Invent. math. 66, 11–33 (1982)

Inventiones mathematicae

Rigidity and Energy

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I. Introduction

Suppose one holds two sticks in the form of a cross in one hand, and places a rubber band in tension around the four ends. When it is released, it comes to rest in the shape of a convex quadrilateral in a plane. It always returns to the same shape, no matter how it is distorted, as long as the ends do not slip, and the rubber band does not break. This is a very simple example of a "rigid" framework of the type we discuss here.

Energy explains why a framework, such as the one above is rigid. The rubber band deforms in such a way as to minimize the total energy in the framework. Only in the final deformed shape will the framework have a minimum energy.

This idea of introducing energy functions is very useful. It can be used to prove a key lemma that was used by Cauchy [5] in 1813 to show that "convex polyhedral surfaces" are rigid. It can explain why some, and perhaps all, of R. Buckminster Fuller's tensegrity structures [10] stay up. When applied to spider webs it shows why they can only take on certain geometric shapes. It also can be used to prove Conjecture 6 of Branko Grünbaum and G.C. Shephard in their "Lectures on lost mathematics" [12]. This conjecture says that if a framework, in the shape of a convex polygon, with rods (sticks) on the boundary and cables inside, is rigid in the plane, so is the framework obtained by reversing the roles of rods and cables.

It is interesting to compare some of these results with the "opening arm" theorem of Axel Schur [18] (See Chern [6] also), which is a very close smooth analogue to Cauchy's lemma. Schur's theorem says that if a convex planar smooth arc (the arm) is opened, that is, it is moved to another position with the same length, but with corresponding points having smaller curvature, then the two ends are moved apart.

Another amusing application is to show that a regular pentagon in 4-space, (a pentagon with all 5 sides equal and all 5 angles between the sides equal) has its angles bounded between 36° and 108°, a comment of O. Bottema in [4].

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* Partially supported by NSF Grant number MCS-7902521

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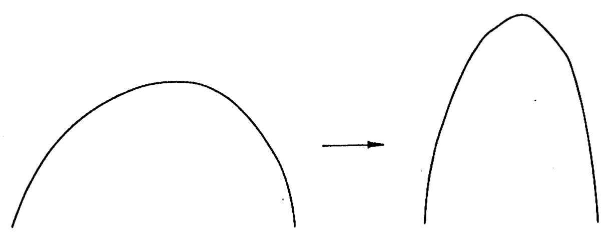

Fig. 1

A great debt is owed to Walter Whiteley for a series of conjectures and questions that were very illuminating [23]. In particular, the statements of Theorem 2 and Theorem 3 here, and much of the general plan of how to prove Grünbaum's Conjecture 6 (Corollary 2 here), were part of his conjectures.

The notation here is copied from Asimow and Roth [1] and Gluck [11]. G represents an abstract finite graph, where each edge is designated as either a *rod*, *cable*, or *strut*. A *realization* of G, also called a *framework*, will be an assignment of a point p_i in $\mathbb{R}^n$ for the i-th vertex of G.

We designate these points by $p=(p_1, \ldots, p_v)$ as one vector in $(\mathbb{R}^n)^v=\mathbb{R}^{nv}$, where v is the number of vertices of G, and $\mathbb{R}^n$ is euclidean n-space. We denote this realization by $G(p)$, regarded as a collection of points and edges in $\mathbb{R}^n$. A *continuous motion* or *flex* of $G(p)$ is a continuous path $p(t)$ in $\mathbb{R}^{nv}$, $p(0)=p$, $0 \leqq t \leqq 1$, such that rods have a fixed length, cables do not increase in length, and struts do not decrease in length. The edges of G (rods, cables, struts) are often called *members*. Note the members of $G(p(t))$ are allowed to cross each other and pass through, even at $t=0$. If $p(t)$ is the restriction of a rigid motion of $\mathbb{R}^n$, then we say the flex is *trivial*. If $G(p)$ has only trivial flexes we say $G(p)$ is *rigid*.

Suppose q in $\mathbb{R}^{nv}$ is another position for the vertices of G. If the rods, cables, and struts of $G(q)$ are the same length, not longer, not shorter respectively than the corresponding rods, cables, and struts of $G(p)$, then we say $G(q)$ is *another embedding* of $G(p)$. Note this is not necessarily a symmetric relation. A *congruence* of $G(p)$ is the restriction of a rigid global motion of $\mathbb{R}^n$ to the vertices of $G(p)$ (allowing reflections). If every other embedding $G(q)$ of $G(p)$ is congruent to $G(p)$, we say $G(p)$ is *uniquely embedded*. Note that if $G(p)$ is uniquely embedded, $G(p)$ is certainly rigid.

As in Gluck [11] and Asimow and Roth [1], and even for cabled, strutted structures, we define a map $f: \mathbb{R}^{nv} \to \mathbb{R}^e$, the *rigidity* map, by

$$f(p_1, \ldots, p_v)=(\ldots, |p_i-p_j|^2, \ldots)$$

where $\{i,j\}$ represents an edge of G, and e is the total number of edges (of all types) of G. A *stress* for $G(p)$ is an assignment of scalars $\omega_{ij}=\omega_{ji}$ for each edge of G such that for all i

$$\sum_j \omega_{ij}(p_i-p_j)=0,$$

where the sum is taken over all vertices j adjacent to i. (We also say $G(p)$ is in *equilibrium with respect to* ω at p_i.)

Often we regard all the ω_{ij}'s as one single vector $\omega=(\ldots,\omega_{ij},\ldots)$ in $\mathbb{R}^e$. A *proper stress* for $G(p)$ is a stress ω such that $\omega_{ij}\geqq 0$ if $\{i,j\}$ is a cable, and $\omega_{ij}\leqq 0$, if $\{i,j\}$ is a strut (no condition for rods). This definition differs slightly from that given in Roth and Whiteley [16], who define a proper stress as above with the extra condition that it be non-zero on all the cables and struts. We wish to allow some of these stresses to be zero.

In Section II we assume that some given framework $G(p)$ is rigid, and we investigate its properties. For later purposes the main object is to show that such a rigid $G(p)$ has a non-zero stress on some cable or strut, assuming $G(p)$ has a cable or strut, Theorem 3. The idea is first to show that the general requirement of rigidity implies that the members can be slackened slightly, and the framework will not move far from its original position, Theorem 1. This allows room to define an energy function with a minimum near the original position. Since the gradient must be zero at this minimum, we are guaranteed a proper stress, non-zero on each cable or strut, Theorem 2. A limiting argument yields Theorem 3.

Section III concentrates on more special frameworks and quadratic energy functions.

After getting acquainted with how quadratic energy functions work with frameworks inspired from spiderwebs, we look at convex polygons. In particular we show how to define a "natural" quadratic energy form in terms of a given stress. We investigate when this form is positive semidefinite of the appropriate nullity. When the convex polygon has cables on the boundary and struts inside, and when it has a proper stress, then it turns out the framework has a positive semi-definite energy form of the right nullity, and so the framework is uniquely embedded, Theorem 5. Grünbaum's Conjecture 6 is an immediate corollary.

In Section IV we discuss how the above results relate to Cauchy's lemma, Schur's theorem, and van der Waerden's theorem.

In a sequel to this paper we hope to explain the relation of these ideas to ideas from engineering and more general energy functions.

II. Implications of Rigidity

We investigate some general properties of a rigid framework. It turns out a great deal can be said.

Suppose one builds a particular framework. In practice it is never possible to get the lengths absolutely accurate, and in any case there is always a little "play". If the framework is infinitesimally rigid, see Gluck [11] or Asimow and Roth [1] for a definition, there is no question that this will not be serious and the ultimate distortion will be very small. The following theorem says that this is true also if the framework is only assumed to be rigid as defined in the introduction. If $G(p)$ is rigid, one consequence is that, if $G(q)$ is another realization, where the rods are the same length, cables not longer, and struts not shorter, then the set of such q's are outside an open set $U_p\subset\mathbb{R}^{nv}$ containing all the realizations congruent to $G(p)$. In other words $G(p)$ is uniquely embedded, if we restrict to those realizations sufficiently close to realizations congruent to $G(p)$. See Connelly [7]. Let us call U_p a *rigidity neighborhood* of p for $G(p)$.

