Jan. 23

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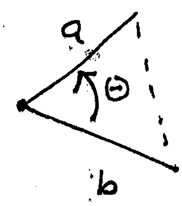
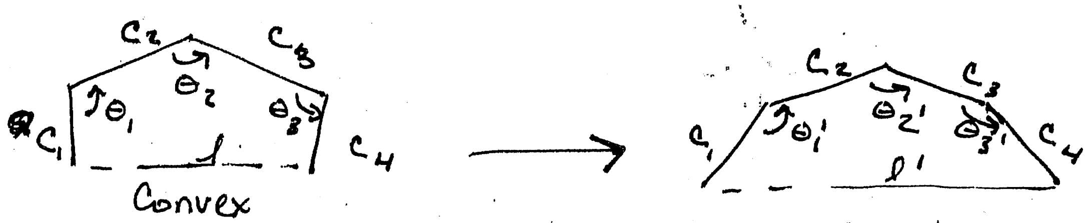
Tensegrity

History:



Greek geometry

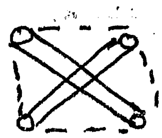
Arm Lemma:



Assume $\theta_1' \geq \theta_1, \theta_2' \geq \theta_2, \ldots \theta_i' \geq \theta_i$

$\Rightarrow l' \geq l$

1948: Ken Snelson: constructed something with cables + stru
B. Fuller - came up with name tensegrity
- tensional Integrity
Iogensen - came up with tensegrities before Snelso

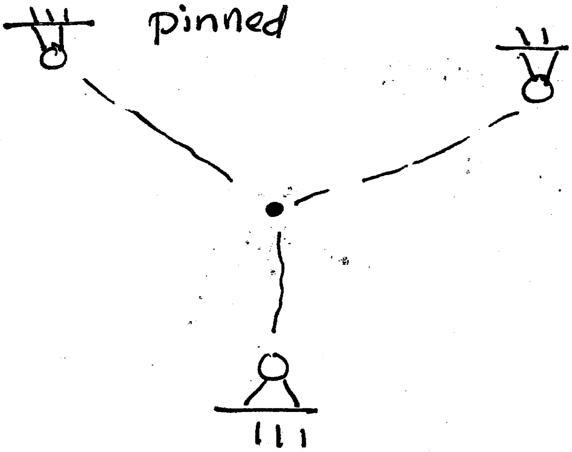


Question: Why do these things stand up?

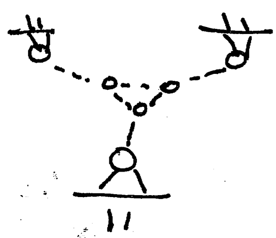
spider webs are an example of a tensegrity

pin some vertices

symbol pinned



or



Engineering

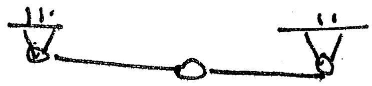
What is the "theory" good for?

- applications
- understanding geometry

1. Structural Engineering
   - Algorithms for computing "rigidity"
   - Bar and Joint framework

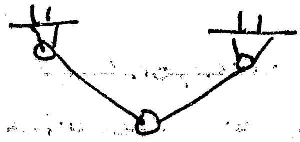
   } 1^{st} order theory

   ⤳ combinatorial questions



Is this rigid?
mathematician - yes
engineer - no

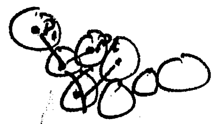
to engineer, the following is rigid because forces can be resolved



(still under applications & Structural Engineering)
- Static rigidity $\Leftrightarrow$ Infinitesimal rigidity } Prestress stability
forces elasticity

2. Computational biology
molecule problem
determine a configuration of atoms (= points) from distance constraints
uniqueness of a configuration with distance constraints.

3. Discrete geometry
- packings and coverings $|p_i - p_j| > d_{ij}$

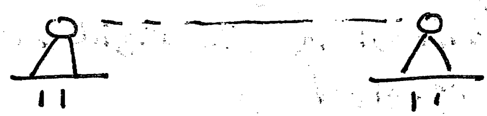


- polyhedra
Steinitz Every 3-connected planar graph comes from a 3 dimensional polytope.

Maxwell - Cremona - relates polytopes to stresses in a planar graph

- distance geometry

Electrical Networks



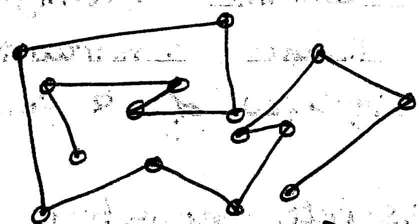
all points lie on line

direct correspondence between spiderwebs on a line + electrical networks

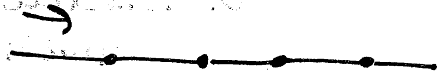
- Algebraic graph theory

G = graph ⟷ Adjacency matrix
Laplacian ⟷ Stress matrices ⟷ M matrix
result of Lovasz has to do with
Colin de Verdier graph invariants

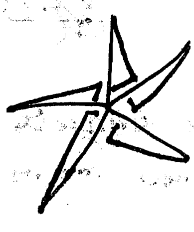
4. Robotics
Robot arms



arm in plane
not convex



Computational geometry



"locked"

can you get an arm that is locked?

turns out it an arm can ~~can~~ be opened up.

What about a cycle? Can it always be made convex?

Rote can have movement only allowing increasing distances

5. Mechanisms : Rigidity conjecture

Is it possible to have embedded triangulated surface that moves?

Cauchy - if convex, then it is rigid

1977 June 3pm found counterexample

1896 R. Bricard

Bellows Conjecture Flexible surface flexed with constant volume

1995 - proved true by I. Sabitov

6. Tension percolation in probability

Jan. 25

homework due Feb. 1

Background
1. linear algebra
   - symmetric matrices
   - quadratic form
2. inverse function theorem
3. basic topology
(4.) representation theory for finite groups

Spectral Theorem
Let $A_{n\times n}$ be an $n\times n$ real symmetric matrix.
Then there is a real orthogonal matrix P $(P^TP = I_{n\times n})$
such that $A = PDP^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Definition: A is positive definite if all $\lambda_i > 0$.
A is positive semi-definite if all $\lambda_i \geq 0$.
A is indefinite if some $\lambda_i > 0$ and another $\lambda_j < 0$.

