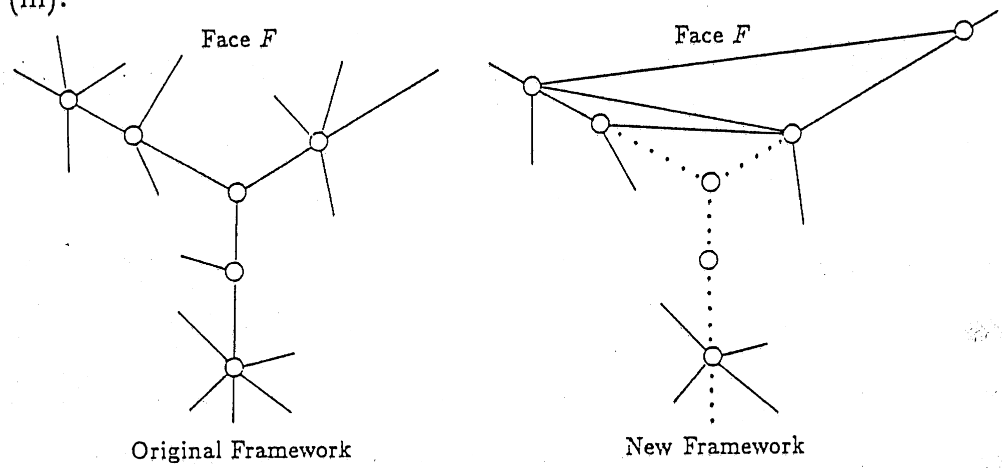


Figure 4.8

A stress of the new framework is given by assigning scalars to its edges in the following ways:

(i) the scalars ω'_{ij} given by Equation (4.5) for the edges of the triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$;

(ii) the original scalars ω_{ij} for the nonzero edges of the original framework that are not on F; and

(iii) the scalar zero for edges or portions of edges of C that have been added to satisfy condition (c).

Equations (4.4) and (4.5) imply that this is a stress of the new framework. The stress is nontrivial since in the original framework there is an edge $\{i, j\}$ not on F for which $\omega_{ij} \neq 0$ (by hypothesis (a)); this edge and its nonzero scalar are present in the new framework. Finally, the nonzero edges of this stressed framework induce a graph G''

with fewer vertices than G' since every vertex of G'' is a vertex of G' and at least one vertex of G', namely the vertex of index zero or two on face F, is not a vertex of G''.

Earlier in the proof we assumed that the affine span of $\mathbf{p}_1, \cdots, \mathbf{p}_n$ is two-dimensional. We now consider the zero- and one-dimensional cases. For these cases, it is useful to recall that in $\mathbb{R}^3$, $(\mathbf{F}_1, \cdots, \mathbf{F}_n)$ is an equilibrium system of forces for $(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ if and only if $\sum \mathbf{F}_i = 0$ and $\sum \mathbf{p}_i \times \mathbf{F}_i = 0$. (See Chapter 2.??.) If $n = 1$ (the zero-dimensional case), then $\mathbf{F}_1 = 0$ which says that $\{\omega_{ij}$: the edge $\{i, j\}$ of G lies on face $F\}$ is a nontrivial stress of the set of edges of $G(\mathbf{p})$ in the face F, contradicting hypothesis (a). If $\mathbf{p}_1, \cdots, \mathbf{p}_n$ are collinear (the one-dimensional case), then $\mathbf{p}_1, \cdots, \mathbf{p}_n$ all lie on a single edge of C. For those $\mathbf{p}_i$ in the interior of this edge, $\mathbf{F}_i$ must be parallel to the edge since $\mathbf{F}_i$ lies in the plane of the face F, and $-\mathbf{F}_i$ which is the sum of the forces on $\mathbf{p}_i$ from edges off of F lies in the plane of a face of C adjacent to face F. Since $\sum \mathbf{F}_i = \sum \mathbf{p}_i \times \mathbf{F}_i = 0$, it follows that all the forces $\mathbf{F}_i$, $1 \leq i \leq n$, are parallel to the edge of C containing $\mathbf{p}_1, \cdots, \mathbf{p}_n$. Then it is not difficult to verify that such an equilibrium force $(\mathbf{F}_1, \cdots, \mathbf{F}_n)$ can be resolved by assigning appropriate scalars to the $n - 1$ edges between adjacent vertices of the set $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$. (Incidentally, this verification is part of the study of infinitesimal rigidity on a line.) To complete the proof in this case, one simply uses these $n - 1$ edges and their accompanying scalars to create the new framework instead of the triangulation of the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ that was used in the two-dimensional case. □

We now prove a number of corollaries of Whiteley's theorem. These corollaries deal primarily with two themes. One is the relationship between the stresses of a convex framework in $\mathbb{R}^3$ and the stresses of its faces. Of course, Whiteley's theorem is one such result. The other main theme deals with the connections between the spatial infinitesimal rigidity of a convex framework and the planar infinitesimal rigidity of its faces. We begin by relating Whiteley's theorem to Alexandrov's rigidity theorem.

Consider a convex framework $G(\mathbf{p})$ in $\mathbb{R}^3$ with convex hull C. If $G(\mathbf{p})$ has no face vertices and its edges are noncrossing, then the set of edges of $G(\mathbf{p})$ in each face F of C extends to a stress free triangulation of F by Exercise 4.1. By Theorem 4.5, the entire extended framework is stress free and thus $G(\mathbf{p})$ is as well, *i.e.*, rank $df_{\mathbf{p}} = e$. In particular, $\mathbf{p}$ is a regular point of f. Our first corollary includes Alexandrov's result regarding the infinitesimal rigidity of triangulated convex polyhedra without face vertices.

Corollary 4.6. *(Alexandrov) Consider a convex framework $G(\mathbf{p})$ in $\mathbb{R}^3$ with convex hull C. Suppose every vertex of $G(\mathbf{p})$ belongs to an edge of C and the edges of $G(\mathbf{p})$ are noncrossing. Then $G(\mathbf{p})$ is rigid (or, equivalently, infinitesimally rigid) in $\mathbb{R}^3$ if and only if every region of $\partial C - G(\mathbf{p})$ is a triangle.*

Proof. We have rank $df_{\mathbf{p}} = e$ and thus $G(\mathbf{p})$ is rigid (or infinitesimally rigid) in $\mathbb{R}^3$ if and only if

$$e = \text{ rank } df_{\mathbf{p}} = 3v - 6 \ .$$

But the usual Euler's formula argument applied to the framework $G(\mathbf{p})$ on the boundary ∂C of C shows that $e = 3v - 6$ if and only if every region of $\partial C - G(\mathbf{p})$ is a triangle, i.e., every face of C has been triangulated. □

A stress $\omega = (\cdots, \omega_{ij}, \cdots)$ of a framework $G(\mathbf{p})$ is simply an element of ker $df_{\mathbf{p}}^T$ where $df_{\mathbf{p}}^T : \mathbb{R}^e \to \mathbb{R}^{3v}$ is the transpose of $df_{\mathbf{p}}$. Thus the vector space S of stresses of $G(\mathbf{p})$ satisfies dim $S = e -$ rank $df_{\mathbf{p}}$. We now adopt notation for use in the remainder of this section. Suppose $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$ whose convex hull is C. For each face F of C, let v_F and e_F be the numbers of vertices and edges of $G(\mathbf{p})$ which belong to the face F, let f_F be the rigidity map for the framework consisting of the set of edges of $G(\mathbf{p})$ which lie on the face F, and let S_F be the subspace of stresses "supported by" F, i.e., the stresses of $G(\mathbf{p})$ with $\omega_{ij} = 0$ for all edges $\{i, j\}$ of $G(\mathbf{p})$ which do not lie on F. Finally, let e_0 be the number of edges of $G(\mathbf{p})$ which are contained in the edges of C and v_0 be the number of face vertices of $G(\mathbf{p})$.

Corollary 4.7. *Let $G(\mathbf{p})$ be a convex framework in $\mathbb{R}^3$ with convex hull C. If the edges of $G(\mathbf{p})$ are noncrossing and each edge of C is a union of edges of $G(\mathbf{p})$, then S is the direct sum of its subspaces $\{S_F : F$ a face of $C\}$.*

Proof. For each face F of C, the set of edges of $G(\mathbf{p})$ which forms the boundary of the convex polygon F is stress free and thus extends to a basis for the rows of $df_F(\mathbf{p})$ where, for an obvious reason, we now let $df_F(\mathbf{p})$ denote the derivative of f_F at $\mathbf{p}$. By Theorem 4.5, the union of these bases is a basis for the rows of $df_{\mathbf{p}}$. A simple counting argument shows that the dimension of S is the sum of the dimensions of the spaces S_F. Note first that

$$\text{rank } df_{\mathbf{p}} = \sum \text{rank } df_F(\mathbf{p}) - e_0 \,.$$

Since $\sum e_F = e + e_0$, we have $\dim S = e - \text{ rank } df_{\mathbf{p}} = e + e_0 - \sum \text{ rank } df_F(\mathbf{p}) = \sum(e_F - \text{ rank } df_F(\mathbf{p})) = \sum \dim S_F$. To complete the proof, it suffices to show that the spaces S_F don't "overlap". That is, we show that $\sum \omega_F = 0$ where each $\omega_F \in S_F$ implies each $\omega_F = 0$. Consider any face F_0 of C. For all edges $[\mathbf{p}_i, \mathbf{p}_j]$ of $G(\mathbf{p})$ which lie on F_0 but are not part of the boundary of F_0, the ij coordinate of ω_F is zero for all F except F_0 and hence for F_0 as well since $\sum \omega_F = 0$. But the set of edges of $G(\mathbf{p})$ which makes up the boundary of F_0 is stress free. Thus the remaining coordinates of ω_{F_0} are also zero and hence $\omega_{F_0} = 0$. □

It is of some interest to observe that the requirement that the edges of $G(\mathbf{p})$ are noncrossing is not essential as long as one restricts the crossings to the interior of faces of C.