Theorem 1. *Let $G(p)$ be rigid in $\mathbb{R}^n$, and U_p a rigidity neighborhood of p for $G(p)$. Let $\varepsilon>0$ be given. Then there is a $\delta>0$ such that when $q\in U_p$ and the following conditions hold*

$$(*)\qquad \begin{aligned} |q_i-q_j|^2&<|p_i-p_j|^2+\delta && \text{for } \{i,j\} \text{ a cable,}\\ |p_i-p_j|^2-\delta<|q_i-q_j|^2&<|p_i-p_j|^2+\delta && \text{for } \{i,j\} \text{ a rod,}\\ |p_i-p_j|^2-\delta<|q_i-q_j|^2& && \text{for } \{i,j\} \text{ a strut,}\end{aligned}$$

there is a rigid congruence of $\mathbb{R}^h$, T, such that

$$|(Tq_1,\ldots,Tq_v)-p|<\varepsilon.$$

Proof. Without loss of generality we assume that some vertex of $G(p)$ (and $G(q)$) is held fixed throughout. The set of $q\in\mathbb{R}^{nv}$ where $G(q)$ is congruent to $G(p)$ is now compact.

Let $E\subset\mathbb{R}^e$ be the set defined by those $(\ldots,e_{ij},\ldots)\in\mathbb{R}^e$ such that

$$\begin{aligned} e_{ij}&\leqq|p_i-p_j|^2 && \text{if } \{i,j\} \text{ is a cable of } G,\\ e_{ij}&=|p_i-p_j|^2 && \text{if } \{i,j\} \text{ is a rod of } G, \text{ and}\\ e_{ij}&\geqq|p_i-p_j|^2 && \text{if } \{i,j\} \text{ is a strut of } G.\end{aligned}$$

Let V_δ be the open neighborhood of E in $\mathbb{R}^e$ defined by e_{ij} replacing $|q_i-q_j|^2$ in (*), for any $\delta>0$. Let $f\colon U_p\to\mathbb{R}^e$ be the restriction of the rigidity map, defined in the introduction, to a rigidity neighborhood defined above. So U_p is a neighborhood of $f^{-1}(E)$ in $\mathbb{R}^{nv}$, restricting attention to only those points with the first vertex, say, fixed.

Let $\varepsilon>0$ be given.

Suppose we cannot find a $\delta>0$ as in the conclusion. Then for each $\delta>0$ there is a point $q(\delta)\in U_p-U_\varepsilon$ such that $f(q(\delta))\in V_\delta$, where U_ε is the ε-neighborhood of $f^{-1}(E)$ in U_p. We may assume cl U_p, the closure of U_p in $\mathbb{R}^{nv}$, is compact, since $f^{-1}(E)$ is compact, since $G(p)$ is rigid, and $f^{-1}(E)$ is the set of $q\in\mathbb{R}^{nv}$ such that $G(q)$ is congruent to $G(p)$ with the first vertex fixed. Also by the local compactness of R^{nv} we may assume $G(p)$ is uniquely embedded when restricted to cl U_p. So there is a sequence of positive δ_i, $i=1,2,\ldots$, converging to 0, such that $q(\delta_i)\in U_p-U_\varepsilon$ for all i. By taking a subsequence if necessary we may assume that $q(\delta_i)$ converges to $q\in(\text{cl } U_p)-U_\varepsilon$. But $f(q(\delta_i))$ converges to a point in E. So $f(q)\in E$. So $q\in f^{-1}(E)$, a contradiction. Thus there is a δ so that $U_\varepsilon\supset f^{-1}(V_\delta)$. In other words if q is within ε of $f^{-1}(E)$, some congruence of $G(q)$ is within ε of $G(p)$, as in the conclusion. This completes Theorem 1.

Suppose one has a rigid framework and one pulls (or pushes) two of the vertices together (or apart). It seems "clear", if our model is to represent physical reality, that the framework will deform slightly and resist this force. The next theorem (suggested by W. Whiteley) says that this is precisely what happens. The cable or strut can be thought of as supplying the force.

Theorem 2. *Let $G(p)$ be a rigid framework in $\mathbb{R}^n$. Let $\varepsilon > 0$ be given. Then there is a $q \in \mathbb{R}^{nv}$ such that $|p-q| < \varepsilon$ and $G(q)$ has a proper stress that is non-zero on all the cables and struts.*

Proof. Let $\delta > 0$ be the δ of Theorem 1 so that its conclusion holds for the ε above.

Let $\omega_{ij}: I \to \mathbb{R}^1$ be smooth (C^2 at least) (energy) functions defined for each edge, where I is an appropriate interval as indicated by the graphs of ω_{ij} below:

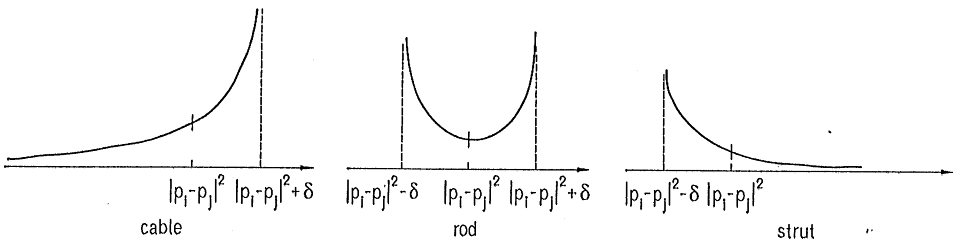

Fig. 2

In each case $\omega_{ij} \to \infty$ at the asymptotes, and cables and struts are monotone with non-zero derivatives. ω_{ij} for a rod has only the one minimum with derivative 0.

Let N be a neighborhood of p in $\mathbb{R}^{nv}$ contained in a rigidity neighborhood and such that (*) holds for $q \in N$. Thus the conclusion of Theorem 1 holds as well. For any $q \in N$ define an energy

$$(**) \qquad E(q) = \tfrac{1}{2} \sum_{ij} \omega_{ij}((q_i - q_i)^2),$$

where the sum is taken over all edges of G. As with Theorem 1 we may assume cl N is compact by fixing one vertex of G. We may extend the definition of E to include the boundary of N by making $E = \infty$ on the boundary. On the extended reals (to include ∞) E is continuous, because if $(q_i - q_j)^2$ is not in the domain of its ω_{ij} it must be at one of the asymptotes, and all nearby defined points will be large. Thus E must have a minimum point $\bar{q} \in N$. By changing $\bar{q}$ by a rigid congruence of $\mathbb{R}^n$ by Theorem 1 we may assume $|\bar{q} - p| < \varepsilon$, and the gradient of E must be 0. Computing n coordinates at a time as usual, we get for this gradient

$$0 = (\ldots, \sum_j \omega'_{ij}((\bar{q}_i - \bar{q}_j)^2)(\bar{q}_i - \bar{q}_j), \ldots),$$

where the sum in the i-th slot is taken over the edges adjacent to the i-th vertex. The proper stress is given by $\omega_{ij} = \omega'_{ij}((\bar{q}_i - \bar{q}_j)^2)$. By the construction of the energy functions it is proper and non-zero on each cable and strut. This finishes Theorem 1.

Lemma 1. *Let G be an abstract framework. Then the set $\{p \in \mathbb{R}^{nv} \mid G(p)$ has a non-zero proper stress$\}$ is closed in $\mathbb{R}^{nv}$.*

Proof. Let $p \in \mathbb{R}^{nv}$ be a limit point of the set S defined in the conclusion. Let $p(i) \in S$ be a sequence of points in $\mathbb{R}^{nv}$ converging to p. Let $\omega(i) \in \mathbb{R}^e$ be the non-zero stress associated to $G(p(i))$. By replacing $\omega(i)$ with $\frac{\omega(i)}{|\omega(i)|}$ if necessary we may assume that $|\omega(i)|=1$. That is $\omega(i)$ is in the unit sphere in $\mathbb{R}^e$. (The equilibrium equations for a stress still hold when all the stresses are multiplied by a constant.) Since the unit sphere is compact, by taking a subsequence if necessary we may assume that $\omega(i)$ converges to a non-zero proper ω. (Note the inequalities defining a proper stress are not strict.) The function which assigns to every $p \in \mathbb{R}^{nv}$, $\omega \in R^e$ the (equilibrium) vector

$$(\ldots, \sum_{jk} \omega_{jk}(p_j - p_k), \ldots)$$

is continuous, and since it is 0 for $p(i)$, $\omega(i)$, and they converge to p, ω, it is 0 for p, ω. Thus $G(p)$ is in equilibrium with respect to ω, which means ω is a non-zero proper stress for $G(p)$. This ends the lemma.