Quadratic form associated to A symmetric is
$Q: \mathbb{R}^n \to \mathbb{R}^1$ such that $Q(v) = v^TAv$.
$= \sum_{i,j} a_{ij} x_i x_j$ where $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Factoid: $\nabla Q(v) = 2Av$

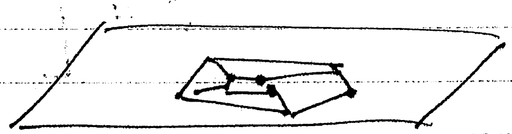
meaning A is positive semi definite.

Factoid: If $Q \geq 0$ and $Q(v) = 0$ then $Av = 0$.

What we will do:

1. Basic theory of tensegrity frameworks, stress matrizes.
   relavent to Snelson-like tensegrities and <u>others</u>.
2. Static rigidity = infinitesimal rigidity → velocity of vertices
   = 1st order rigidity

   resolutions of external forces
3. 3D Theory
   - Cauchy's theorem: 3D polyhedra are "rigid"
     → Dehn → AD Alexandrov → W. Whiteley
     → Connelly
4. Prestress stability
5. Maxwell - Cremona correspondence



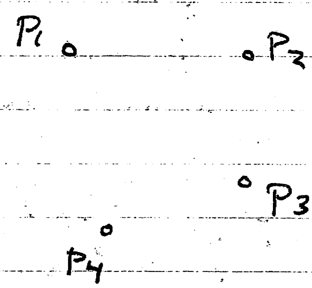
Is there 3D polytope corresponding to polytope in plane.
And there is some stress associated with it.

6. Flexible surfaces - embedded surface which flexes
   - Bellows conjecture
7. Lovasz' Theory of Colin de Verdiere number of a graph
   ↔ stress matrices
8. Symmetric tensegrities
   - many examples
   - representation theory

Notation:

finite **Configuration** $\equiv (p_1, \ldots, p_n)$

$p_i \in \mathbb{E}^d \qquad p_i = \begin{bmatrix} x_{1_i} \\ x_{2_i} \\ \vdots \\ x_{n_i} \end{bmatrix}$



$\mathbb{E}^d$ stands for d-dim Euclidean space. Yes, this is $\mathbb{R}^d$, we use $\mathbb{E}$ since we will be using Euclidean metric.

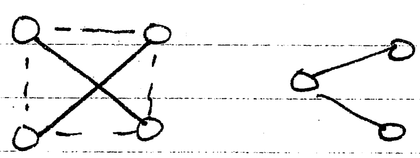
2 types of points:
- pinned
- unpinned

G = graph finite, undirected without loops or multiple edges each vertex here corresponds to a point in configuration

3 types of edges

cable	o - - - o	certain length or less
bar	o——o	certain length
strut	o══o	certain length or more

Tensigrity framework
$G(p)$



Bar framework - all members are bars
Spider Web - all members are cables
spider webs only interesting if there are pinned vertices

circle - sphere packing is an example of all struts

Fix a tensegrity graph G, and let p, q be two configurations for G. We say $G(p)$ dominates $G(q)$ if

write $G(p) \geq G(q)$

$$|p_i - p_j| \left\{ \begin{matrix} \geq \\ = \\ \leq \end{matrix} \right\} |q_i - q_j| \text{ for } (i,j) \text{ a } \left\{ \begin{matrix} \text{cable} \\ \text{bar} \\ \text{strut} \end{matrix} \right\}$$

Example $G = K_n$ = complete bar framework.

def. of congruence: ~~$K_p(n)$~~ $K_n(p) \overset{\leq}{\geq} K_n(q)$ then we say
p and q are congruent configurations.
(meaning all pairwise distances the same). Write $p \sim q$.

We say $G(p)$ is globally rigid if
$G(p) \geq G(q) \Rightarrow p \sim q$ i.e. p is congruent to q.
(p & q configurations in E^d.)

Jan. 30

http://www.math.cornell.edu/~connelly/762.html
links to notes, texts, problems (latex)

do first set of exercises - end of 2.8
Due Thurs. next week - Feb. 8

due this Thurs. Feb 1 - exercises on quadratic forms

Tools:

Stress $G(p)$ = tensegrity

Def: A stress for G is an assignment of scalars $w_{ij} = w_{ji}$ to each member $\{i,j\}$

real number

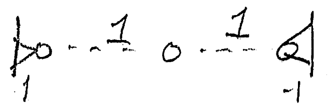
When appropriate, we will say $w_{ij} = w_{ji} = 0$ when $\{i,j\}$ is not a member of G.

We think of $\omega = (\ldots w_{ij} \ldots)$ = row vector in $\mathbb{R}^e$ where $e = \#$ members of G

We say ω is a proper stress if
$w_{ij} = w_{ji}$ is ≥ 0 for $\{i,j\}$ = cable
≤ 0 strut
There is no condition for $\{i,j\}$ = bar.

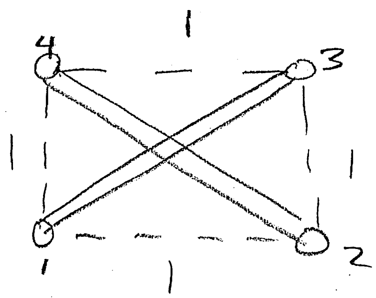
We say that a stress for $G(p)$ is an equilibrium stress (or a self-stress) if for each non-pinned vertex i, of G, $\sum_j \overset{\text{stress}}{w_{ij}} (p_j - p_i) = \overset{\text{vectors}}{0}$.

Examples:



Exercise: calculate a non-zero equilibrium stress for a cycle $(p_1, \ldots, p_n)$ in $\mathbb{E}^1$ where $p_i \neq p_{i+1} \ldots$

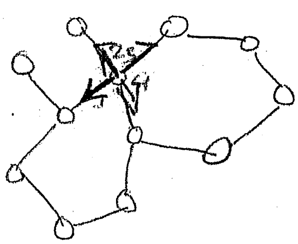
Calculate $\omega_{i,i+1}$?

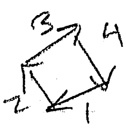


then $\omega_{31} = -1$

since $\underbrace{\omega_{21}}_{=1}\begin{bmatrix}1\\0\end{bmatrix} + \underbrace{\omega_{41}}_{=1}\begin{bmatrix}0\\1\end{bmatrix} + \omega_{31}\begin{bmatrix}1\\1\end{bmatrix} = \{$

Graphical Statics

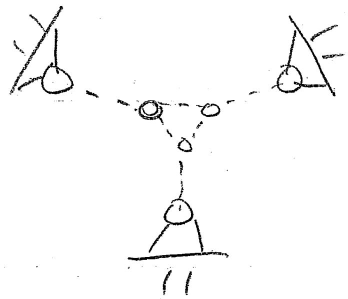




closes in on itself

create the polygons for each vertex - fit together as tiling

Why equilibrium stresses?
special case: all $\omega_{ij} > 0$
Call this a <u>spider web</u>.
want there to be pinned vertices in order for it to be interesting.