Definition 4.3. A convex framework with convex hull C has *crossing interior edges* if the intersection of any pair of edges is empty or consists of a single point which is either an endpoint of both edges or lies in the interior of a face of C.

This definition allows two types of actual crossings – ones in which the point of intersection is an endpoint of neither edge and ones in which the point of intersection is an endpoint of one edge but not the other. Note that according to our definitions, noncrossing edges are crossing interior edges. Example 4.3 shows the need for prohibiting crossings on edges of C.

Example 4.3. Consider the tetrahedron shown in Figure 4.9 in which the "front" vertical edge is the single unbroken segment $[\mathbf{p}_1, \mathbf{p}_2]$. In the two faces adjacent to this edge, place the stressed portion of the framework found in Example 4.2. Then the crossings of the resulting framework occur at the points $\mathbf{p}_3$ and $\mathbf{p}_4$ (for instance, $[\mathbf{p}_1, \mathbf{p}_2] \cap [\mathbf{p}_3, \mathbf{p}_5] = \{\mathbf{p}_3\}$), neither of which is in the interior of a face of the tetrahedron. Then the portion of the framework from Example 4.2 supports a stress, but each face of this framework is stress free.

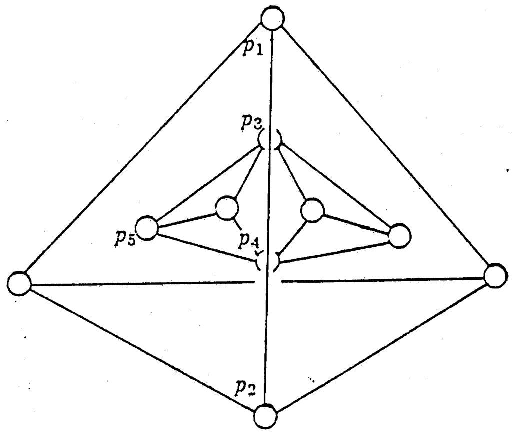


Figure 4.9

Corollary 4.8. *Suppose $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$ with crossing interior edges and C is its convex hull. If every edge of C is a union of edges of $G(\mathbf{p})$, then the stress space S of $G(\mathbf{p})$ is the direct sum of its subspaces $\{S_F : F$ a face of $C\}$.*

Proof. The idea is to simply "break" edges at crossings, forming a new framework $G'(\mathbf{p}')$ to which the previous corollary applies. More precisely, if $\mathbf{p}_0$ is a point of intersection of edges of $G(\mathbf{p})$ which is not a vertex of $G(\mathbf{p})$, let $\mathbf{p}_0$ be a vertex of $G'(\mathbf{p}')$ and break each edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $G(\mathbf{p})$ through $\mathbf{p}_0$ into the two edges $[\mathbf{p}_0, \mathbf{p}_i]$ and $[\mathbf{p}_0, \mathbf{p}_j]$. If $\mathbf{p}_0$ is a point of intersection of edges of $G(\mathbf{p})$ which is a vertex of $G(\mathbf{p})$, then break each edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $G(\mathbf{p})$ with $\mathbf{p}_0$ in its interior into the two edges $[\mathbf{p}_0, \mathbf{p}_i]$ and $[\mathbf{p}_0, \mathbf{p}_j]$. Then $G'(\mathbf{p}')$, the resulting framework, is a convex framework in $\mathbb{R}^3$ with noncrossing edges.

Suppose $\omega = (\cdots, \omega_{ij}, \cdots)$ is a stress of $G(\mathbf{p})$. The stress ω leads in a natural way to a stress ω' of the framework $G'(\mathbf{p}')$ with the property that for each "broken" edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $G(\mathbf{p})$, we have

$$\omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \omega'_{0i}(\mathbf{p}_i - \mathbf{p}_O) = \omega'_{0j}(\mathbf{p}_O - \mathbf{p}_j) \ .$$

In physical terms, $[\mathbf{p}_O, \mathbf{p}_i]$ and $[\mathbf{p}_O, \mathbf{p}_j]$ exert opposite forces at $\mathbf{p}_O$ which are equal in magnitude to the force exerted by $[\mathbf{p}_i, \mathbf{p}_j]$ on its endpoints $\mathbf{p}_i$ and $\mathbf{p}_j$. By Corollary 4.7, ω' is a sum of stresses ω'_F of the faces F of $G'(\mathbf{p}')$. Since the crossings only occur in the interior of faces, each ω'_F leads naturally (by removing the breaks) to a stress ω_F of the face F of $G(\mathbf{p})$. (In general, one cannot simply remove breaks and create a new stress. However, one can in this case because ω' arose by breaking edges.) Clearly we have that $\omega = \sum \omega_F$. The argument at the end of the proof of Corollary 4.7 gives the uniqueness of the representation of ω as a sum of facial stresses. □

One final comment about the relationship between the stresses of a convex framework and the planar stresses of its faces seems appropriate. Since a "stress free" theorem is easily seen to be equivalent to a "direct sum" theorem, it is clear that Corollary 4.8 implies a version of Theorem 4.5 in which crossing interior edges are allowed (but, of course, conditions (a) and (c) of Theorem 4.5 remain unchanged).

Even to a casual observer, it appears likely that there is some connection between the spatial rigidity of a convex framework and the planar rigidity of its faces. Perhaps the most natural question is: for a convex framework in $\mathbb{R}^3$, does the planar rigidity (or infinitesimal rigidity) of each face guarantee the spatial rigidity (or infinitesimal rigidity) of the entire convex framework? Supporting evidence is provided by Corollary 4.6 which implies that for a convex framework in $\mathbb{R}^3$ with no face vertices and noncrossing edges, the framework is rigid in $\mathbb{R}^3$ if and only if every face is rigid in $\mathbb{R}^2$, and the same result holds for infinitesimal rigidity.

A set of unpublished but widely circulated notes, entitled "Lectures on Lost Mathematics", by Branko Grünbaum and G.C. Shephard had a significant impact on developments in the study of rigidity during the late 1970's. Their observations and conjectures inspired much of the work for convex frameworks during that time. Although many of their conjectures turned out to be correct (and, in some cases, quite difficult to establish), their record with two conjectures about the planar rigidity of faces implying the spatial rigidity of the convex framework was not so good. Their first conjecture of this kind (Conjecture 4, Grünbaum & Shephard (1975)) states that a convex framework is rigid in $\mathbb{R}^3$ whenever each face is rigid in $\mathbb{R}^2$. (Actually both their conjectures are formulated for cabled frameworks; we here state the versions for bar frameworks.) Their later conjecture (Sample Conjecture, Grümbaum & Shephard (1978)) simply adds the requirement that all the edges of the convex hull be bars of the framework. A counterexample to both conjectures is given by attaching a vertex of degree two (a "floppy triangle") to one face of a framework given by a tetrahedron. However, the fact that this example is not three-connected seems to violate at least the spirit of both conjectures. We will soon find three-connected counterexamples to both conjectures (see Corollary 4.12 and Example 4.5.) Despite the existence of these counterexamples, there is a good deal that can be said about the connections between the spatial rigidity of a framework and the planar rigidity of its faces. However, the precise conditions under which these connections are valid are of a somewhat delicate nature.

If $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$ which has a face vertex, then it is easy to describe a nontrivial infinitesimal flex of $G(\mathbf{p})$ – simply "push" (or assign a nonzero vector to) the face vertex in a direction orthogonal to the face. The following result says that all the nontrivial infinitesimal flexes of $G(\mathbf{p})$ arise in this way provided that each face of $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^2$. Connelly (1980) introduces the techniques and proves a version of this useful and important result. Incidentally, it is worthwhile to note that the theorem does not require that the edges of the convex hull be present in the framework and there is no restriction whatsoever on the nature of the crossings of edges.

Theorem 4.9. *Suppose that $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$, C is the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_v\}$, and v_0 is the number of face vertices of $G(\mathbf{p})$. If the set of vertices and edges of $G(\mathbf{p})$ in each face F of C forms an infinitesimally rigid framework in $\mathbb{R}^2$, then*

$$ker\ df_{\mathbf{p}} = T_{\mathbf{p}} \oplus N_{\mathbf{p}}$$

where $N_{\mathbf{p}}$ is the v_0-dimensional space of infinitesimal flexes orthogonal to faces.