Theorem 3. *Let $G(p)$ be a rigid framework with a cable or strut. Then $G(p)$ has a non-zero proper stress.*

Proof. By Theorem 2 for every $\varepsilon > 0$ there is a q such that $|p-q| < \varepsilon$ and $G(q)$ has a proper stress, non-zero on the cable or strut. By Lemma 1, $G(p)$ then has a non-zero proper stress.

Remark 1. Although Theorem 2 guarantees that the stress will be non-zero on all the cables and struts of the nearby framework, when the limit process is applied in Lemma 1 this may no longer be the case. Consider the following example where the preferred cable has no stress in the limit position:

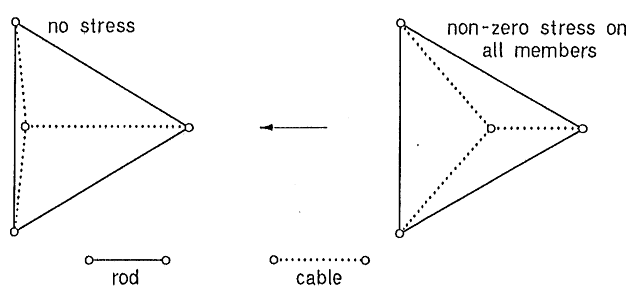

Fig. 3

Remark 2. In many cases Theorem 3 can be proved by other methods. For instance when $G(p)$ is infinitesually rigid (with a cable or strut), then the non-zero proper stress can be found using the results of Roth and Whiteley [16]. In fact the conclusion of Theorem 2 holds, but for $q=p$. In other words $G(p)$ "resolves" the stresses in the cables and struts without deforming. However, I do not see how to obtain Theorem 3 in case $G(p)$ is not infinitessimally rigid by these techniques.

III. Quadratic Energy Functions

We have already seen how energy functions were used to find the stresses for Theorem 2. We now use a stress to create an energy function. It turns out that in many cases the energy has minima that are simple enough to describe completely. This is then used to show the framework is rigid, since the non-rigid motions that preserve the minimum energy are not permissible flexes for the framework.

In order to get acquainted with this technique we consider a particularly simple case first. Suppose $G(p)$ is a framework where a subset of the vertices, B, is held fixed, while the other vertices are allowed to move freely. One might have some rigid framework behind the scenes which includes the vertices of B. All the (other) edges of G are cables, and each vertex of G not in B is connected to B by a sequence of cables. By using an appropriate energy function the following is easy.

Theorem 4. *Let ω be a stress for $G(p)$ with each $\omega_{ij}>0$, where we only require equilibrium at the vertices of G not in B. Then (assuming the vertices of B are held fixed) $G(p)$ is rigid.*

Proof. Define the energy of $G(q)$,

$$E(q)=\tfrac{1}{2}\sum_{i,j}\omega_{ij}(q_i-q_j)^2,$$

where the sum is taken over only the cables of G, and $q\in\mathbb{R}^{nv}$. E is a quadratic function, where the vertices of B that enter in the formula are taken as constants. Note also $\nabla E(q)$, the gradient of E at q, is an affine linear function of q.

$$\nabla E(q)=(\ldots,\sum_{j}\omega_{ij}(q_i-q_j),\ldots).$$

We claim that E has a unique minimum at p. If q is any point other than p, then $E((1-t)p+tq)$ is a quadratic polynomial in t that approaches infinity as t approaches infinity, and whose derivative at $t=0$ is

$$\nabla E(p)\cdot(q-p)=0.$$

$\nabla E(p)=0$ by the equilibrium condition for ω. If some vertex of $G(p)$ (not in B) is different from the corresponding vertex of $G(q)$ it will travel to ∞ in $(1-t)$ $p+tq$ as $t\to\pm\infty$. Since every vertex is connected to a vertex of B by a series of cables some cable must get arbitrarily large for large t. Since all the $\omega_{ij}>0$ this means that E must also get large. Thus p is the unique minimum for E.

Now it is clear $G(p)$ is rigid. If $G(q)$ is any other realization and $|q_i-q_j|\leqq|p_i-p_j|$ for every cable $\{i,j\}$, then $E(q)\leqq E(p)$. Since p is the unique minimum $p=q$, as was desired.

Remark 3. Theorem 4 can be viewed as explaining the rigidity of spider webs. Their thin threads are attached to some rigid object, and then they are stretched until there is a proper stress.

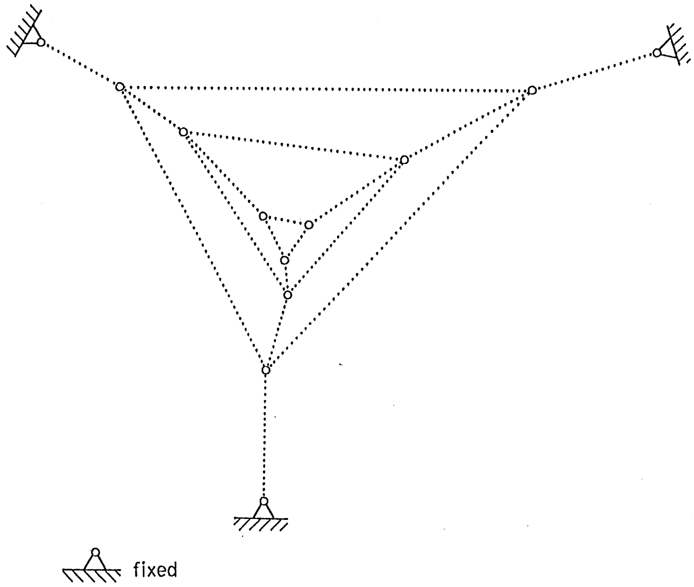

Fig. 4

In fact if such a framework is rigid, we can regard the vertices of B being held rigid by some infinitesimally rigid rod framework $G_1(p)$ with the minimum number of rods. Then the cables and vertices of the spider web are attached to $G_1(p)$ to get $G_2(p)$ containing $G_1(p)$. $G_1(p)$ cannot have a proper stress by itself, see Gluck [11] or Asimow and Roth [1] for instance. Thus some of the cables of $G_2(p)$ connected to B have a proper non-zero stress by Theorem 3. If just these cables and vertices are added to $G_1(p)$, Theorem 4 applies to show that this subframework of $G_2(p)$ is rigid. If this is not all of $G_2(p)$, include the new vertices into B and repeat the above argument. Eventually all of the vertices of $G_2(p)$ will be included. Thus all frameworks of the form where Theorem 4 may apply (without a positive stress perhaps), if they are rigid, are rigid because of several applications of Theorem 4. This is also related to the higher order rigidity of rod frameworks discussed in Connelly [7].

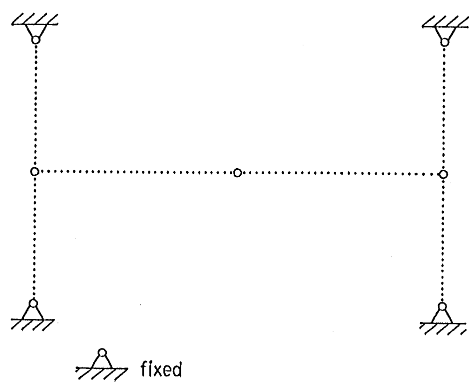

Fig. 5

For example the rod or cable framework of Fig. 5 is rigid by two applications of Theorem 4, and as a rod framework would seem to be "third order" rigid.