Important idea:

Energy: Start with any stress for graph G.
Define stress-energy for any configuration
$q = (q_1, \dots, q_n)$, $q_i \in \mathbb{E}^d$
by $\sum_{i<j} \omega_{ij} |q_i - q_j|^2 = E_\omega(q)$

ω_{ij}: constants q_i, q_j: variables pinned vertices are constants

E_ω is a quadratic function
(if no pinned vertices, it is a quadratic form)
$E_\omega : \mathbb{E}^k \to \mathbb{E}^1$ $k = \#$ of variable vertices

What are critical points of E_ω?

Lemma: A configuration p is a critical point for E_ω if and only if p is in equilibrium with respect to the stress ω.

Proof: Let p be a critical point for E_ω.
Let p' be any direction, $p_i' = 0$ for pinned vertices
We calculate

$$E_\omega(p + tp') = \sum_{i<j} \omega_{ij} (p_i - p_j + t(p_i' - p_j'))^2$$

$$= \sum_{i<j} \omega_{ij} (p_i - p_j)^2 + 2t \sum_{i<j} \omega_{ij} (p_i - p_j)(p_i' - p_j')$$

$$+ t^2 \sum_{i<j} (p_i' - p_j')^2 \omega_{ij}$$

At $t=0$ $\frac{dE_\omega}{dt} = 2\sum \omega_{ij}(p_i - p_j)\cdot(p_i' - p_j') = 0.$

Let p' be 0 except in one non-pinned coordinate.

$2\sum_j \omega_{ij}(x_i - x_j) = 0$ for the x-coordinate

Doing this for all unpinned vertices we get the equilibrium condition.
Conversely, these p''s are a basis for all possible p''s and so the equilibrium condition implies a critical point.

Spider webs $\omega_{ij} > 0$

<u>The Spiderweb Theorem</u>: Let G be a spiderweb graph where each nonpinned vertex is connected by a chain of cables to a pinned vertex. Suppose $G(p)$ is in equilibrium with respect to a nonzero proper equilibrium stress. Then $G(p)$ is globally rigid.

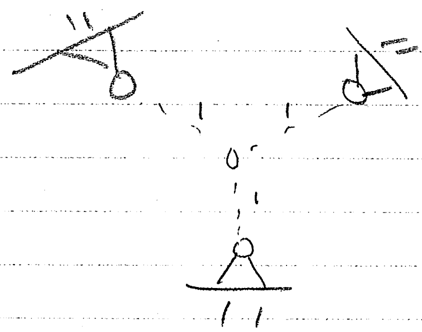
<u>Proof</u>: Use the principle of least work. First we show that p is a global minimum point for E_ω. Before that we show there must be at least one global min point for E_ω.
Call this point q_{min}. q_{min} is also critical point.
$E_\omega(q + tq') = E_\omega(q) + 0 + t^2 E_\omega(q') > E_\omega(q)$
> 0 unless $q' = 0$
So q is unique global minimum
Similarly p is the unique global min. So $p = q$.

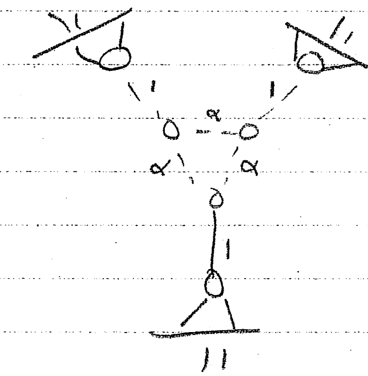
Suppose $G(q) \le G(p)$

Then $E_\omega(q) = \sum_{i<j} \overset{\text{positive}}{w_{ij}} (q_i - q_j)^2 \le \sum_{i<j} w_{ij} (p_i - p_j)^2$

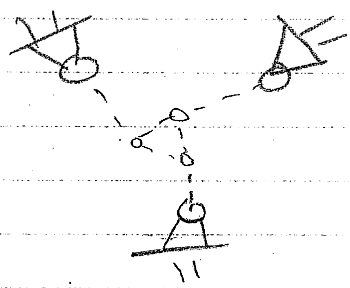
$= E_\omega(p)$

But p is the global min of E_ω so $p = q$.
So $G(p)$ is globally rigid.

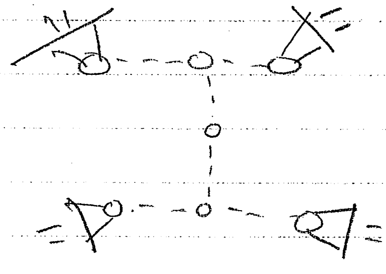




both are globally rigid



not globally rigid



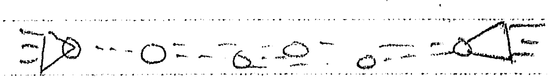
does not have a proper (on all members) nonzero equilibrium stress but it is globally rigid

Feb. 1

We were discussing spider webs

Ex: dream catchers

Spider web on a line

 $\longleftrightarrow$ network of registers

pinned vertices → voltage potential applied

at each non-pinned vertex
$$\sum_j \omega_{ij}(p_j - p_i) = 0$$

Cable with stress ↔ conductance $= \frac{1}{\text{resistance}}$

Equilibrium condition ⟺ Kircher's law of current = current out

Lemma: Let $G(p)$ be any tensegrity framework with an equilibrium stress ω. Let q be an affine image of the configuration p, i.e. there is a $A_{n\times n}$ matrix, $b \in E^d$, $q_i = Ap_i + b$. Then $G(q)$ is also in equilibrium with respect to ω.

Proof: Check equilibrium equation.
$$\sum_j \omega_{ij}(q_j - q_i) = \sum_j \omega_{ij}(Ap_j + b - Ap_i - b)$$
$$= \sum_j \omega_{ij}(A(p_j - p_i))$$
$$= A\left(\sum_j \omega_{ij}(p_j - p_i)\right) = 0.$$

Homogeneous stresses
(no pinned vertices)
G = any tensigrity graph
ω = (proper) stress for G
Define an energy (potential) function as before
$$E_\omega(q) = \sum \omega_{ij}(q_i - q_j)^2$$

E_ω is a quadratic form (no linear or constant term). What is the matrix of E_ω with respect to a standard basis of $\mathbb{E}^{nd}$.