Proof. Let $A = \{i : \mathbf{p}_i$ lies on an edge of $C\}$ and $B = V - A = \{i : \mathbf{p}_i$ is a face vertex of $G(\mathbf{p})\}$. We show that ker $df_{\mathbf{p}}$, the space of infinitesimal flexes of $G(\mathbf{p})$, is the direct sum of the six-dimensional space $T_{\mathbf{p}}$ of trivial infinitesimal flexes and the v_0-dimensional space $N_{\mathbf{p}} = \{\mathbf{u} = (\mathbf{u}_1, \cdots, \mathbf{u}_v) \in \mathbb{R}^{3v} : i \in A$ implies $\mathbf{u}_i = 0$ and $i \in B$ implies $\mathbf{u}_i$ is orthogonal to the face of C containing $\mathbf{p}_i\}$. Clearly $T_{\mathbf{p}} + N_{\mathbf{p}} \subset$ ker $df_{\mathbf{p}}$ since both are subspaces of ker $df_{\mathbf{p}}$. Using the description of elements of $T_{\mathbf{p}}$ as $\mathbf{r} \times \mathbf{p}_i + \mathbf{t}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{R}^3$ (see Chapter 2.7), it is easy to show that $T_{\mathbf{p}} \cap N_{\mathbf{p}} = \{O\}$ so $T_{\mathbf{p}} + N_{\mathbf{p}} = T_{\mathbf{p}} \oplus N_{\mathbf{p}}$. Thus it suffices to show that ker $df_{\mathbf{p}} \subset T_{\mathbf{p}} + N_{\mathbf{p}}$. For this purpose, we consider a new convex framework $G'(\mathbf{p}')$ in $\mathbb{R}^3$ whose vertices are $\{\mathbf{p}_i : i \in A\}$ and whose edges are obtained by any triangulation (by noncrossing edges) of the faces of C (using only the vertices in A). For each face F of C, let P_F be the orthogonal projection of $\mathbb{R}^3$ onto the (two-dimensional subspace parallel to the) face F.

Suppose $\mathbf{u} \in$ ker $df_{\mathbf{p}}$. We now construct $\mathbf{w} \in T_{\mathbf{p}}$ so that $\mathbf{u} - \mathbf{w} \in N_{\mathbf{p}}$. For each face F of C, since $\mathbf{u} \in$ ker $df_{\mathbf{p}}$, we have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$ and hence $(\mathbf{p}_i - \mathbf{p}_j) \cdot (P_F\mathbf{u}_i - P_F\mathbf{u}_j) = 0$ for all edges $\{i, j\}$ of $G(\mathbf{p})$ on F. Since the set of vertices and edges of $G(\mathbf{p})$ on F is infinitesimally rigid in the plane by hypothesis, we then have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (P_F\mathbf{u}_i - P_F\mathbf{u}_j) = 0$ for all $\mathbf{p}_i$ and $\mathbf{p}_j$ on F, even those $\mathbf{p}_i$ and $\mathbf{p}_j$ not joined by an edge, by Proposition 2.30. Therefore we have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$ for all edges $\{i, j\}$ of the new framework $G'(\mathbf{p}')$. Since the new framework is infinitesimally rigid in

$\mathbb{R}^3$ by Alexandrov's theorem (Corollary 4.6), the infinitesimal flex $\{\mathbf{u}_i : i \in A\}$ of $G'(\mathbf{p}')$ is trivial. Thus there exist vectors $\mathbf{t}$ and $\mathbf{r}$ in $\mathbb{R}^3$ such that $\mathbf{u}_i = \mathbf{t} + \mathbf{r} \times \mathbf{p}_i$ for every $i \in A$. Now we simply apply this infinitesimal motion to all the vertices of $G(\mathbf{p})$ to get $\mathbf{w}$. That is, let $\mathbf{w} = (\mathbf{t}+\mathbf{r}\times\mathbf{p}_1, \cdots, \mathbf{t}+\mathbf{r}\times\mathbf{p}_v) \in T_\mathbf{p} \subset \ker df_\mathbf{p}$. Clearly $\mathbf{u}-\mathbf{w} \in \ker df_\mathbf{p}$ and $\mathbf{u}_i - \mathbf{w}_i = 0$ for all $i \in A$. Consider any face vertex $\mathbf{p}_j$ of $G(\mathbf{p})$, say $\mathbf{p}_j$ belongs to the face F of C. Then the previous projection argument applied to $\mathbf{u} - \mathbf{w}$ gives

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (P_F(\mathbf{u}_j - \mathbf{w}_j) - P_F(\mathbf{u}_i - \mathbf{w}_i)) = 0$$

for all i such that $\mathbf{p}_i$ lies on F, even if $\mathbf{p}_i$ and $\mathbf{p}_j$ are not joined by an edge. But for all $i \in A$, $\mathbf{u}_i - \mathbf{w}_i = 0$ and thus $(\mathbf{p}_j - \mathbf{p}_i) \cdot P_F(\mathbf{u}_j - \mathbf{w}_j) = 0$ for all $i \in A$ for which $\mathbf{p}_i$ lies on F. Therefore $P_F(\mathbf{u}_j - \mathbf{w}_j) = 0$ which says $\mathbf{u}_j - \mathbf{w}_j$ is orthogonal to F. Thus $\mathbf{u} - \mathbf{w} \in N_\mathbf{p}$. □

Note that Theorem 4.9 says that $\dim \ker df_\mathbf{p} - \dim T_\mathbf{p}$, which we call the number of *infinitesimal degrees of freedom* of $G(\mathbf{p})$, equals v_0. In particular, for triangulated convex frameworks, the number of infinitesimal degrees of freedom is given by the number of face vertices (although such frameworks are always rigid by the rigidity theorem of Connelly (1980)).

A useful way to interpret Theorem 4.9 is that if $\mathbf{u}$ is an infinitesimal flex of a convex framework which has faces that are infinitesimally rigid in the plane, then $\mathbf{u}$ must be a trivial infinitesimal flex on the set of edge vertices of the framework. That is, there exists an infinitesimal congruence $\mathbf{x} \to V_\mathbf{x}$ of $\mathbb{R}^3$ such that $\mathbf{u}_i = V_{\mathbf{p}_i}$ for all $i \in A$. By subtracting a trivial infinitesimal flex from $\mathbf{u}$, one obtains an infinitesimal flex that is zero on the edge vertices of the framework (as in the proof of Theorem 4.15).

The following corollary, which generalizes Corollary 4.6, requires no assumption about the nature of the crossings of edges and does not require that the edges of the convex hull be present in the framework. However, it does prohibit the presence of face vertices.

Corollary 4.10. *Let $G(\mathbf{p})$ be a convex framework in $\mathbb{R}^3$ with convex hull C and suppose every vertex of $G(\mathbf{p})$ belongs to an edge of C. If the set of vertices and edges of $G(\mathbf{p})$ in each face of C forms an infinitesimally rigid framework in the plane, then $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^3$.*

Proof. Since $v_0 = 0$, we have ker $df_{\mathbf{p}} = T_{\mathbf{p}}$ by Theorem 4.9 and thus $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{R}^3$. □

Theorem 4.9 and Corollary 4.10 lead to versions of what Whiteley (1984) calls "substitution principles" for convex frameworks. (Actually one could view the key step in the proof of Whiteley's theorem in which a face was rebuilt as a primitive kind of substitution principle since it involves substituting a retriangulated new face for an original face.) However, a more substantial substitution principle says that if for the facial frameworks of an infinitesimally rigid convex framework in $\mathbb{R}^3$ one substitutes planar frameworks (utilizing the same vertices) that are infinitesimally rigid in $\mathbb{R}^2$, then the resulting framework is still infinitesimally rigid in $\mathbb{R}^3$. In fact, one can even introduce additional vertices on the edges of the convex hull and then substitute planar frameworks on this augmented set of vertices. All this is very easy to show since if the original framework is infinitesimally rigid in $\mathbb{R}^3$ it has no face vertices and thus the resulting framework is infinitesimally rigid in $\mathbb{R}^3$ by Corollary 4.10. One immediate consequence of this substitution principle is that the "weak" form of Alexandrov's theorem (which says that a triangulated convex framework whose vertices are precisely the extreme points or "natural" vertices of the convex hull is infinitesimally rigid in $\mathbb{R}^3$) implies the "strong" form of Alexandrov's theorem (which allows the presence of edge vertices). Or, by appealing to Theorem 4.9 rather than its corollary, one can also introduce face vertices, substitute for the faces planar infinitesimally rigid frameworks on this augmented set of vertices, and conclude that the number of infinitesimal degrees of freedom of the resulting framework is equal to the number of face vertices. Moreover, as Whiteley (1984) observes, neither the convexity of the polyhedron nor even the convexity of its faces is essential here. One can extend these substitution principles to certain nonconvex "polyhedral frameworks with faces" (see Section 4 of Whiteley (1984) for the definition of an appropriate class of frameworks) either by Connelly's infinitesimal motions argument (as in the proof of Theorem 4.9) or by Whiteley's statics argument (as in Theorem 4.1 of Whiteley (1984)). In this setting, the infinitesimal rigidity of the original framework will play the role that Alexandrov's theorem does in the proof of Theorem 4.9. Finally, the reader should note that beginning with Theorem 4.9 we have drifted toward infinitesimal kinematics and away from statics. This shift in point of view comes from our desire to prove Connelly's rigidity theorem which is stated in the language of infinitesimal motions.