We consider yet another type of energy function. Let $\omega \in \mathbb{R}^e$ be a proper stress for $G(p)$. Then for $q \in \mathbb{R}^{nv}$

$$E(q)=\tfrac{1}{2}\sum_{ij}\omega_{ij}(q_i-q_j)^2$$

will be called the *energy form associated to the stress* ω, for $G(p)$. Note that unlike before, where some of the vertices are considered as constants, now $E(q)$ is a homogeneous quadratic form in nv variables. The following are some properties:

A. $\nabla E(p)=0$

B. $\nabla E(q)$ is a linear function of q.

C. $E(p)=0$

D. The 0 set of ∇E, the critical points of E, are invariant under affine linear transformations.

I.e., if $T: \mathbb{R}^n \to \mathbb{R}^n$ is affine linear and we define $T(p)=(Tp_1, \ldots, Tp_v)$, then $E(T(p))=0$ and $\nabla E(T(p))=0$.

A. is just the statement that ω is a stress for $G(p)$.

B. follows since $E(q)$ is a quadratic form.

For C. suppose $E(p)\neq 0$. Then $E(tp)=t^2E(p)$, t real, and $E(tp)$ is not constant. But $\frac{d}{dt}E(tp)=\nabla E(tp)\cdot p=0$, by B. So $E(tp)$ is constant, a contradiction. So $E(p)=0$.

For D. consider an affine linear function $T: \mathbb{R}^n \to \mathbb{R}^n$, where $Tx=Lx+b$, where L is linear, and b is a constant vector. Then

$$\begin{aligned}\nabla E(Tp) &= (\ldots, \sum_j \omega_{ij}(Tp_i-Tp_j), \ldots)\\ &= (\ldots, \sum_j \omega_{ij}(Lp_i-Lp_j), \ldots)\\ &= (\ldots, L(\sum_j \omega_{ij}(p_i-p_j)), \ldots)\\ &= 0.\end{aligned}$$

Thus ω is a stress for $Tp=(Tp_1, \ldots, Tp_v)$ as well as p. Note by C. $E(Tp)=0$ as well.

In order to discuss E more efficiently, we shall condense things a little bit. Although $E(q)$ is a quadratic form in nv variables, the matrix corresponding to E has a certain redundancy. Note that

$$\omega_{ij}(p_i-p_j)^2=\omega_{ij}p_i^2+\omega_{ij}p_j^2-2\omega_{ij}p_i\cdot p_j.$$

Define a symmetric matrix Ω where the i, j-th entry is $-\omega_{ij}$ if $i\neq j$, and $\sum_k \omega_{ik} = \sum_k \omega_{kj}$ if $i=j$, where $\omega_{ij}=\omega_{ji}$. We will call Ω the *stress matrix* associated to the stress $\omega=(\ldots, \omega_{ij}, \ldots)\in\mathbb{R}^e$. If we were to regard the p_i's as formal symbols, then $\frac{1}{2}\Omega$ would be the matrix of E. This is not be confused with the Ω matrix in Bolker and Roth [3].

Precisely, however, if Q is the quadratic form associated to Ω, so $Q(x) = x^t \Omega x$, (x^t is x transpose) for $x \in \mathbb{R}^v$, then

$$E(q) = \sum_{k=1}^{n} Q(p \cdot \bar{e}_k), \quad \text{where} \quad \bar{e}_i = (\overbrace{e_i, \ldots, e_i}^{v \text{ times}}), \quad i = 1, \ldots, n,$$

and e_i is the standard i-th basis vector (with i-th coordinate 1, the rest 0) in $\mathbb{R}^n$. So the matrix of E is just "n copies" of Ω. Note E is positive semi-definite if and only if Ω is. Ω is never definite and has nullity at least 1, since $(1, \ldots, 1)$ is always in the null space of Ω. The row and column sums must be 0 by the definition of Ω.

Example 1. Consider the following framework $G(p)$, where p_1, p_2, p_3, p_4 are the vertices of a square in the plane. The outside edges are cables, and the two diagonals are struts. A stress of 1 on the cables and -1 on the struts is a proper stress for $G(p)$.

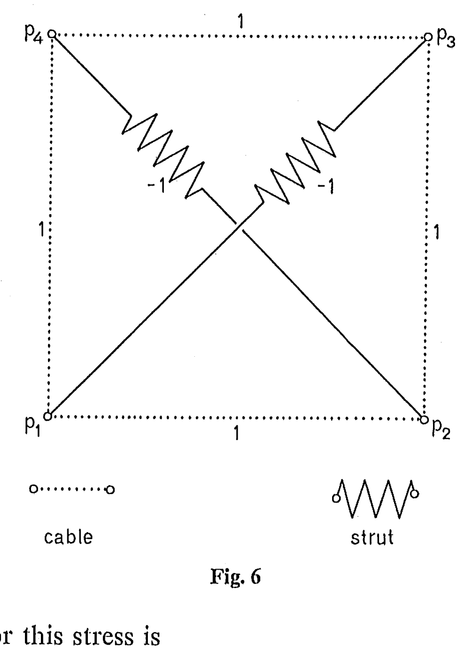

Fig. 6

The stress matrix for this stress is

$$\Omega = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Note Ω is positive semi-definite with nullity 3. Recall that to show a matrix is semi-definite it is sufficient to find $v-n$ variables and show that the $k \times k$

submatrices for $k \leqq v-n$, obtained by using the first (or last) k of the $n-v$ variables, have positive determinants, where n is the nullity. Here since the diagonal entries are positive, we have Ω semi-definite. The importance of this will appear later.

Example 2. This is a generalization of the previous example where the four vertices of $G(p)$ are more generally situated. Since we can change the position of p by an affine linear map and not change the stress ω, we can assume that three of the four points are in a special position. Accordingly we assume $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (a, b)$, $p_4 = (0, 1)$. In order for $G(p)$ to represent a *convex* polygon we assume also $a+b>1$. (See Fig. 7.)

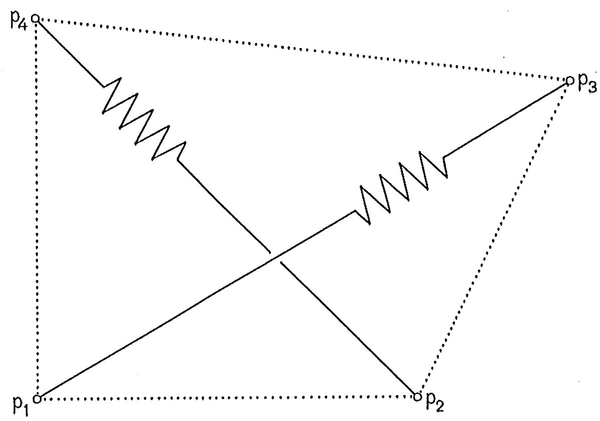

Fig. 7

Since the stress is only determined up to a scaling factor we assume $\omega_{13} = -1$. The equilibrium equations then give the following stress matrix.

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} a+b-1 & -a & 1 & -b \\ -a & \dfrac{a^2}{a+b-1} & \dfrac{-a}{a+b-1} & \dfrac{ab}{a+b-1} \\ 1 & \dfrac{-a}{a+b-1} & \dfrac{1}{a+b-1} & \dfrac{-b}{a+b-1} \\ -b & \dfrac{ab}{a+b-1} & \dfrac{-b}{a+b-1} & \dfrac{b^2}{a+b-1} \end{pmatrix} \end{array} = \Omega.$$

Note the two diagonal stresses, ω_{13}, ω_{24}, are always negative, the side stresses are always positive, and Ω is positive semidefinite of nullity 3.

Example 3. Let the vertices of $G(p)$ be the corners of a unit cube in $\mathbb{R}^3$ with cables along the edges and struts along the 4 main body diagonals.

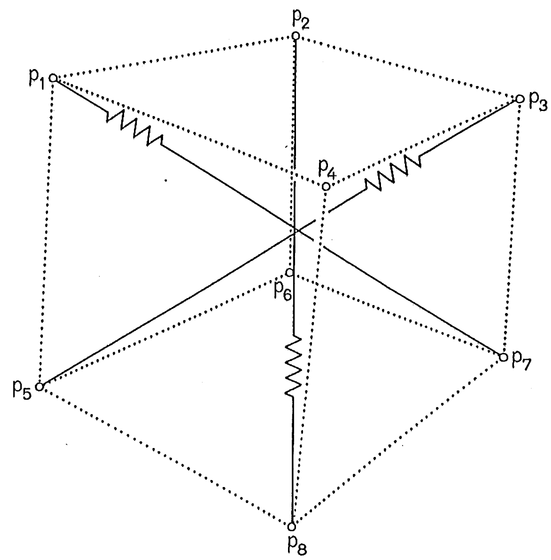

Fig. 8

Then if the cable stresses are 1, the strut stresses are -1. So the stress matrix is

$$
\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \end{array} \\
\left(\begin{array}{rrrrrrrr}
2 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\
-1 & 2 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & -1 & 2 & -1 & 1 & 0 & -1 & 0 \\
-1 & 0 & -1 & 2 & 0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 & 2 & -1 & 0 & -1 \\
0 & -1 & 0 & 1 & -1 & 2 & -1 & 0 \\
1 & 0 & -1 & 0 & 0 & -1 & 2 & -1 \\
0 & 1 & 0 & -1 & -1 & 0 & -1 & 2
\end{array}\right)
\end{array} = \Omega .
$$

It turns out this matrix has nullity 4 and is positive semi-definite also.