Note

$$\underbrace{\underbrace{[x_i \; x_j] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[x_i - x_j \quad -x_i + x_j]} \begin{bmatrix} x_i \\ x_j \end{bmatrix}}_{x_i(x_i - x_j) + x_j(x_j - x_i)} = (x_i - x_j)^2$$

$$= (x_i - x_j)(x_i - x_j) = (x_i - x_j)^2$$

Define: $$\Omega_{ij} = \begin{array}{c} \begin{array}{cccc} & i & j & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array}$$

positive semi-definite

Note $[x_1, \ldots, x_n] \, \Omega_{ij} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_i - x_j)^2$

Note: $\Omega_{ij} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Also, Ω_{ij} is symmetric

Define $\Omega = \sum_{i<j} \omega_{ij} \Omega_{ij}$ = stress matrix associated to the stress ω.

$$[x_1, \dots, x_n] \, \Omega \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i<j} \omega_{ij}(x_i - x_j)^2$$

Note: $\Omega \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \Leftrightarrow$ row + columns of Ω are 0
(diagonal) entries determined by off-diagonal

ij^{th} entry of Ω is $-\omega_{ij}$
ij^{th} entry of Ω is 0 for nonmembers of G

The Kronecker product of 2 matrices A, B is
(or tensor product)

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & \\ a_{21}B & & & \\ \vdots & & & \\ & & & a_{mn}B \end{bmatrix}$$

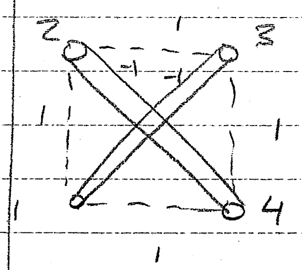
Claim: The matrix of E_ω is $\Omega \otimes I^d$.

$E_\omega \geq 0$ iff $\Omega \geq 0$.
positive semi-definite

column vectors
Let $P = [p_1, \dots, p_n]_{d \times m} \equiv$ configuration matrix

$$\hat{P} = \begin{bmatrix} p_1 & \dots & p_n \\ 1 & & 1 \end{bmatrix}$$

$$\hat{P}\,\Omega = \left[\dots, -\sum_j \omega_{ij}(p_j - p_i), \dots\right] = 0$$ iff p is in equilibrium with respect to ω



stress

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = \Omega \geq 0$$

positive semi-defin

If $G(p)$ is in equilibrium with respect to a stress ω, what are all other configurations q that are *also* in equilibrium with respect to ω?
Any q that is an affine image of p will have the same equilibrium stress. Are there any others?

We say a configuration is *universal* with respect to an equilibrium configuration if any other configuration q in equilibrium with respect to ω is an affine image of p.

Feb. 6

Joe

Jim Saxe: Global rigidity problem is strongly NP-complete

partition problem

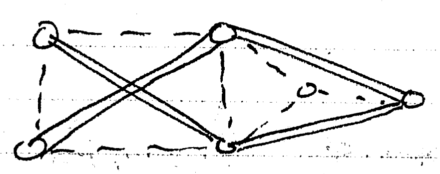
[redacted]

Principle of least work:

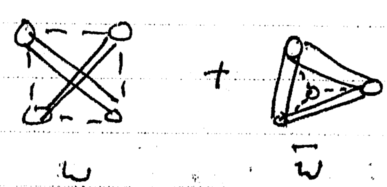
If E_ω is at a minimum for configuration p, then any other configuration q with $G(q) \leq G(p)$ is such that $E_\omega(q) = E_\omega(p)$ and all members of $G(p)$ and $G(q)$ have same length.

$$E_\omega(q) = \sum_{i<j} \omega_{ij} (q_i - q_j)^2$$

We say $G(p)$ is unyielding if when $G(q) \leq G(p)$ then $|q_i - q_j| = |p_i - p_j|$ for all members $\{i,j\}$ of G.



unyielding
not globally rigid
and not universal with respect to the proper stress $\omega + \bar{\omega}$



Let ω be a stress. Does there exist a universal configuration for ω?

Affine geometry:

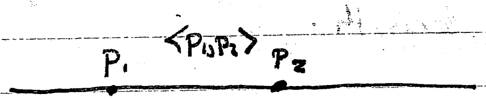
$p_i \in E^d$

$$\hat{p} = \begin{bmatrix} p_1 & \dots & p_n \\ 1 & \dots & 1 \end{bmatrix}$$ augmented configuration matrix

~~Fact: dimension of affine~~

The affine span of $p_1, \dots, p_n$ is

$$\langle p_1, \dots, p_n \rangle = \left\{ \sum_{i=1}^{n} \lambda_i p_i \;\middle|\; \sum_{i=1}^{n} \lambda_i = 1 \right\}$$



Factoid: = linear space + constant vector

$$\text{linspan}\left\{ \langle p_1, \dots, p_n \rangle + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} = \text{linear span of } \begin{bmatrix} p_1 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} p_n \\ 1 \end{bmatrix}$$

dimension of linear span of $\hat{p}_1, \dots, \hat{p}_n$ = column rank of $\hat{P}$

$$\hat{p}_1, \dots, \hat{p}_n \;=\; \begin{bmatrix} p_1 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} p_n \\ 1 \end{bmatrix}$$

Fact: $\operatorname{rank} \hat{P} = \dim\{\text{affine span of } p_1, \dots, p_n\} + 1$

Recall that if $p = (p_1, \dots, p_n)$ is in equilibrium with respect to a stress ω, then

$$\hat{P}\Omega = 0.$$

$$\hat{p} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & & \cdots & y_n \\ 1 & & \cdots & 1 \end{bmatrix}$$

$$[x_1, \ldots, x_n]\,\Omega = 0$$

$$\vdots$$

$$[1, \ldots, 1]\,\Omega = 0$$

So we can add rows to $\hat{p}$ to be sure that rows of $\hat{p}$ span Kernel of Ω.