Our next example both illustrates the use of Corollary 4.10 and shows that its converse fails.

Example 4.4. Consider a cube with a diagonal edge added across each of its six faces. After adding an additional crossing diagonal edge to two adjacent faces, delete the common edge of these two faces (shown by the dotted line in "front" in Figure 4.10). By Corollary 4.10, $G(\mathrm{p})$ is infinitesimally rigid in $\mathbb{R}^3$ and thus rank $df_{\mathrm{p}} = 3v - 6$. Since there exists a stress of $G(\mathrm{p})$ which is nonzero on the edges of the two doubly braced faces of the cube, we can delete any single edge of one of these two faces and the resulting framework is both stress free and infinitesimally rigid in $\mathbb{R}^3$. Since one face of the resulting framework is infinitesimally flexible (in fact, flexible) in $\mathbb{R}^2$, the converse of Corollary 4.10 is false.

Of course, the difficulty is again caused by the absence of an edge of C, and once this is remedied, the converse of Corollary 4.10 is indeed valid.

Corollary 4.11. *Let $G(\mathrm{p})$ be a convex framework in $\mathbb{R}^3$ with convex hull C. Suppose $G(\mathrm{p})$ has crossing interior edges and every edge of C is a union of edges of $G(\mathrm{p})$. If $G(\mathrm{p})$ is infinitesimally rigid in $\mathbb{R}^3$, then the set of vertices and edges of $G(\mathrm{p})$ in each face of C forms an infinitesimally rigid framework in the plane.*

Proof. Clearly the infinitesimal rigidity of $G(\mathrm{p})$ in $\mathbb{R}^3$ implies that $G(\mathrm{p})$ has no face

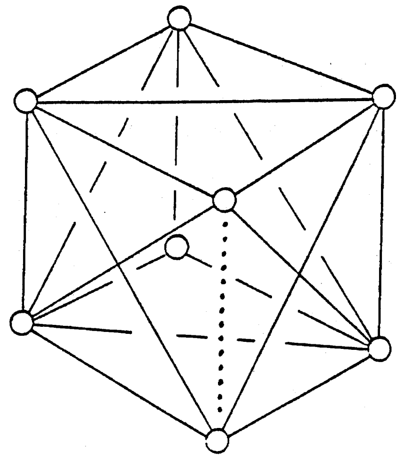


Figure 4.10

vertices. We apply Euler's formula to the graph G' induced by the edges of $G(\mathbf{p})$ which are contained in the edges of C. Let e_0 be the number of such edges. Let f_0 be the number of regions of $\partial C - G'$, i.e., the number of faces of C, and let v_n be the number of vertices of $G(\mathbf{p})$ which are incident with exactly n edges of G'. For each face F of C, the set of edges of $G(\mathbf{p})$ which forms the boundary of F is stress free and hence extends to a basis for the rows of $df_F(\mathbf{p})$. By Corollary 4.8, the union of these bases is a basis for the rows of $df_{\mathbf{p}}$. Thus rank $df_{\mathbf{p}} = \sum$ rank $df_F(\mathbf{p}) - e_0$. Since rank $df_F(\mathbf{p}) \leq 2v_F - 3$ for every face F of C, we have

$$\text{rank } df_{\mathbf{p}} = \sum \text{rank } df_F(\mathbf{p}) - e_0 \leq \sum(2v_F - 3) - e_0 =$$

$$2\sum nv_n - 3f_0 - e_0 = 4e_0 - 3f_0 - e_0 = 3(e_0 - f_0) = 3(v-2) = 3v - 6$$

with equality if and only if rank $df_F(\mathbf{p}) = 2v_F - 3$ for every face F of C. Since rank $df_{\mathbf{p}} = 3v - 6$ by Proposition 2.35, we have rank $df_F(\mathbf{p}) = 2v_F - 3$ for every F. Therefore the set of vertices and edges of $G(\mathbf{p})$ in each face of C is infinitesimally rigid in $\mathbb{R}^2$, again by Proposition 2.35. □

Note that by combining Corollaries 4.10 and 4.11 (so the convex framework must have no face vertices, crossing interior edges and each edge of the convex hull present), one obtains a result that says the entire framework is infinitesimally rigid in $\mathbb{R}^3$ if and only if every face is infinitesimally rigid in $\mathbb{R}^2$. Our final corollary deals with frameworks for which both conjectures of Grünbaum & Shephard always fail, i.e., the faces are rigid in the plane and yet the entire framework is flexible in $\mathbb{R}^3$.

Corollary 4.12. *Consider a convex framework $G(\mathbf{p})$ in $\mathbb{R}^3$ with convex hull C such that $G(\mathbf{p})$ has crossing interior edges and every edge of C is a union of edges of $G(\mathbf{p})$. If the set of vertices and edges of $G(\mathbf{p})$ in each face of C is both infinitesimally rigid in the plane and stress free, then $\mathbf{p}$ is a regular print of the rigidity map f (and hence $G(\mathbf{p})$ is flexible in $\mathbb{R}^3$ whenever $v_O > 0$).*

Proof. By Corollary 4.8, $G(\mathbf{p})$ is stress free so rank $df_{\mathbf{p}} = e$ and thus $\mathbf{p}$ is a regular point of f. Therefore $f^{-1}(f(\mathbf{p}))$ is a smooth manifold near $\mathbf{p}$ of dimension $3v -$ rank $df_{\mathbf{p}}$. And $G(\mathbf{p})$ will be flexible in $\mathbb{R}^3$ whenever this manifold has dimension greater than six, the dimension of the smooth manifold $H_{\mathbf{p}}$ of points congruent to $\mathbf{p}$. But Theorem 4.9 implies that

$$3v - \text{ rank } df_{\mathbf{p}} - 6 = \text{ dim ker } df_{\mathbf{p}} - 6 = v_O \ . \quad \square$$

Example 4.2 (continued). For the framework $G(\mathbf{p})$ shown in Figure 4.6, we have rank $df_{\mathbf{p}} = 3v - (v_O + 6) = 10$ by Theorem 4.9. But max $\{$ rank $df_{\mathbf{q}} : \mathbf{q} \in \mathbb{R}^{3v}\} = 11$ by Theorem 4.3 since clearly there exists $\mathbf{q} \in \mathbb{R}^{3v}$ such that $G(\mathbf{q})$ is the framework given by a convex polyhedron in $\mathbb{R}^3$ in the sense of Definition 4.1. Therefore $\mathbf{p}$ is not a regular point of the rigidity map. This shows the necessity of the assumption that every edge of C is a union of edges of $G(\mathbf{p})$ in the previous corollary.

Our next example illustrates a typical use of Corollary 4.12 and gives a flexible three-connected framework for which all the faces are rigid in the plane.

Example 4.5. Consider a cube with diagonal edges added across five of its six faces. Add a face vertex to the sixth face and join it by edges to three of the four vertices of the face. The resulting framework $G(\mathbf{p})$ is shown in Figure 4.11. By Corollary 4.12, $G(\mathbf{p})$ is flexible in $\mathbb{R}^3$.

Consider a convex framework $G(\mathbf{p})$ with convex hull C and crossing interior edges such that $v_O \neq 0$, $e_F = 2v_F - 3$ for every face F of C (so the "count" is correct on each

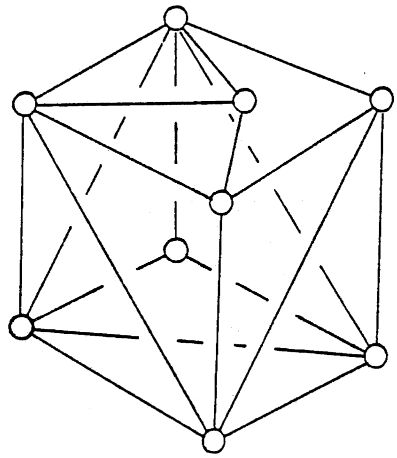


Figure 4.11

face) and every edge of C is a union of edges of $G(\mathbf{p})$. By Corollary 4.12, $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$ only if some face of C is infinitesimally flexible in $\mathbb{R}^2$! The following example shows how this phenomenon can occur. Incidentally, the role that stresses play in rigid but not infinitesimally rigid frameworks is becoming clearer (see Chapter 7).