The stress matrix is only distantly related to a $3v \times 3v$ matrix used by structural engineers called a stiffness matrix. (See Martin [15] or Langhaar [13] for instance.) Our stress matrix (or rather its energy matrix) assumes that all the members are perfectly elastic springs with rest position at 0 length. The positive stresses would correspond to "spring constants", but the negative stresses would be as if the rest position of that member were at ∞. One virtue of our approach is that the forces are a linear function of the position, whereas in the engineering set-up a quadratic approximation is used instead of the "true" energy to obtain the forces as a linear function of the displacements from the rest position. This will be discussed more fully in the sequel.

The significance of the nullity of the stress matrix is explained in the following lemma.

Lemma 2. *Suppose $G(p)$ in $\mathbb{R}^n$ has a stress ω, and the affine span of $p_1, \ldots, p_v$ in $\mathbb{R}^n$ is k dimensional. Then the nullity of Ω is $\geqq k+1$. Conversely given Ω and $n \geqq$ (nullity of Ω)$+1$, there is $q=(q_1, \ldots, q_v) \in \mathbb{R}^{nv}$ such that the dimension of the affine span of $q_1, \ldots, q_v$ is (nullity of Ω)$+1$. Furthermore, we can choose q so that p_i is the orthogonal projection of q_i onto the affine span of $p_1, \ldots, p_v$.*

Proof. Without loss of generality we assume $p_v=0$, and by renumbering if necessary, that $p_1, \ldots, p_k$ are linearly independent, where k is the dimension of the affine span of $p_1, \ldots, p_k$. By property D above we assume also that $p_i=e_i$, $i=1, \ldots, k$, where e_i is the i-th standard basis vector for $\mathbb{R}^n$. So the vectors

$$(***) \qquad (p_1 \cdot e_1, \ldots, p_v \cdot e_i) \in \mathbb{R}^v, \quad i=1, \ldots, k,$$

are in the null space of Ω again by D (projection onto the line through e_i is affine), and they are clearly linearly independent in $\mathbb{R}^v$. Since the last coordinate is 0, $(1, \ldots, 1)$ together with the above vectors are also independent. Since we have found $k+1$ independent vectors in the null space of Ω, we have shown that the nullity of Ω is $\geqq k+1$.

For the rest we suppose $n \geqq$(nullity of Ω)$+1$ and $p_i \in R^k$. Let $l=$nullity of Ω. Find $\bar{q}_j=(q_{1j}, \ldots, q_{vj}) \in R^v$, $j=1, \ldots, l-k-1$ such that the vectors of (***), $(1, 1, \ldots, 1)$, and $\bar{q}_1, \ldots, \bar{q}_{l-k-1}$ span the null space of Ω. Then $q_i=(p_i \cdot e_1, \ldots, p_i \cdot e_k, q_{i1}, q_{i2}, \ldots, q_{i,l-k-1}) \in \mathbb{R}^{l+1}$, for $i=1, \ldots, v$, provide points $q=(q_1, \ldots, q_v) \in \mathbb{R}^{(l+1)v}$ such that $G(q)$ has the same stress ω as $G(p)$. Note $G(q)$ projects orthogonally onto $G(p)$. ($q_v=0$ still.)

The basic idea used here is that if $p_1, \ldots, p_v$ are regarded as column vectors and are put together to form a matrix, then the rows of this matrix are elements of the null space of Ω, assuming that $G(p)$ has the stress ω. This finishes the lemma.

Note that this lemma gives us a geometric method of investigating the null space of Ω. Namely if we can find the highest dimensional space in which G has a realization with stress ω, then the nullity of Ω is just one more than the dimension of this realization.

More precisely, Lemma 2 provides a correspondence from null spaces of stress matrices Ω to realizations of a graph G with maximum affine span, modulo affine linear maps.

The following is a very simple result that will be useful later.

Lemma 3. *Let $A(t)$, $0 \leqq t \leqq 1$, be a v by v symmetric matrix, where the entries are continuous functions of t, and the null space of $A(t)$ is constant for all t. If $A(1)$ is positive semi-definite, then so is $A(0)$.*

Proof. Restricting $A(t)$ to a complement of the null space, it is positive semi-definite at $A(1)$ and so for all t. Thus $A(0)$ is positive semi-definite.

We now turn our attention to convex polygons in the plane. Let $(p_1, \ldots, p_v)=p$, where $p_i \in \mathbb{R}^2$, and each p_i is on the boundary of a convex polygon, cyclicly ordered and distinct. If every p_i can be separated from the others by a straight line, we say p is *strictly convex.*

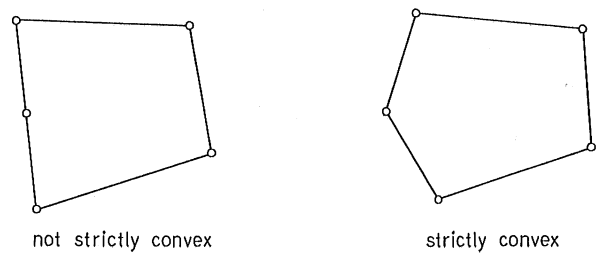

Fig. 9

As in Connelly [7] we define the notion of a *Cauchy polygon*. This is a framework $G(p)$ in the plane, where the vertices $p_1, \ldots, p_v$, in order, form a strictly convex polygon in the plane, the edges $\{i, i+1\}$, $i=1, 2, \ldots, v$ are cables, and $\{i, i+2\}$, $i=1, \ldots, v-2$ are struts (indices modulo v).

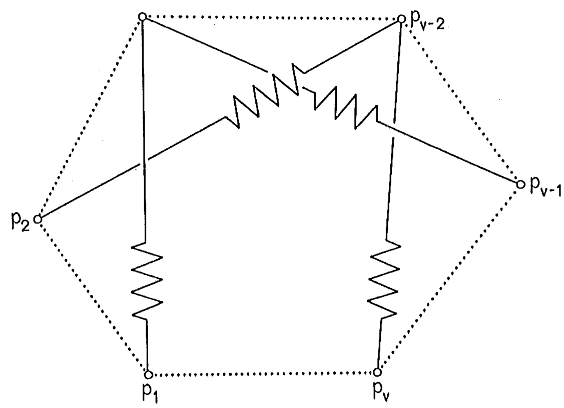

Fig. 10

It is known by Whiteley and Roth [16] that such a framework has a nonzero proper stress ω. We simply extend their proof to show that Ω is positive semi-definite of nullity 3.

Lemma 4. *Any Cauchy polygon $G(p)$ has a proper stress with a positive semi-definite stress matrix of nullity* 3.

Proof. We proceed by induction on v, the number of vertices of G. We start at $v=4$. Example 2 is a Cauchy polygon and by property D, all Cauchy polygons with $v=4$ satisfy the statement of this lemma. So we assume that all Cauchy polygons with v or fewer vertices have a positive semi-definite stress matrix of nullity 3. We wish to show the same statement for $v+1$.