If the rows of $\hat{p}$ do span the kernel of Ω then we claim that $(p_1, \ldots, p_n) = p$ is universal.
corresponding configuration

Suppose $\bar{p}$ is in equilibrium with respect to ω.
So $\hat{\bar{p}}\,\Omega = 0$ $\quad \hat{\bar{p}} = \begin{pmatrix} \bar{x}_1 & \cdots & \bar{x}_n \\ \bar{y}_1 & \cdots & \bar{y}_n \\ 1 & \cdots & 1 \end{pmatrix}$

$$[\bar{x}_1, \ldots, \bar{x}_n] = a_{11}[x_1 \ldots x_n] + a_{12}[y_1 \ldots y_n] + \ldots + b_1[1 \ldots 1]$$

$$[\bar{y}_1, \ldots, \bar{y}_n] = a_{21}[x_1 \ldots x_n] + a_{22}[y_1 \ldots y_n] + b_2[1 \ldots 1]$$

$$[1 \ldots 1] = 0\,[\quad] + \ldots + 1\,[1 \ldots 1]$$

Define $\begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & & b_2 \\ 0 & \cdots & & 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$

So $\bar{p} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \hat{p}$

$$= \begin{bmatrix} A_{d' \times d} & b_{d \times 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & \cdots & p_n \\ 1 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A p_1 + b & \cdots & A p_n + b \\ 1 & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{p}_1 & \cdots & \bar{p}_n \\ 1 & & 1 \end{bmatrix}$$

So $\bar{p}_i = A p_i + b$

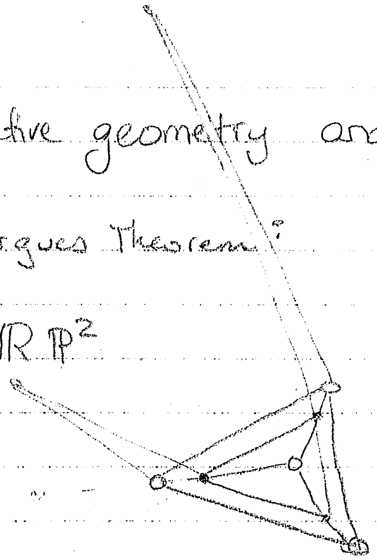
So the configuration $\bar{p}$ is an affine image of p. So p is universal if $\dim \langle p_1, \ldots, p_n \rangle$ is maximal.

Feb. 8

Projective geometry and homogeneous coordinates

Desargues Theorem:

$\mathbb{RP}^2$



4 points
no 3 in a line

3 points lie in same line

$\mathbb{F}$ = Field $\rightsquigarrow \mathbb{FP}^n$ = projective plane
homogeneous coordinates

we'll do this for $\mathbb{F} = \mathbb{R}$ – but you could do this for any field.

$\mathbb{R} \rightsquigarrow \mathbb{RP}^d$ = d-dimensional projective space.

Start in $\mathbb{R}^{d+1}$

A "point" in $\mathbb{RP}^d$ is a line through the origin in $\mathbb{R}^{d+1}$

$w, v \in \mathbb{R}^{d+1}$, $v \sim w$ if there is a scalar $c \neq 0$ s.t. $cv = w$.

A "line" in $\mathbb{RP}^d \Leftrightarrow$ plane through origin

$[v]$ "point" $[\ell]$ line if line through v is in the plane ℓ in $\mathbb{R}^{d+1}$

homogeneous coordinates

Let's look at $\mathbb{RP}^2$. $= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$

If $z \neq 0$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$

If $z = 0$ $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \leftrightarrow$ points at ∞

Conics in $\mathbb{RP}^2$

in affine coordinate in $\mathbb{P}^2$

Equation for conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

so conics determined by 6 constants up to a multiple.

In homogeneous coordinates this becomes.

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0, \quad z = 1.$$

Think quadratic forms.

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$v^T X v = 0$

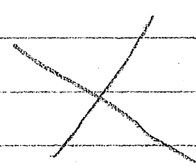
X = symmetric matrix

Interesting X's are those that are <u>indefinite</u> ← eigenvalues: at least one negative + one positive

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}$$ circle at $z = 1$

semi-definite $\leftrightarrow$ lines

(positive) or (negative)



$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \text{ - a point.}$$

What determines a conic in $\mathbb{RP}^2$? $\mathbb{RP}^1$?

In $\mathbb{RP}^1$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to \begin{matrix} x^2-1=0 \\ x=\pm 1 \end{matrix} \qquad \text{two points}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad x=0 \qquad \text{one point}$$

also whole space + empty space.
So 2 points determine conic in $\mathbb{RP}^1$ (not counting whole space).

In $\mathbb{RP}^2$

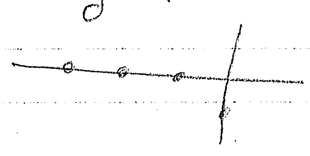
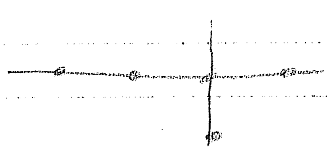
Suppose $p_1, \dots, p_5$ are 5 points in $\mathbb{RP}^2$. Is there a proper conic through these 5 points?
We can solve

$$p_i^T X_{3\times 3} p_i = 0 \quad \text{for the coefficients of } X.$$

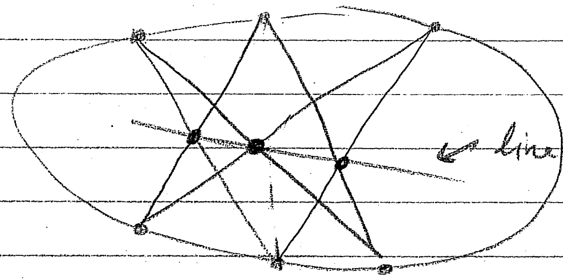
unknowns are the entries of X.
These are 5 linear equations in 6 unknowns.

There always is a nonzero solution.

If the 5 points are such that they are all distinct, no 3 lie on a line, then there is a unique X up to scaling that solves $p_i^T X p_i = 0$.

 or 

Pascal's Theorem



Tensegrities:

Let $p=(p_1,\dots,p_n)$ be a configuration in $\mathbb{E}^d$. What are those configurations $q=(q_1,\dots,q_n)$ such that q is an affine image of p and $G(q)\sim G(p)$. (i.e. member lengths of $G(p)$ + $G(q)$ are the same. for $\{i,j\}$ member of G, $|p_i-p_j|=|q_i-q_j|$.)