Example 4.6. Let $G(\mathbf{p})$ be the convex framework in $\mathbb{R}^3$ shown in Figure 4.4 which consists of a tetrahedron with a triangle inside its base. A simple geometrical argument (using the collinearities introduced by the location of the three vertices of the triangle in the base) shows that $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$. However, if the face vertices in the base are located so that the base is infinitesimally rigid in the plane (which almost always happens), then $G(\mathbf{p})$ is flexible in $\mathbb{R}^3$ (with three "degrees of freedom") by Corollary 4.12.

4.4. Connelly's Rigidity Theorem.

Connelly (1980) shows that any framework given by a triangulation of a convex polyhedron is rigid in $\mathbb{R}^3$. In other words, if $G(\mathbf{p})$ is a convex framework in $\mathbb{R}^3$ with convex hull C such that the edges of $G(\mathbf{p})$ are noncrossing and every region of $\partial C - G(\mathbf{p})$ is a triangle (we refer to such a framework as a *triangulated convex framework*), then $G(\mathbf{p})$ is rigid in $\mathbb{R}^3$.

For the most part, Sections 4.2 and 4.3 deal with infinitesimal rigidity for convex frameworks; both the statements of the results and the techniques used in their proofs involve rank, dimension and other standard notions from linear algebra. By and large, we establish rigidity by establishing infinitesimal rigidity in these sections. However, Connelly's rigidity theorem is *not* a result about infinitesimal rigidity since a triangulated convex framework has, by Theorem 4.9, as many infinitesimal degrees of freedom as it has face vertices. Thus the rigidity of triangulated convex framework is not a consequence of standard "first order" techniques. Following Connelly, we incorporate into our attack on the problem "second order" information which arises from twice differentiating the equations that express the constancy of edge lengths (just as the first order conditions $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ arise from the first derivative of these equations).

Suppose $G(\mathbf{p})$ is a framework in $\mathbb{R}^d$ and f is its rigidity map. If $\mathbf{x}(t) = (\mathbf{x}_1(t), \cdots, \mathbf{x}_v(t))$ is a smooth path satisfying $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(t) \in f^{-1}(f(\mathbf{p}))$ for all $t \in [0,1]$, then

$$|\mathbf{x}_i(t) - \mathbf{x}_j(t)|^2 = |\mathbf{p}_i - \mathbf{p}_j|^2$$

for all edges $\{i,j\}$ of G and all $t \in [0,1]$. Differentiating and evaluating at $t = 0$, one obtains

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0 \tag{4.6}$$

for all edges $\{i,j\}$ of G where $\mathbf{p}'_i = \mathbf{x}'_i(0)$. Differentiating again gives

$$(\mathbf{p}'_i - \mathbf{p}'_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) = 0 \tag{4.7}$$

for all edges $\{i,j\}$ of G where $\mathbf{p}_i'' = \mathbf{x}_i''(0)$. With this as motivation, one is in a position to define second-order rigidity. If $\mathbf{p}' = (\mathbf{p}_1', \cdots, \mathbf{p}_v') \in \mathbb{R}^{vd}$ and $\mathbf{p}'' = (\mathbf{p}_1'', \cdots, \mathbf{p}_v'') \in \mathbb{R}^{vd}$ (where it should be noted that $\mathbf{p}'$ and $\mathbf{p}''$ need not now arise as the derivatives of a smooth motion $\mathbf{x}(t)$) satisfy Equations (4.6) and (4.7) for all edges of G, then $(\mathbf{p}', \mathbf{p}'')$ is said to be a *second-order flex* of $G(\mathbf{p})$.

Definition 4.4. A bar framework $G(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^{vd}$, is *second-order rigid* in $\mathbb{R}^d$ if for every second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $G(\mathbf{p})$, the infinitesimal (or first-order) flex $\mathbf{p}'$ is trivial.

Obviously it is straight-forward to define n^{th}-order flexes and n^{th}-order rigidity in an analogous fashion. We next prove that second-order rigidity implies rigidity. Whether or not n^{th}-order rigidity implies rigidity for $n \geq 3$ is unknown (although it is certainly suspected that this is the case). See Chapter 7 for examples of second-order rigid frameworks.

Theorem 4.13. *(Connelly) If a bar framework $G(\mathbf{p})$ is second-order rigid in $\mathbb{R}^d$, then $G(\mathbf{p})$ is rigid in $\mathbb{R}^d$.*

Proof. Suppose $G(\mathbf{p})$ is second-order rigid in $\mathbb{R}^d$. We show $G(\mathbf{p})$ is rigid in $\mathbb{R}^d$ by using Definition 3 of rigidity (Chapter 2.10). Consider any analytic flex $\mathbf{x}(t)$ of $G(\mathbf{p})$; this means that $\mathbf{x} : [0,1] \to \mathbb{R}^{vd}$ is an analytic path satisfying $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(t) \in f^{-1}(f(\mathbf{p}))$ for all $t \in [0,1]$. We verify that $\mathbf{x}(t)$ is congruent to $\mathbf{p}$ for all $t \in [0,1]$ by showing that the distance between all pairs of vertices remains constant (so $\mathbf{x}(t) \sim \mathbf{p}$ for all $t \in [0,1]$ by Proposition 2.7). For this, consider $m, n \in V$ and let $g(t) = |\,\mathbf{x}_m(t) - \mathbf{x}_n(t)\,|^2$ for all $t \in [0,1]$. Since g is analytic, the fact that it is constant follows from showing that all of its derivatives vanish at $t = 0$.

We first prove that $g'(0) = 0$. Let $\mathbf{p}' = \mathbf{x}'(0)$ and $\mathbf{p}'' = \mathbf{x}''(0)$. Since $\mathbf{x}(t) \in f^{-1}(f(\mathbf{p}))$ for all t, differentiating twice and evaluating at $t = 0$ (as was done to obtain Equations 4.6 and 4.7) gives

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i' - \mathbf{p}_j') = 0$$

and

$$(\mathbf{p}_i' - \mathbf{p}_j') \cdot (\mathbf{p}_i' - \mathbf{p}_j') + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i'' - \mathbf{p}_j'') = 0$$

for all edges $\{i, j\}$ of G. Since $G(\mathbf{p})$ is second-order rigid, $\mathbf{p}'$ is trivial and thus there exists a congruent motion $t \to h_t$ from $[0,1] \to H$ such that $h_0 = I$ and $\mathbf{p}' = \frac{d}{dt}h_t(\mathbf{p})|_{t=0}$. The latter equation is, of course, an abbreviation for

$$(\mathbf{p}_1', \cdots, \mathbf{p}_v') = \left(\frac{d}{dt}h_t(\mathbf{p}_1)\bigg|_{t=0}, \cdots, \frac{d}{dt}h_t(\mathbf{p}_v)\bigg|_{t=0} \right) .$$

Now define the smooth path $\mathbf{y} : [0,1] \to \mathbb{R}^{vd}$ by

$$\mathbf{y}(t) = h_t^{-1}\mathbf{x}(t)$$

where h_t^{-1} is the inverse of h_t in H. Then $\mathbf{y}(0) = \mathbf{x}(0) = \mathbf{p}$ and $\mathbf{y}'(0) = \frac{d}{dt}h_t^{-1}(\mathbf{p})|_{t=0} + \mathbf{x}'(0) = \mathbf{p}' - \mathbf{p}' = \mathbf{O}$ since $\frac{d}{dt}h_t^{-1}(\mathbf{p})|_{t=0} = -\frac{d}{dt}h_t(\mathbf{p})|_{t=0}$. Since each h_t^{-1} is a congruence, $g(t) = |\,\mathbf{y}_m(t) - \mathbf{y}_n(t)\,|^2$ so the fact that $\mathbf{y}'(0) = \mathbf{O}$ gives $g'(0) = 0$.

It seems more illuminating to show how one obtains $g''(0) = 0$ than to present the formal induction argument. The idea is to use the path $\mathbf{y}$ just obtained to construct another path $\mathbf{z}$ with $\mathbf{z}(0) = \mathbf{p}$ and $\mathbf{z}'(0) = \mathbf{z}''(0) = \mathbf{O}$. Let $\mathbf{q}' = \mathbf{y}''(0)$ and $\mathbf{q}'' = \mathbf{y}''''(0)/3$. Since $\mathbf{y}(t) \in f^{-1}(f(\mathbf{p}))$ for all t, differentiating $|\,\mathbf{y}_i(t) - \mathbf{y}_j(t)\,|^2$ four times and evaluating at $t = 0$ gives

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i' - \mathbf{q}_j') = 0$$

and

$$(\mathbf{q}_i' - \mathbf{q}_j') \cdot (\mathbf{q}_i' - \mathbf{q}_j') + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i'' - \mathbf{q}_j'') = 0$$

for all edges $\{i, j\}$ of G. (Note that these two equations arise from the second and fourth derivatives of $|\,\mathbf{y}_i - \mathbf{y}_j\,|^2$ because $\mathbf{y}'(0) = \mathbf{O}$.) Thus $(\mathbf{q}', \mathbf{q}'')$ is a second-order flex of $G(\mathbf{p})$ so $\mathbf{q}'$ is trivial. This means there exists a (twice continuously differentiable) congruent motion $t \to k_t$ from $[0,1] \to H$ such that $k_0 = I$ and $\mathbf{q}' = \frac{d}{dt}k_t(\mathbf{p})|_{t=0}$. Now let $\mathbf{z} : [0,1] \to \mathbb{R}^{vd}$ be the smooth path defined by

$$\mathbf{z}(t) = k_{t^2/2}^{-1}\mathbf{y}(t) .$$

Then $\mathbf{z}(0) = \mathbf{y}(0) = \mathbf{p}$, $\mathbf{z}'(0) = \mathbf{y}'(0) = \mathbf{O}$ and the (messy) computation of the second derivative of $\mathbf{z}(t)$ shows $\mathbf{z}''(0) = \mathbf{y}''(0) - \mathbf{q}' = \mathbf{O}$. Since each $k_{t^2/2}^{-1}$ is a congruence, we have $g(t) = |\,\mathbf{z}_m(t) - \mathbf{z}_n(t)\,|^2$ and therefore $g''(0) = 0$ since $\mathbf{z}'(0) = \mathbf{z}''(0) = \mathbf{O}$.