Let $G(p)=G(p_1, \ldots, p_{v+1})$ be a given Cauchy polygon with $v+1$ vertices. Let $G'(p_1, \ldots, p_{v-1}, p_{v+1})$ be the Cauchy polygon associated to the same vertices with p_v deleted. Let $G''(p_{v-2}, p_{v-1}, p_v, p_{v+1})$ be the Cauchy polygon associated to the "last" four vertices of G starting at $v-2$. Let Ω' and Ω'' be the positive semi-definite stress matrices of nullity 3 associated to

$G'(p_1, \ldots, p_{v-1}, p_v)$ and $G''(p_{v-2}, p_{v-1}, p_v, p_{v+1})$ respectively. Let $\tilde{G}'$ be the graph obtained by adding a single vertex $\{v\}$ and no edges to G'. Let $\tilde{\Omega}'$ be the $v+1$ by $v+1$ symmetric matrix associated to $\tilde{G}'(\tilde{p})$, where $\tilde{p}=(p_1, \ldots, p_{v-1}, \tilde{p}_v, p_{v+1})$ and $\tilde{p}_v$ is arbitrary. We insist that $\tilde{G}'(\tilde{p})$ have the "same" stress as $G'(p_1, \ldots, p_{v-1}, p_{v+1})$ so that $\tilde{\Omega}'$ is simply obtained by adding a row and column of zeros to Ω'.

Similarly define $\tilde{\Omega}''$ as the $v+1$ by $v+1$ symmetric matrix obtained by adding $v-3$ rows and columns of zeros to Ω''. Note that for $\tilde{\Omega}''$, $\omega''_{v+1,v-2}>0$, and for $\tilde{\Omega}'$, $\omega'_{v+1,v-2}<0$. (Recall that the off-diagonal matrix entries are minus these stresses.)

From the proof of Lemma 2 it is clear that the null space of $\tilde{\Omega}'$ can be naturally identified with the set of affine images of $(p_1, \ldots, p_{v-1}, \tilde{p}'_v, p_{v+1})$, with the v-th vertex $\tilde{p}'_v$ chosen arbitrarily. On the other hand, the null space of $\tilde{\Omega}''$ can be identified with the affine images of $(\tilde{p}''_1, \ldots, \tilde{p}''_{v-3}, p_{v-2}, p_{v-1}, p_v, p_{v+1})$, with the first $v-3$ vertices chosen arbitrarily. It is clear that the intersection of these two sets is the set of affine images of $p=(p_1, \ldots, p_{v+1})$.

The key point is that

$$\omega''_{v+1,v-2}\Omega' - \omega'_{v+1,v-2}\Omega''=\Omega$$

is a stress matrix for $G(p)$. (*See Fig.* 11).

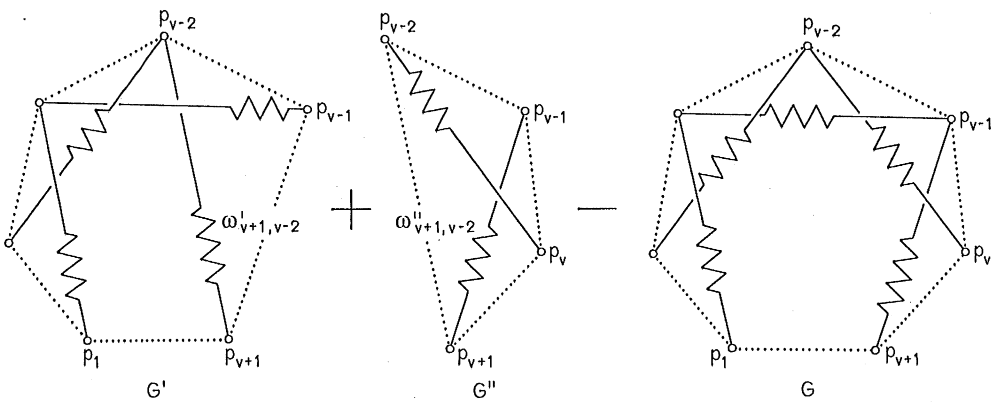

Fig. 11

Call ω the stress for the stress matrix Ω. By the scaling $\omega_{v+1,v-2}=0$. Except for the stress $\omega_{v+1,v-1}$, all the other stresses of ω are the sum of stresses of the same sign. However,

$$\omega_{v=1,v} = -\omega'_{v+1,v-2}\,\omega''_{v-1,v}>0$$

$$\omega_{v,v+1} = -\omega'_{v+1,v-2}\,\omega''_{v,v+1}>0.$$

If $\omega_{v,v-2}\geqq 0$ also, since the final polygon $G(p)$ is convex at p_v, $G(p)$ could not be in equilibrium at p_v. (See Lemma 6.2 of Roth and Whiteley [16].) Thus $\omega_{v,v-2}<0$. Thus Ω is a stress matrix (coming from a *proper* stress) for $G(p)$, a Cauchy polygon.

Since each of $\tilde{\Omega}'$ and $\tilde{\Omega}''$ is positive semi-definite, so is Ω. Thus the null space of Ω is the intersection of the null spaces of $\tilde{\Omega}'$ and $\tilde{\Omega}''$, which is identified with the set of affine images of p, by the argument above. Thus Ω has nullity 3. This finishes the lemma.

Many special cases of polygonal frameworks can be shown to have such stress matrices. However, a large class can be dealt with once one knows that for a given p there is some matrix Ω, with possibly more entries non-zero than desired, that has the correct nullity and definiteness.

Theorem 5. *Let $G(p)$ be a framework in the plane, where p is a convex polygon with cables on the boundary, struts inside. Suppose $G(p)$ has a proper non-zero stress ω. Then the stress matrix Ω for ω has nullity* 3 *and is positive semi-definite.*

Proof. We show first that the nullity of Ω is exactly 3. By Lemma 2 the nullity is at least 3, and if the nullity is greater than 3, then there is a $G(\bar{p})$ in $\mathbb{R}^3$, with ω as a stress also, that projects orthogonally onto $G(p)$ in $\mathbb{R}^2$, and the affine span of $G(\bar{p})$ is 3 dimensional. Let H be the convex hull of the vertices of $G(\bar{p})$ in $\mathbb{R}^3$. Each natural face or facet of H, the 2 dimensional intersection of H with a support plane, projects in a one-to-one fashion into $\mathbb{R}^2$. Consider just the top faces of H, the facets seen from $+\infty$. These project to give a decomposition of the convex polygon in $\mathbb{R}^2$ as a convex cell complex. Since H is 3 dimensional there will be some edge e, in the interior of the polygon, which is the projection of a top edge $\bar{e}$ of H, a one-dimensional facet of H. e separates the polygon and it is easy to see that some strut of $G(p)$ must cross e in order for there to be a proper equilibrium stress ω. This is because if there were no such strut it would be possible to increase e slightly, keeping all the cables of fixed length and increasing any other possible strut length. This will change (decrease) the energy in such a way that even the first derivative is <0. But this is not possible at a critical point, i.e., when $G(p)$ is in equilibrium. Thus some strut of $G(p)$ must cross e in its interior.

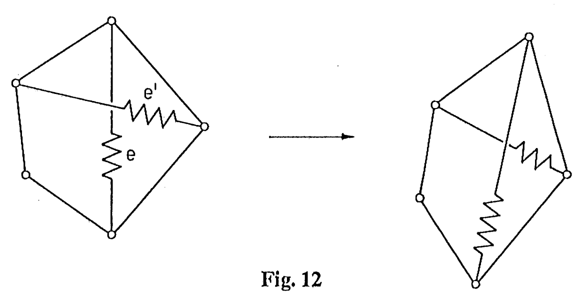

Fig. 12

Let e' be a strut of $G(p)$ which crosses e. Let $\bar{e}'$ be the corresponding strut in $G(\bar{p})$. Note that neither endpoint of $\bar{e}'$ is one of the endpoints of $\bar{e}$. e separates the vertices of G into two sets V_1 and V_2, where the endpoints of e correspond to $V_1 \cap V_2$. Regard $G(\bar{p})$ as "hinged" along $\bar{e}$ and consider the motion of $G(\bar{p})$ that simply flattens out this hinge, moving V_1 and V_2 each as a rigid set. Note that this increases the length of $\bar{e}'$ and every other such crossing strut, even so that their first derivatives are >0. So the first derivative of the energy of G at $\bar{p}$ is negative, and as before this is impossible if $G(\bar{p})$ is in equilibrium with stress ω. (Note that the effect of these motions is to flatten out the $G(\bar{p})$ framework into the plane each time decreasing the energy.) Thus the nullity of Ω is precisely 3 by Lemma 2.