$q_i = Ap_i+b$ $\qquad$ A is $d\times d$

So $|p_i-p_j| = |Ap_i+b-Ap_j+b|$

$= |A(p_i-p_j)|$

So $(p_i-p_j)^2 = [A(p_i-p_j)]^2$ (dot product)

$(p_i-p_j)^T(p_i-p_j) = [A(p_i-p_j)]^T A(p_i-p_j)$

$= (p_i-p_j)^T A^TA(p_i-p_j)$

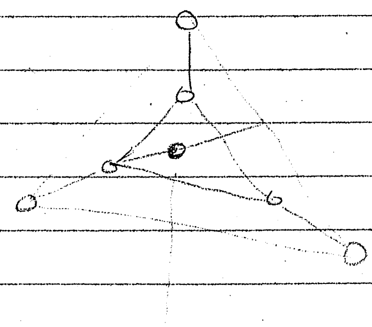
$$(p_i-p_j)^T\underbrace{[I-A^TA]}_{\text{quadratic form}}(p_i-p_j) = 0$$

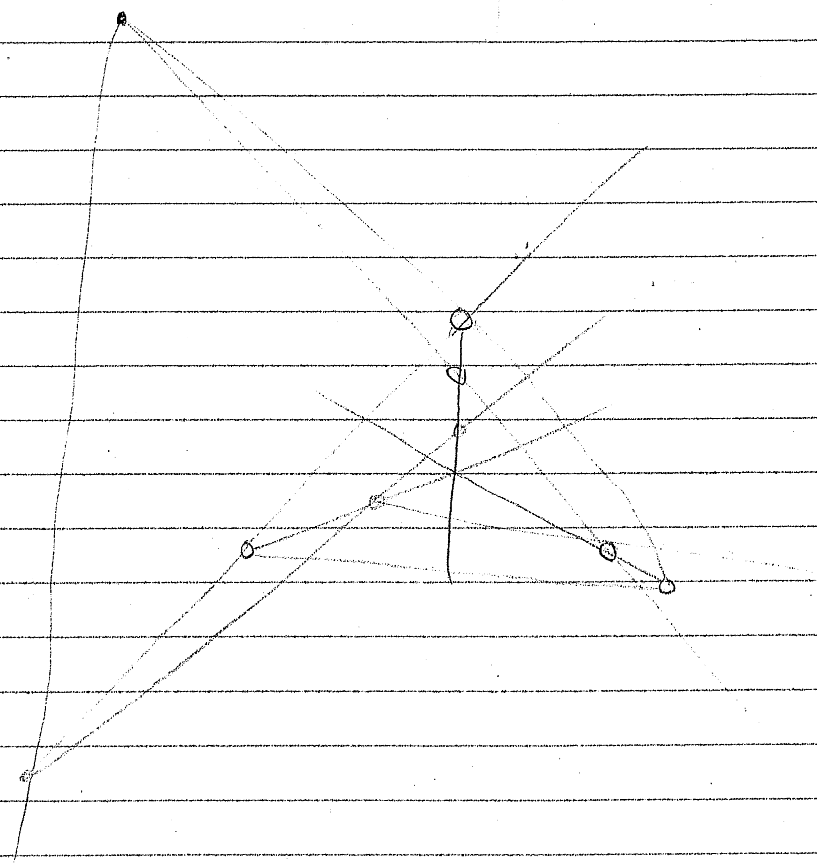
So these "directions" $\{(p_i-p_j)\}$ lie on a conic $\Sigma = I-A^TA$

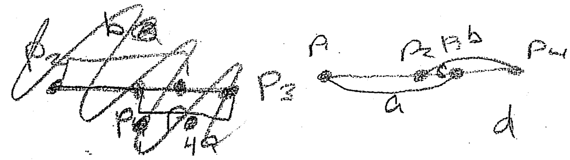
Conversely, if they lie on a conic Σ, then $(p_i-p_j)^T\Sigma(p_i-p_j)=0$.

Look at $I - \varepsilon X$. For ε small enough, $I - \varepsilon I$ is positive definite. (follows from exercise.)

to take square root of positive definite matrix, diagonalize + take square roots.







$a+b=c+d$

$a-c+b-d=0$

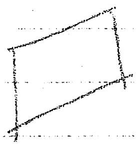
~~[illegible]~~ Feb 13

Let $v_1, \ldots, v_k \in \mathbb{E}^d$, all nonzero. We say they lie on a <u>conic at ∞</u> if there is a symmetric nonzero matrix X such that $v_i^T X v_i = 0$ for $i = 1, \ldots, k$.
($[v_i]$ are homogeneous coordinates for points in $\mathbb{R}P^{d-1}$.)

<u>Lemma</u>: There is an affine/ noncongruence map $p \mapsto Ap$, $p \in \mathbb{E}^d$ such that $|v_i| = |Av_i|$ iff $v_1, \ldots, v_k$ lie on a conic at ∞.
<u>Proof</u>: If such an A = linear map ~ matrix exists, then $\bar{X} = I - A^T A$ provides the conic. Suppose $v_i^T \bar{X} v_i = 0$, $\bar{X} \neq 0$, symmetric. Then there is an $\varepsilon > 0$ such that $I - \varepsilon \bar{X}$ is positive definite.
By the spectral theorem, there is an orthogonal matrix P such that $P(I - \varepsilon X)P^T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}$, $\lambda_i > 0$

Define $A = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \; \sqrt{\lambda_d} \end{bmatrix} P$

Then $A^T A = P^T \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_d} \end{bmatrix} P = I - \varepsilon \bar{X}$



So $|Av_i|^2 = v_i^T A^T A v_i = v_i^T v_i = |v_i|^2$ $\square$

<u>Fundamental theorem of Stressed Tensegrities</u>:

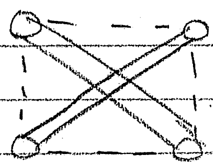
Let $G(p)$ be a tensegrity in $\mathbb{E}^d$ with a (non-zero) proper equilibrium stress ω (no pinned vertices).
Suppose
(i) The configuration p is universal with respect to ω (rank Ω $= n-d-1$, Ω = stress matrix)

If satisfies conditions of theorem: Super stable.

(ii) The stress matrix Ω is positive semidefinite.
(iii) The "stressed" matrix directions do not lie at a conic at ∞.
Then $G(p)$ is globally rigid in any $\mathbb{E}^{d'} \supset \mathbb{E}^d$.
(say $G(p)$ is universally globally rigid)
<u>Proof</u>:
Suppose $G(q) \leq G(p)$.
(ii) and the principle of least work implies that every stress member $\{i,j\}$ (i.e. $\omega_{ij} \neq 0$) is such that
$|q_i - q_j| = |p_i - p_j|$, and
$E_\omega(q) = E_\omega(p) = 0$.
Then $G(q)$ is also in equilibrium with respect to ω.