The completion of the proof via induction is left to the (dedicated) reader. □

The proof of Theorem 4.13 can be interpreted in the following simple way. Suppose that $G(\mathbf{p})$ is second-order rigid and $\mathbf{x} : [0,1] \to f^{-1}(f(\mathbf{p}))$ is an infinitely differentiable flex of $G(\mathbf{p})$. Then for each n there exists an "equivalent" infinitely differentiable flex $\mathbf{y} : [0,1] \to f^{-1}(f(\mathbf{p}))$ such that

$$\mathbf{y}'(0) = \mathbf{y}''(0) = \cdots = \mathbf{y}^{(n)}(0) = \mathbf{O} .$$

where "equivalent" means that for every pair m, n of vertices and every t

$$|\,\mathbf{x}_m(t) - \mathbf{x}_n(t)\,| = |\,\mathbf{y}_m(t) - \mathbf{y}_n(t)\,| .$$

In Theorem 4.13 $\mathbf{x}(t)$ was chosen to be analytic and consequently the distance between each pair of vertices could never change.

The main theorem of Connelly (1980) is that any triangulated convex framework is second-order rigid (and thus rigid). The following exercise will be used in the proof.

Exercise 4.2. Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ be three noncollinear points in $I\!R^3$ and suppose $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3)$ and $\mathbf{p}'' = (\mathbf{p}''_1, \mathbf{p}''_2, \mathbf{p}''_3)$ satisfy Equation (4.7) for $1 \le i, j \le 3$. Then $\mathbf{p}'$ and $\mathbf{p}''$ can be extended to the convex hull (indeed, the entire affine span) of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ in such a way that Equation (4.7) is satisfied for all pairs of points. To accomplish this, if $\mathbf{p}_0 = \sum_{i=1}^{3} \lambda_i \mathbf{p}_i$ where $\sum_{i=1}^{3} \lambda_i = 1$, let $\mathbf{p}'_0 = \sum_{i=1}^{3} \lambda_i \mathbf{p}'_i$ and $\mathbf{p}''_0 = \sum_{i=1}^{3} \lambda_i \mathbf{p}''_i$. Then one can verify that for all $\mathbf{p}_0$ and $\mathbf{q}_0$ in the plane of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$,

$$(\mathbf{p}'_0 - \mathbf{q}'_0) \cdot (\mathbf{p}'_0 - \mathbf{q}'_0) + (\mathbf{p}_0 - \mathbf{q}_0) \cdot (\mathbf{p}''_0 - \mathbf{q}''_0) = 0 .$$

The proof of the second-order rigidity of triangulated convex frameworks involves examining the behavior of a second-order flex on each face of the underlying convex polyhedron. Thus the following lemma focuses on convex polygons (the faces). Consider coplanar points $\mathbf{p}_1, \cdots, \mathbf{p}_v$ in $\mathbb{R}^3$ and let C be the convex hull of $\{\mathbf{p}_1, \cdots, \mathbf{p}_v\}$. We say that a framework $G(\mathbf{p})$, $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_v)$, is a *triangulated convex polygon* if the edges of $G(\mathbf{p})$ form a triangulation of this planar convex set C.

Lemma 4.14. *Suppose $G(\mathbf{p})$ is a triangulated convex polygon in $\mathbb{R}^3$ with convex hull C and $(\mathbf{p}', \mathbf{p}'')$ satisfies Equation (4.7) for all edges of G. If $\mathbf{p}'_i = 0$ for every vertex $\mathbf{p}_i$ which lies on an edge of C, then $\mathbf{p}'_i = 0$ for all vertices of $G(\mathbf{p})$.*

Proof. The proof involves four steps. We first show that $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) = 0$ for any two adjacent vertices $\mathbf{p}_i$ and $\mathbf{p}_j$ of the convex hull C. (Of course, by a vertex of C we mean an extreme point or "natural" vertex of the convex set C.) Next we show that $(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}''_k - \mathbf{p}''_0) \leq 0$ for any interior point $\mathbf{p}_0$ of C (which need not even be a vertex of $G(\mathbf{p})$) and any vertex $\mathbf{p}_k$ of C. Then we show that, in fact, $(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}''_k - \mathbf{p}''_0) = 0$ for all such $\mathbf{p}_0$ and $\mathbf{p}_k$. Finally, we show that $\mathbf{p}'_0 = 0$ for all interior points $\mathbf{p}_0$ of C (so $\mathbf{p}'_i = 0$ for all vertices $\mathbf{p}_i$ of $G(\mathbf{p})$ which lie in the interior of C). Three of these four steps involve applying Equation (4.7) to a sequence of points along a line segment in the convex set C.

Step 1. Suppose that $[\mathbf{p}_1, \mathbf{p}_n]$ is an edge of C and $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are the consecutive vertices of $G(\mathbf{p})$ on this edge of C (as shown in Figure 4.12). Since $\mathbf{p}'_i = 0$ for $1 \leq i \leq n$ by hypothesis, $\mathbf{p}''_i - \mathbf{p}''_{i+1}$ is orthogonal to $\mathbf{p}_i - \mathbf{p}_{i+1}$ (and hence $\mathbf{p}_1 - \mathbf{p}_n$) for $1 \leq i < n$. Therefore their sum $\mathbf{p}''_1 - \mathbf{p}''_n$ is orthogonal to $\mathbf{p}_1 - \mathbf{p}_n$.

Step 2. Consider any interior point $\mathbf{p}_0$ of C (where $\mathbf{p}_0$ need not be a vertex of $G(\mathbf{p})$) and let $\mathbf{p}_k$ be any vertex of the convex hull C. If $[\mathbf{p}_0, \mathbf{p}_k]$ happens to be an edge of $G(\mathbf{p})$, then $(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}''_k - \mathbf{p}''_0) \leq 0$ by Equation (4.7). We now show this inequality holds even if $[\mathbf{p}_0, \mathbf{p}_k]$ is not an edge of $G(\mathbf{p})$. Let $\mathbf{q}_1, \cdots, \mathbf{q}_m$ be the successive points of intersection of the open line segment $(\mathbf{p}_0, \mathbf{p}_k)$ with the edges of $G(\mathbf{p})$ (as shown in Figure 4.12). In case that an edge of $G(\mathbf{p})$ intersects $(\mathbf{p}_0, \mathbf{p}_k)$ in a segment (that is, an edge of the triangulation happens to be part of $(\mathbf{p}_0, \mathbf{p}_k)$), only the endpoints of the

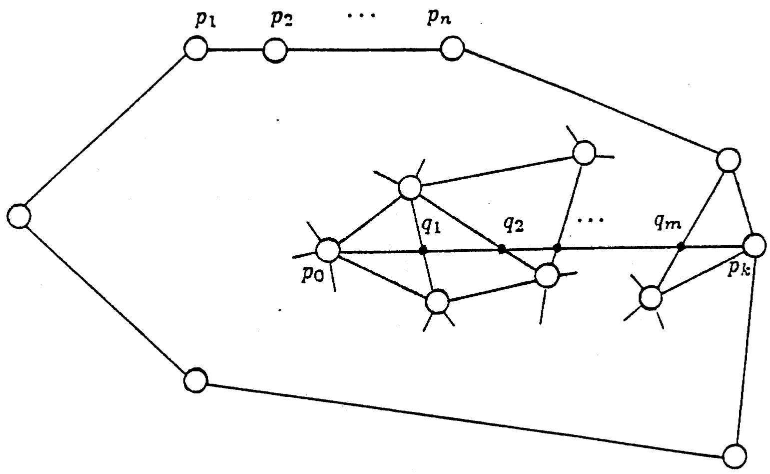


Figure 4.12

segment of intersection appear in $\mathbf{q}_1, \cdots, \mathbf{q}_m$. Letting $\mathbf{q}_0 = \mathbf{p}_0$, $\mathbf{q}_{m+1} = \mathbf{p}_k$, and using Exercise 4.2 to extend $\mathbf{p}'$ and $\mathbf{p}''$ to the points $\mathbf{q}_0, \cdots, \mathbf{q}_m$, we have

$$(\mathbf{q}'_{i+1} - \mathbf{q}'_i) \cdot (\mathbf{q}'_{i+1} - \mathbf{q}'_i) + (\mathbf{q}_{i+1} - \mathbf{q}_i) \cdot (\mathbf{q}''_{i+1} - \mathbf{q}''_i) = 0$$

for $0 \leq i \leq m$. Thus for $0 \leq i \leq m$, $(\mathbf{q}_{i+1} - \mathbf{q}_i) \cdot (\mathbf{q}''_{i+1} - \mathbf{q}''_i) \leq 0$ and hence $(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{q}''_{i+1} - \mathbf{q}''_i) \leq 0$ since $\mathbf{p}_k - \mathbf{p}_0$ is a positive multiple of each $\mathbf{q}_{i+1} - \mathbf{q}_i$. Summing gives

$$(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}''_k - \mathbf{p}''_0) \leq 0 \ .$$