To show that Ω is positive semi-definite we proceed as follows. Let $\Omega(1)$ be the stress matrix for a Cauchy polygon with vertices p. Consider the stress matrix $\Omega(t)=(1-t)\Omega+t\Omega(1)$, $0\leqq t\leqq 1$. By the above each Ω has nullity 3 since it is a stress matrix with a proper stress (allowing any internal edge to have a negative, strut stress). By Lemma 4, $\Omega(1)$ is positive semidefinite. By the above argument, $\Omega(t)$ has nullity 3 for all $0\leqq t\leqq 1$. By Lemma 3, $\Omega(0)=\Omega$ is also positive semi-definite. This completes Theorem 5.

The following was conjectured by Walter Whiteley.

Corollary 1. *Let $G(p)$ be as in Theorem 5, with a proper, non-zero stress. Then $G(p)$ is uniquely embedded in $\mathbb{R}^n$, for $n\geqq 2$. Thus $G(p)$ is rigid in $\mathbb{R}^n$.*

Proof. Let $G(p')$ be any other realization of $G(p)$, with cables not longer, struts not shorter. By Theorem 5 defining the energy form for the stress ω for $G(p)$, we know $E(p')\geqq E(p)=0$. If the inequality is strict then some strut must have decreased in length or some cable increased. Thus $E(p')=E(p)=0$. Since E is semi-definite (because Ω is) p' is in equilibrium with respect to ω as well.

We claim there is an affine linear function $T\colon R^2\to R^2$ such that $Tp=p'$. We can certainly arrange that $Tp_i=p_i'$ for 3 non-colinear p_i of p, and if this is the case, all the other $Tp_i=p_i'$ as well. Otherwise we could take some coordinate of Tp_i-p_i' and add it to p_i in the e_3 direction to get an equilibrium $\bar{p}$ not lying in a plane, contradicting Lemma 2. Thus $Tp_i=p_i'$ for all $i=1,\ldots,v$.

Now we must show T is a rigid motion of $\mathbb{R}^n$. We can clearly assume $n=2$. Let $p_1,\ldots,p_v$ be the vertices of $G(p)$ written in clockwise cyclic order, and we may assume $G(p)$ is strictly convex; each p_i is at a corner. Recall from linear algebra that $T(S^1)-T(0)$ is an ellipse centered at the origin, where $S^1=\{x\in\mathbb{R}^2 \mid |x|=1\}$. Those points where the ellipse is outside S^1 correspond to directions where T is expanding, $|T(x)-T(0)|>|x|$; similarly those points of the ellipse inside S^1 come from points where T is contracting, $|T(x)-T(0)|<|x|$.

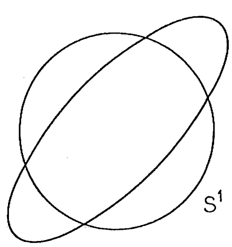

Fig. 13

Thus there are 4 intervals, symmetric about 0, where T is expanding and contracting alternately, or 2 of the same type with 2 points separating them, or just one region, or no regions of either type (no expanding or contracting, a rigid motion).

Consider the lines from 0 through $p_{i-1}-p_i$ and p_i-p_{i+1}. Since they correspond to cable directions $|Tp_i-Tp_{i+1}|\leqq|p_i-p_{i+1}|$, $i=1,\ldots,v$, these lines must be inside or on the boundary of the contracting regions. Since $G(p)$ is in equilibrium for each p_i, there must be some strut p_i-p_j, $j\neq i-1$, $i+1$, and the line through 0 and p_i-p_j must be between the lines through $p_{i-1}-p_i$ and $p_{i+1}-p_i$ in a clockwise sense.

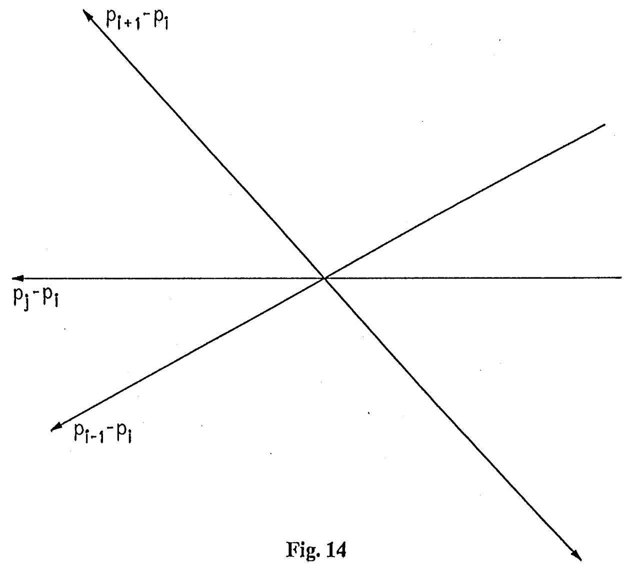

Fig. 14

Thus the clockwise interval from $p_{i-1}-p_i$ to $p_{i+1}-p_i$ must contain a vector that is non-contracting.

Thus none of these clockwise intervals can be contained in one of the contracting intervals. But as we proceed around the convex polygon each time adding one of the clockwise intervals onto the next we eventually cover the whole circle exactly once. (This is a version of the tangent map.) But after 2 steps we cannot continue jumping from one contracting interval to the other without overlapping. Then there can be no contracting or expanding intervals, and T is rigid motion as desired. Thus $G(p)$ is uniquely embedded.

Remark 4. Even without Theorem 5, Corollary 1 applies to any polygonal framework once it is known that it has a proper stress with a positive semi-definite stress matrix Ω of nullity 3. In particular Corollary 1 applies directly to Cauchy polygons using Lemma 4, without the need to use Theorem 5. In the next section we show how the result that Cauchy polygons are uniquely embedded can be used to prove a key lemma in Cauchy's original paper [5], his Theorem II. Cauchy's original proof was found to be inadequate, and there have been several replacement proofs and comments since then.

Corollary 2 (Grünbaum's conjecture). *Let $\bar{G}(p)$ be a framework coming from a convex polygon, but with rods on the boundary and cables on the inside. Let G be the graph with cables replacing rods from $\bar{G}$ and struts replacing cables from $\bar{G}$. If $\bar{G}(p)$ is rigid in $\mathbb{R}^2$, then $G(p)$ is uniquely embedded in $\mathbb{R}^n$, for any $n\geqq 2$.*

Proof. By Theorem 3, $\bar{G}(p)$ must have a proper non-zero stress ω. It is clear that $\omega_{ij}<0$ on the boundary cables. Then $-\omega$ is a proper non-zero stress for $G(p)$. By Corollary 1 $G(p)$ is uniquely embedded in $\mathbb{R}^n$.

Remark 5. In [12] Grünbaum and Shephard remark that the converse is false.

An analogue of Theorem 5 should hold in dimension 3. The above proof does not seem to extend, but a few examples indicate that the stress matrix "coming from" convex polyhedra in $\mathbb{R}^3$ are semi-definite of nullity 4 as would be desired. For instance the framework of Example 3 is of this type. It is also easy to check that 3 cable (line) directions and 4 strut directions cannot be separated by an ellipse in the projective plane of such directions. Thus the argument of Corollary 1 works, and the framework is uniquely embedded in $\mathbb{R}^n$ for $n \geqq 3$. It is interesting to observe that Branko Grünbaum has an example of a 3-dimensional polyhedron with cables on the boundary, struts inside, with a proper non-zero stress, with the cables forming a 3-connected graph, but which is not rigid in $\mathbb{R}^3$. The flex, however, extends to an affine motion of $\mathbb{R}^3$, so there is still hope for the analogue of Theorem 5 and a rigidity theorem, if there are enough cables and struts to stop such non-rigid affine motions. This would give some version of Whiteley's conjectures for dimension 3, see Whiteley [23].

IV. The Relation to Related Results

As mentioned in Remark 4, Lemma 4 and Corollary 1 alone can be used to show the Cauchy polygons are uniquely embedded. This in turn can be used to show the following lemma of Cauchy [5], his Theorem II.