But by (i) p is universal, so q is an affine image of p.
By (iii) that affine image is a congruence, i.e. $p \sim q$. $\square$

Example:



$\Omega_{4\times4}$ has 3-dim kernel

$$\sim \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & & 0 & \\ 0 & & & \lambda \end{pmatrix}$$

So we need $\lambda \geq 0$ for matrix to be positive semi-definite

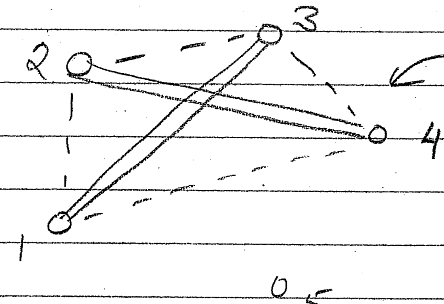
Key here is we have $d+2$ vertices.

Feb. 15

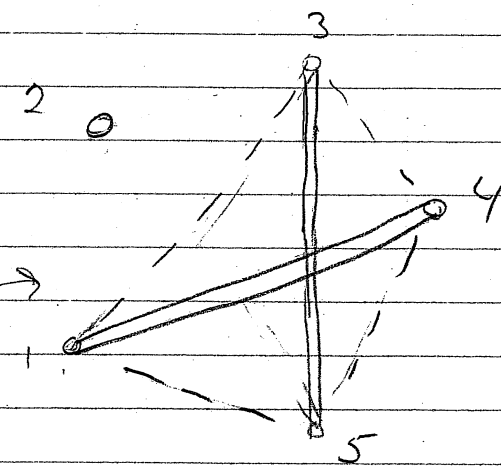
Example Cauchy polygons

Definition: A Cauchy polygon $C_n(p_1, \ldots, p_n)$ is a tensegrity where $p_1, \ldots, p_n$ are vertices of a (strictly) convex planar polygon in order, and $\{i, i+1\}$ $i = 1, \ldots, n \pmod n$ are cables in C_n and $\{i, i+1\}$ $i = 1, \ldots, n-2$ are the struts.

Proposition Cauchy polygons are super stable.
Proof: Assume the base case $n = 4$.
(Finish this in a bit)
Use induction on n.
Show the idea for $n = 5$ assuming $n = 4$.



has stress matrix Ω_1



has stress matrix Ω_2

So in $\Omega = \Omega_1 + \Omega_2$, $\omega_{14} = 0$

What is sign of ω_{13} in Ω

In order to have equilibrium

$\Omega = \Omega_1 + \Omega_2$

$\Omega_1 \geq 0 \quad \Omega_2 \geq 0$

* Need to show: $\Omega \geq 0$ sum of 2 positive-semi-definite so $\Omega \geq 0$.

* Need to show: $\omega \leftrightarrow \Omega$ is <u>universal</u>, i.e. $\operatorname{rank} \Omega = 5-(2+1) = 2$.

Suppose q is universal for ω.

$(\Omega \otimes I^d) q = 0 \Leftrightarrow q^T (\Omega \otimes I^d) q = 0$

$\Leftrightarrow q^T (\Omega_1 \otimes I^d) q = 0 = q^T (\Omega_2 \otimes I^d) q$

so q is in equilibrium with respect to $\Omega_1 + \Omega_2$

So $\underset{\text{affine dim}}{\dim} \langle q_1, \dots, q_4 \rangle = 2 = \dim \langle q_2, q_3, q_4, q_5 \rangle$

The spans overlap in q_1, q_2, q_3, q_4.

So the whole span must be 2-dimensional.

So q is universal

So dim affine span of q is 2, so p must also be universal.

The number of stressed cable + strut directions is $\geq 3 > 2$.

So $C_5(p)$ is superstable

$\rightarrow$ same argument for $C_n(p)$ from $C_{n-1}(p)$.

<u>Invariance</u>:

Let Ω be a stress matrix for an equilibrium stress for a tensegrity $G(p)$.

A projective transformation of $\mathbb{RP}^d$ is given by

$$\begin{bmatrix} p_i \\ 1 \end{bmatrix} \rightarrow A \begin{bmatrix} p_i \\ 1 \end{bmatrix} = \begin{bmatrix} q_i \\ \lambda_i \end{bmatrix}$$ in homogeneous coordinates for $\mathbb{RP}^d$.

In affine coordinates $p_i \mapsto q_i / \lambda_i \quad (\lambda_i \neq 0)$.

Equilibrium : $\hat{p}\,\Omega = 0 = \begin{bmatrix} p_1 & \cdots & p_n \\ 1 & & 1 \end{bmatrix} \Omega$

$A = (d+1) \times (d+1)$ matrix

$A\hat{P} = \left[A\begin{bmatrix} p_1 \\ 1 \end{bmatrix}, \ldots, A\begin{bmatrix} p_n \\ 1 \end{bmatrix} \right] = \begin{bmatrix} q_1 & \cdots & q_n \\ \lambda_1 & & \lambda_n \end{bmatrix}$

$\hat{Q} = \begin{bmatrix} q_1/\lambda_1 & \cdots & q_n/\lambda_n \\ 1 & & 1 \end{bmatrix}$

Let $D_{n\times n} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} =$ $n \times n$ diagonal matrix

$A\hat{P}D^{-1}D\Omega D = A\underbrace{\hat{P}\Omega}_{=0} D = 0$

Call $\Omega' = D\Omega D$ = new stress matrix

$A\hat{P}D^{-1} = \hat{Q}$ = new configuration

signature of Ω' is the same as Ω
+'s and −'s and 0's

$\lambda_i \neq 0$

The i,j entry of Ω' is $\lambda_i\lambda_j(-\omega_{ij})$
where $(-\omega_{ij}) = ij$ entry of Ω.

So the sign of the new stress is $\pm$ sign of ω_{ij} depending on sign of $\lambda_i\lambda_j$

Feb. 20

homework:

problem ~~[illegible]~~ 3:

$|p_i - p_{i+1}| = |p_{i-1} - p_i| \quad \forall i$

indices taken mod 5

From last week's homework problem 2a

flexible	rigid	flexible
rigid	flexible	rigid
flexible	rigid	flexible

boundaries are rigid except for ~~[illegible]~~ vertices which are rigid

---

Fact: Let ~~[illegible]~~ $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \dots \lambda_n(\Omega)$ ~~[illegible]~~ be the real eigenvalues of a symmetric $n \times n$ matrix Ω. Then each λ_i is a continuous function of the entries of Ω.