Step 3. The fact that each of these inner products equals zero follows in an interesting way from properties of a stress of a new triangulated convex polygon with underlying convex set C. Let's now label the vertices of C by $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_m$ and let $\mathbf{p}_0$ be any interior point of C. As shown in Figure 4.13, let $H(\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_m)$ be the framework with vertices $\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_m$ and edges $[\mathbf{p}_0, \mathbf{p}_k]$ for $1 \leq k \leq m$ and $[\mathbf{p}_k, \mathbf{p}_{k+1}]$ for $1 \leq k \leq m$ (with $\mathbf{p}_{m+1} = \mathbf{p}_1$). Since this framework has $m+1$ vertices and $2m$ edges (so $e > 2v - 3$ for this framework), it has a nontrivial stress

$$\omega = (\cdots, \omega_{0k}, \cdots, \omega_{k(k+1)}, \cdots) \ .$$

At every boundary vertex $\mathbf{p}_k$, $1 \leq k \leq m$, we have

$$\omega_{(k-1)k}(\mathbf{p}_k - \mathbf{p}_{k-1}) + \omega_{0k}(\mathbf{p}_k - \mathbf{p}_0) + \omega_{k(k+1)}(\mathbf{p}_k - \mathbf{p}_{k+1}) = \mathbf{O}$$

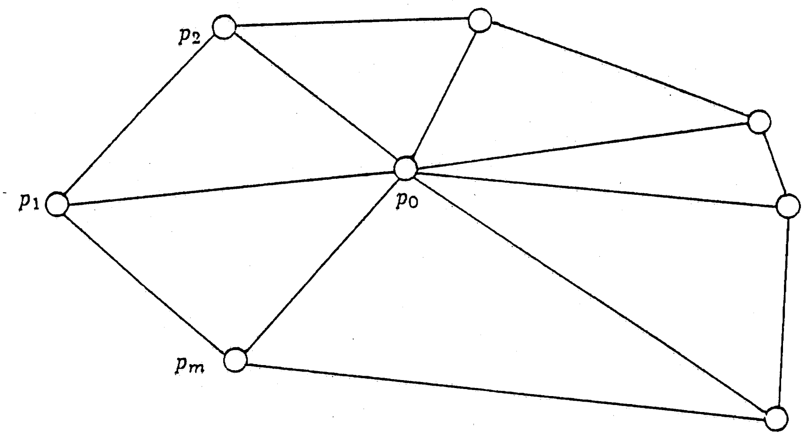


Figure 4.13

which implies that $\omega_{(k-1)k}$ and $\omega_{k(k+1)}$ have one sign while ω_{0k} has the opposite sign. (This simple observation can be considered a two-dimensional version of the index two separation argument in the proof of Lemma 4.2. It is at this point that the convexity of C is essential.) Therefore, in the stress ω all of the scalars are nonzero, those $\omega_{k(k+1)}$ corresponding to boundary edges having one sign and those ω_{0k} corresponding to interior edges having the opposite sign. Letting f_H be the rigidity map of H and expanding

$$\omega \; df_H(\mathbf{p}_0, \cdots, \mathbf{p}_m) \begin{pmatrix} \mathbf{p}_0'' \\ \vdots \\ \mathbf{p}_m'' \end{pmatrix} = 0 \; ,$$

one finds the scalar equation

$$\sum_{k=1}^{m} \omega_{k(k+1)}(\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot (\mathbf{p}_{k+1}'' - \mathbf{p}_k'') + \sum_{k=1}^{m} \omega_{0k}(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}_k'' - \mathbf{p}_0'') = 0 \; .$$

By Step 1, each inner product in the first sum vanishes. Since each ω_{0k} has the same sign and each $(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}_k'' - \mathbf{p}_0'') \leq 0$, we conclude that for $1 \leq k \leq m$

$$(\mathbf{p}_k - \mathbf{p}_0) \cdot (\mathbf{p}_k'' - \mathbf{p}_0'') = 0 \; .$$

Step 4. We consider any interior point $\mathbf{p}_0$ of C and show $\mathbf{p}_0' = 0$. Let $\mathbf{p}_k$ be any vertex of C. As in Step 2 of the proof, let $\mathbf{q}_1, \cdots, \mathbf{q}_m$ be the points of intersection of $(\mathbf{p}_0, \mathbf{p}_k)$ with the edges of $G(\mathbf{p})$ (as shown in Figure 4.12). Letting $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{q}_{m+1} = \mathbf{p}_k$, we have

$$(\mathbf{p}_k - \mathbf{q}_i) \cdot (\mathbf{p}_k'' - \mathbf{q}_i'') = 0 \quad \text{for} \quad 0 \leq i \leq m$$

by Step 3 and therefore both $\mathbf{p}_k'' - \mathbf{q}_i''$ and $\mathbf{p}_k'' - \mathbf{q}_{i+1}''$ are orthogonal to $\mathbf{q}_{i+1} - \mathbf{q}_i$ for $0 \le i \le m$. Thus their difference $\mathbf{q}_{i+1}'' - \mathbf{q}_i''$ is also orthogonal to $\mathbf{q}_{i+1} - \mathbf{q}_i$ so Equation (4.7) gives

$$(\mathbf{q}_{i+1}' - \mathbf{q}_i') \cdot (\mathbf{q}_{i+1}' - \mathbf{q}_i') = 0$$

for $0 \le i \le m$. Therefore we have

$$0 = \mathbf{p}_k' = \mathbf{q}_{m+1}' = \mathbf{q}_m' = \cdots = \mathbf{q}_0' = \mathbf{p}_0'$$

which completes the proof of Lemma 4.14. □

The framework $H(\mathbf{p}_0, \cdots, \mathbf{p}_m)$ shown in Figure 4.13 can be treated as a tensegrity framework by replacing each of the interior edges of H by a cable (which provides an upper bound for the distance between a point of vertices rather than fixing this distance). From this point of view (which Connelly (1980) uses extensively), our stress argument in Step 3 establishes the infinitesimal rigidity of the tensegrity framework. Chapter 7 deals with tensegrity frameworks in great detail.

Lemma 4.14 and Theorem 4.9 lead directly to a proof of Connelly's rigidity theorem.

Theorem 4.15. *(Connelly) Triangulated convex frameworks in $\mathbb{R}^3$ are second-order rigid in $\mathbb{R}^3$.*

Proof. Let $G(\mathbf{p})$ be a triangulated convex framework in $\mathbb{R}^3$ and suppose that $(\mathbf{p}', \mathbf{p}'')$ is a second-order flex of $G(\mathbf{p})$. Since the set of vertices and edges of $G(\mathbf{p})$ on each face of the convex hull C forms an infinitesimally rigid framework in $\mathbb{R}^2$, the set $df_{\mathbf{p}}$ of infinitesimal flexes of $G(\mathbf{p})$ is the direct sum of $T_{\mathbf{p}}$ and $N_{\mathbf{p}}$ by Theorem 4.9. Therefore there exists $\mathbf{u} \in T_{\mathbf{p}}$ and $\mathbf{w} \in N_{\mathbf{p}}$ such that

$$\mathbf{p}' = \mathbf{u} + \mathbf{w} \ .$$

Since $\mathbf{w}_i = \mathbf{O}$ for all $i \in A$ (where $A = \{j : \mathbf{p}_j$ lies on an edge of $C\}$), we have that $\mathbf{p}_i' = \mathbf{u}_i$ for all $i \in A$, i.e., $\mathbf{p}'$ is trivial on the vertices in A. Of course our goal is to show that $\mathbf{p}'$ is trivial on all the vertices and this is precisely where Lemma 4.14 enters the picture.