Lemma 5. *If, in a convex planar or spherical polygon ABCDEFG, all the sides AB, BC, CD, ..., FG, with the exception of only AG, are assumed invariant, one may increase or decrease simultaneously the angles B, C, D, E, F included between these same sides; the variable side AG increases in the first case, and decreases in the second.*

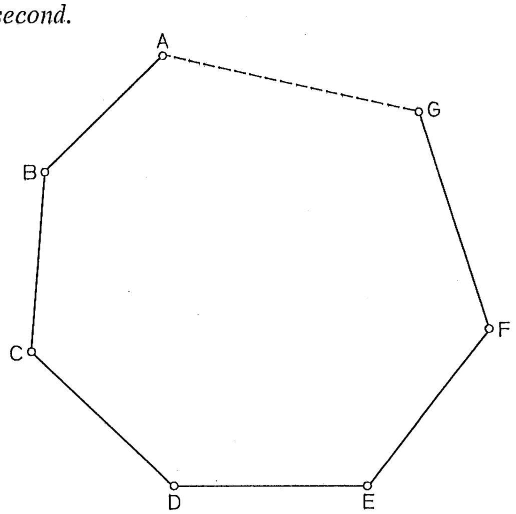

Fig. 15

We have taken liberties and removed G from the list of angles allowed to increase or decrease.

Apparently it was pointed out by Steinitz [20], more than a hundred years after Cauchy, that Cauchy's proof has some gaps. The crux of the problem was that, in the induction that Cauchy described, the polygon, between the original one and the one with varied angles, may not be convex; the induction process inadvertantly breaks down. (See Grünbaum and Shephard [12] for further comments on the corrections to Cauchy's paper.)

Of course Cauchy's proof works quite well if the varied polygon is "close" to the original. This naturally is enough to get a weaker result about the rigidity of convex polyhedra, but later it is clear Cauchy had in mind that any two "isometric" convex polyhedra were congruent.

Steinitz published a "correct" proof of Cauchy's lemma [20], but it was reasonably detailed and complicated. Stoker [21] much later gave another proof much in the spirit of Cauchy's very natural idea, but of course Cauchy's "pitfall" is somewhat of an annoyance. Lyusternik [14] gives a proof somewhat like Steinitz's. Even more recently I.J. Schoenberg and S.K. Zaremba [17] describe an elegant "simple" proof that involves the "trick" of choosing a reference point appropriately in the middle of the arc AG.

It is especially interesting to compare Cauchy's lemma with a theorem of Axel Schur [18]. See especially Chern [6] pages 35–39. Schur's theorem is an almost exact analogue of Cauchy's lemma in the smooth case; in fact since it is stated in the piecewise-smooth case, also, at least in the plane, it is a generalization of Cauchy's lemma. Schur assumes that the original arc AG is smooth (or piecewise-smooth), the varied arc has the same length (each segment has the same length), and that the curvature of corresponding points on the varied arc decreases (and the angle between tangents to the curve at the corners decreases). Then the length AG decreases.

It is especially intriguing that the proof of this in Chern [6] involves choosing an internal reference point and estimating a certain integral. Schoenberg and Zoremba's [17] proof is almost exactly the same as Chern's.

With this background in mind we propose fearlessly yet another, essentially a fourth, proof of Cauchy's lemma.

Proof of Lemma 5. Since the original polygon and the varied one are both convex, it is no loss in generality to assume that the angles B, C, ..., F increase; then we show that AG increases. The advantage of looking at just this case (at least in the plane) is that the varied polygon does not have to be convex, or for that matter even stay in the plane. It can pop out to 3-space or n-space if it likes.

First we do the case when the original polygon is planar. Regard the sides AB, BC, ..., FG, *and* AG as cables, since they do not increase in length. Since the angles $B, \ldots, F$ now increase, and their sides stay the same length, the lengths AC, BD, ..., EG also increase, so we regard these as struts. Notice this is what we called a Cauchy polygon. Thus if AG decreases, we have another embedding of the original Cauchy polygon, which is impossible. Thus AG must increase, and can only stay the same if all the other angles and sides stay the same.

For the spherical case consider the cone over the spherical polygon from the center of the sphere. Since the polygon is assumed to be spherically convex, it lies in a hemisphere, and there is a flat plane that separates the center of the sphere and the spherical polygon. This plane intersects the cone in a planar polygon. If one side and the angles of the spherical polygon are increased, then the planar polygon corresponds to another polygon with corresponding side and angles increased. However, the varied planar polygon may not be planar; but this does not matter, since our Cauchy polygon is allowed to vary into 3-space. Thus AG must increase as before.

Another application of the polygons of Theorem 5 being uniquely embedded is to a comment of 0. Bottema, [4] about a theorem of van der Waerden [22]. Van der Waerden's theorem says that if a pentagon with equal sides and equal angles between the sides is embedded in 3-space, then it is planar. There have been subsequently several proof of this, see S. Šmakal [19], and we do not intend to add to them. Bottema makes the comment, however, that if such a polygon embeds in any euclidean space (which might as well be 4-space), and θ is the angle between the sides, then $36° \leqq \theta \leqq 108°$. In fact he shows that if θ is in this range, the polygon embeds in 4-space, and is planar if and only if one of the equalities hold.

That θ is bounded between 36° and 108° follows immediately from the fact that Cauchy polygons are uniquely embedded, even allowing them to pop up to 4-space. The lower bound follows from comparing the given space pentagon with a planar regular pentagon whose diagonals are the same lengths as (and correspond to) the sides of the space pentagon. The upper bound follows from comparing the given space pentagon with a planar regular pentagon whose diagonals are the same length as (and correspond to) the diagonals of the space pentagon. Clearly there are many generalizations along these lines.

In [16] Roth and Whiteley remark that there is a convex polygonal framework $G(p)$, with rods on the boundary and cables inside such that for a nearby realization $G(p')$, $G(p')$ is infinitesinally rigid, but $G(p)$ is flexible. This provided an answer to a question of Grünbaum and Shephard in [12], page 2.13.

Using Theorem 2 and Corollary 1 we can also provide several such examples. For instance any of the following frameworks from Grünbaum and Shephard [12] or Roth and Whiteley [16] provide such a $G(p)$. The vertices all form a regular hexagon.

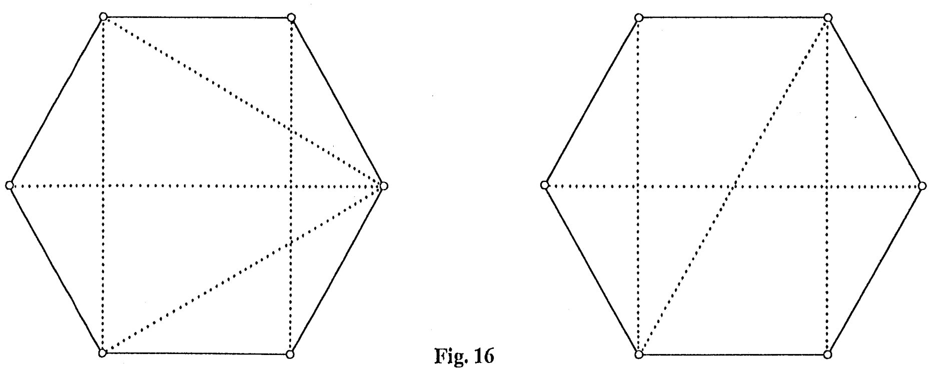

Fig. 16

Each of these $G(p)$ have the following properties:

(1) If all the members of G are replaced by rods to get $\bar{G}$, then $\bar{G}(p)$ is infinitesinally rigid.

(2) $G(p)$ has a proper non-zero stress: Let the graph G' be obtained by changing all rods of G to cables and all cables of G to rods. Then by Corollary 1, $G'(p)$ is rigid.

(3) $G(p)$ is flexible. (This is stated for the graphs of Fig. 3 in Grünbaum and Shephard [12].)

For any graph $G(p)$ satisfying (1), (2), (3) there is a p' close to p such that $G(p')$ is infinitesinally rigid. This is because by Theorem 2, since $G'(p)$ is rigid (by (2)), there is a p' close to p such that $G'(p')$ has a proper stress ω that is non-zero on all members.

Then $-\omega$ is a proper stress for $G(p')$. Also by (1), if p' is close enough to p, $\bar{G}(p')$ is infinitesinally rigid (see Gluck [11] or Asimow and Roth [1]). By the main theorem of Roth and Whiteley [16] this implies $G(p')$ is infinitesinally rigid.

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Oblatum 20-XI-1980 & 2-X-1981