However, to apply Lemma 4.14, one needs to get from the second-order flex $(\mathbf{p}', \mathbf{p}'')$ with $\mathbf{p}'$ trivial on the vertices in A to a second-order flex $(\mathbf{q}', \mathbf{q}'')$ with $\mathbf{q}'$ zero on the vertices in A. This is straightforward for the first-order part, but the second-order part is a little tricky. Since $\mathbf{p}'$ is trivial on the vertices in A, there exists a skew-symmetric 3×3 matrix S and a vector $\mathbf{t} \in I\!R^3$ such that

$$\mathbf{p}'_i = S\mathbf{p}_i + \mathbf{t}$$

for all $i \in A$. (It seems a little more convenient to use the skew-symmetric rather than the cross product description of trivial infinitesimal flexes at this point. See Chapter 2.7.) Letting $\mathbf{q}'_i = \mathbf{p}'_i - (S\mathbf{p}_i + \mathbf{t})$ for all vertices i, it is very easy to verify that for all edges $\{i, j\}$ of G

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}'_i - \mathbf{q}'_j) = 0 \ .$$

Letting $\mathbf{q}''_i = \mathbf{p}''_i - 2S\mathbf{p}'_i + S^2\mathbf{p}_i$ for all vertices i, a somewhat longer but still entirely elementary calculation shows that for all edges $\{i, j\}$ of G I like it.

$$(\mathbf{q}'_i - \mathbf{q}'_j) \cdot (\mathbf{q}'_i - \mathbf{q}'_j) + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}''_i - \mathbf{q}''_j) = 0 \ .$$

Thus $(\mathbf{q}', \mathbf{q}'')$ is a second-order flex of $G(\mathbf{p})$ with $\mathbf{q}'_i = \mathbf{O}$ for all $i \in A$. By Lemma 4.14, $\mathbf{q}'_i = \mathbf{O}$ for all i and thus $\mathbf{p}'$ is trivial on all the vertices of the framework. □

It seems fitting to end the chapter with a more substantial exercise due to Connelly and an interesting conjecture due to Whiteley. The conjecture is merely the bar framework version of Whiteley's Conjecture 7.?? for tensegrity frameworks. It is, in fact, yet another modification of Conjecture 4 of Grünbaum & Shephard (1975).

Conjecture. For a convex framework in $I\!R^3$ with no edge vertices and no face vertices (so each vertex is a "natural" vertex), if each face is rigid in $I\!R^2$ then the entire framework is rigid in $I\!R^3$.

What about 2nd order rigid in each face?

Of course, when the edges are noncrossing or the faces are infinitesimally rigid in $\mathbb{R}^2$ it is easy to establish the result using Corollary 4.6 or Corollary 4.10.

The following exercise extends Theorem 4.15 to the case in which "holes" are allowed in the faces of a convex surface in $\mathbb{R}^3$. It represents a slight generalization of results of Connelly (1980).

Exercise 4.3.

(a) Consider a convex polygon with vertices $\mathbf{p}_1, \cdots, \mathbf{p}_v$ in $\mathbb{R}^2$. For $1 < i < v$, let $\overline{\mathbf{p}}_i$ be a point in the interior of the triangle formed by $\mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}$ and join $\overline{\mathbf{p}}_i$ to these three points by edges (as shown in Figure 4.14). Show that the resulting framework is infinitesimally rigid in $\mathbb{R}^2$ and has a nonzero stress with coefficients of one sign for all the boundary edges of the polygon and the opposite sign for all the interior edges of the polygon. (We refer to such a framework as a *collar.*)

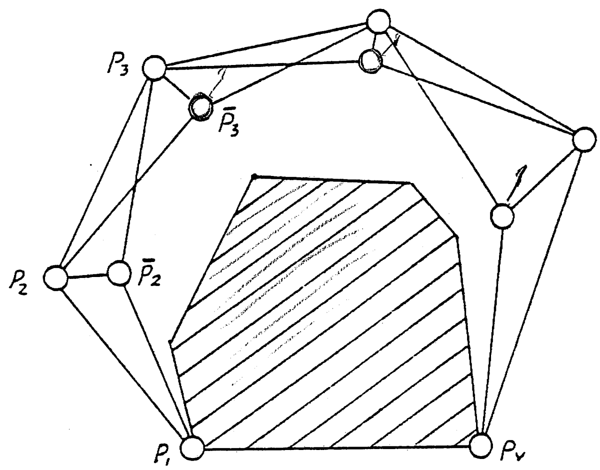


Figure 4.14

(b) Consider a framework $G(\mathbf{p})$ in $\mathbb{R}^3$ obtained from a planar convex polygon by removing the interior of a convex polygon (the "hole") and then triangulating the resulting set. Show that if the hole can be enclosed in a collar (Figure 4.14 illustrates an instance in which the boundary of the hole intersects the boundary of the polygon, but it may happen that the boundary of the hole lies entirely in the interior of the polygon), then Lemma 4.14 holds for the framework $G(\mathbf{p})$.

(c) Consider a convex framework in $\mathbb{R}^3$ with convex hull C which has holes of the type considered in (b) in some of its faces. Prove that if the intersection of the boundary of each hole with the boundary of its face is either empty, a (natural) vertex of C, or a (natural) edge of C having no vertices of $G(\mathbf{p})$ in its interior, then the convex framework is second-order rigid in $\mathbb{R}^3$.

(d) Find examples where the boundary of a hole intersects the boundary of its face in an edge vertex or in an edge of C containing an edge vertex and the framework is flexible in $\mathbb{R}^3$.

Bibliography.

Alexandrov (1958)

A.D. Alexandrov, **Konvexe Polyeder**, Akademie-Verlag, Berlin, 1958.

Asimow & Roth (1978)

L. Asimow and B. Roth, *The rigidity of graphs*, **Trans. Amer. Math. Soc.** 245 (1978), 279-289.

Asimow & Roth (1979)

L. Asimow and B. Roth, *The rigidity of graphs II*, **J. Math. Anal. Appl.** 68 (1979), 171-190.

Cauchy (1813)

A.L. Cauchy, *Sur les polygones et les polyèdres*, **J. École Polytechnique**, XVIe, Cahier, Tome IX (1813), 87-98.

Cohn-Vossen (1936)

S.E. Cohn-Vossen, *Die Verbiegung von Flächen im Grossen*, **Fortschr. Math. Wiss.** 1 (1936), 33-76.

Connelly (1978)

R. Connelly, *A counterexample to the rigidity conjecture for polyhedra*, **Inst. Hautes Études Sci. Publ. Math.** 47 (1978), 333-338.

Math Monthly version better

Connelly (1979)

R. Connelly, *The rigidity of polyhedral surfaces*, **Math. Mag.** 52 (1979), 275-283.

Connelly (1980)

R. Connelly, *The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces*, **Adv. in Math.** 37 (1980), 272-299.

Dehn (1916)

M. Dehn, *Über der Starrheit konvexer Polyeder*, **Math. Ann.** 77 (1916), 466-473.

Efimov (1957)

N.W. Efimov, *Flächenverbiegung im Grossen*, Akademie-Verlag, Berlin, 1957.

Gluck (1975)

H. Gluck, *Almost all simply connected closed surfaces are rigid*, Lecture Notes in Math., no. 438, **Geometric Topology**, Springer-Verlag, Berlin, 1975, 225-239.

Grünbaum & Shephard (1975)

B. Grünbaum and G.C. Shephard, *Lectures on lost mathematics*, notes, University of Washington, Seattle, 1975.

Grünbaum & Shephard (1978)

B. Grünbaum and G.C. Shephard, *Rigidity of polyhedra, frameworks, and cabled frameworks*, Abstract 760-D3, **Notices Amer. Math. Soc.** 25 (1978), A-642.

Hadamard (1907)

J. Hadamard, *Erreurs des mathematiciens*, **Interméd. Math.** 14 (1907), 31.

Heath (1956)

T.L. Heath, **The Thirteen Books of Euclid's Elements**, Dover, New York, 1956.

Lebesgue (1909)

H. Lebesgue, *Démonstration complète du théorème de Cauchy sur l'égalité des polyèdres convexes*, **Interméd. Math.** 16 (1909), 113-120.

Legendre (1794)

A.M. Legendre, **Éléments de Géométrie**, Paris, 1794.

Liebmann (1900)

H. Liebmann, *Über die Verbiegung der geschlossenen Flächen positiver Krümmung*, **Math. Ann.** 53 (1900), 81-112.

Lyusternik (1963)

L.A. Lyusternik, **Convex Figures and Polyhedra**, Dover, New York, 1963.

Pogorelov (1973)

A.V. Pogorelov, **Extrinsic Geometry of Convex Surfaces**, Transl. Math. Monographs 35, Amer. Math. Soc., 1973.

Spivak (1979)

M. Spivak, **A Comprehensive Introduction to Differential Geometry**, Volume V, Publish or Perish, Berkeley, 1979.

Steinitz & Rademacher (1934)

E. Steinitz and H. Rademacher, **Vorlesungen über die Theorie der Polyeder**, Springer, Berlin, 1934.

Stoker (1968)

J.J. Stoker, *Geometrical problems concerning polyhedra in the large*, **Comm. Pure Appl. Math.** 21 (1968), 119-168.

Weyl (1917)

H. Weyl, *Über die Starrheit der Eiflächen und konvexen Polyeder*, **S.-B. Preuss. Akad. Wiss.** (1917), 250-266.

Whiteley (1984)

W. Whiteley, *Infinitesimally rigid polyhedra I*, **Trans. Amer. Math. Soc.** 285 (1984), 431-465.