Proof outline #1:

$$\lambda_i = \min_{L^i} \max_{i} \left\{ \frac{v^T \Omega v}{v^T v} \;\middle|\; v \in L^i \right\}$$

L^i = i-dim linear subspace

$$\{ v^T \Omega v \mid v \in S^{i-1} \}$$

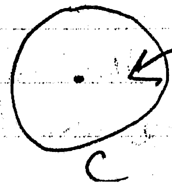
unit sphere

$\square$

Proof outline #2:

The complex roots of a polynomial $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$ counted with multiplicities vary continuously with $a_{n-1}, \dots, a_0$

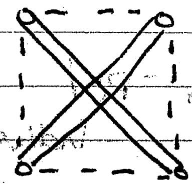
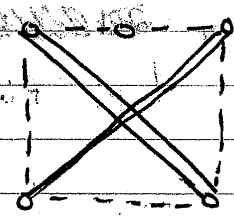
Apply this to characteristic polynomial of Ω.



$$\#\text{roots in } C = \frac{1}{2\pi i} \int_C \frac{p'(z)}{p(z)}\, dz$$

$\square$

Application

 $\rightarrow$ 

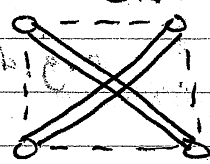
$0 \le \Omega_{4\times4}$ $\qquad\qquad$ $\Omega_{5\times5}$

$\Omega_{4\times4}$ has one positive eigenvalue, 4.

↖ positive semi-definite

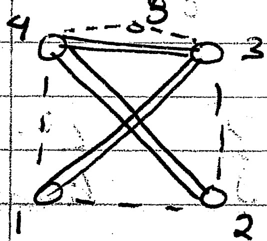
Wants to know $\Omega_{5\times5} \ge 0$ since $\Omega_{4\times4} \ge 0$.

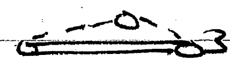
$$0 \le \begin{bmatrix} \Omega_{4\times4} & 0 \\ 0 & 0 \end{bmatrix}$$



this vertex sits in 3-sp

$$\sim \begin{bmatrix} 4 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}$$





$$\begin{bmatrix} \Omega_{4\times4} & 0 \\ 0 & 0 \end{bmatrix} + \cancel{\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{bmatrix}$$

Ω_2 $\qquad$ ↖ has rank 1

Call $\begin{pmatrix} \Omega_{4\times4} & 0 \\ 0 & 0 \end{pmatrix} = \Omega_1$

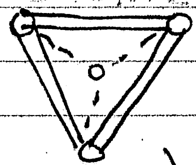
Sum ≥ 0 since it is sum of 2 positive semi-def matrices.

$\operatorname{rank}(\Omega_1 + \Omega_2) = 2$

$3 = \dim\ker(\Omega_1 + \Omega_2) = \dim\ker \Omega_1$

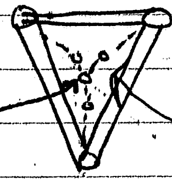
$\Omega_1 + t\,\Omega_2 \qquad 0 \le t \le 1$ $\qquad$ let's say that at $\Omega_1 + \Omega_2$ we have cancellation

Example:



$\Omega \geq 0$

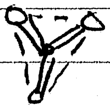
rank $\Omega = 1$ dimker $\Omega = 3$



$\Omega \geq 0$

dim ker $\Omega = 3$

by same argument as in last example

Ω'  Add this tensegrity in there

rank $\Omega' = 1$

not positive semi-definite!

instead negative semi-definite

$\Omega' \leq 0$

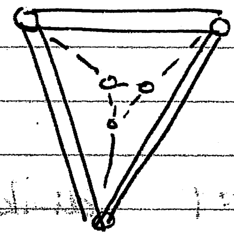
(Any thing with affine span plane must have at least 3 eigenvalues)

look at $\Omega + t\Omega'$

has 3 eigenvalues $= 0$.

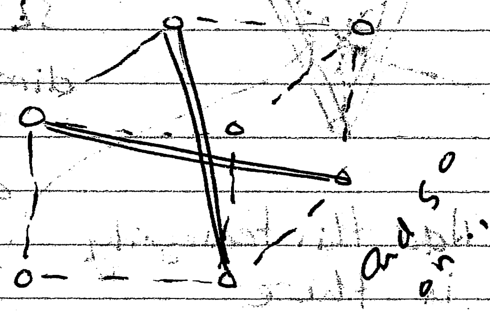
At $t = t_0$ the stresses on cancel, because the three vectors have only the one dependency.

dim ker$(\Omega + t\Omega')$ stays constant until $t = t_0$. Then the central vertex ~~must have~~ is not connected to any of the other vertices and the universal configuration is 3-dimensional. The dimension of the ker $\Omega + t\Omega')= 4$.



is super-stable

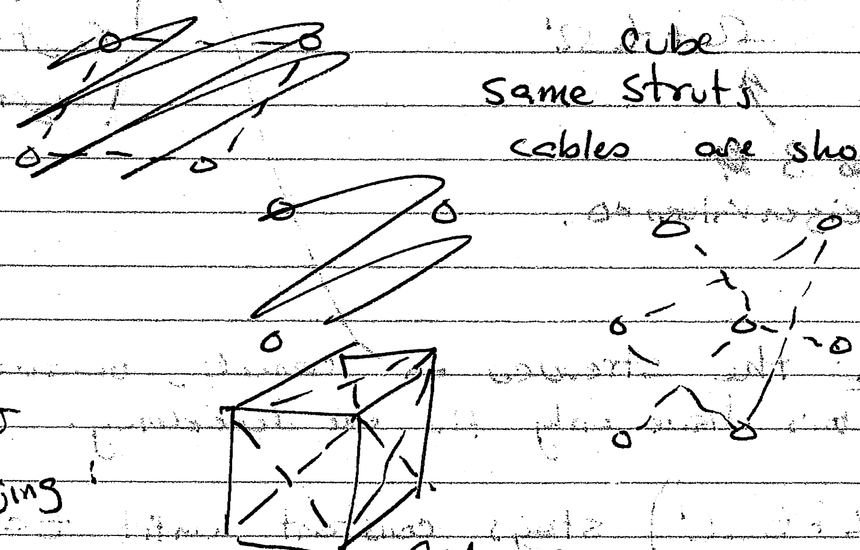
Example:



cube

struts are long diagonals

stress matrix Ω is positive semi-definite
rank $= 8 - 4 = 4$



cube
same struts
cables are short diagonals

and so on...

better drawing:

and so on...

(lines are not actually there)

interesting: cables are not connected

Claim: Ω_1 must have a negative eigenvalue.

this is, because we could take the 2 non-connected sets of cable + make them far apart.

So it is not globally rigid.

This is because cable subgraph is not connected. (whole graph is connected through struts + cables).

So if cable subgraph is not connected then there is a negative eigenvalue.

So $(1-t)\Omega_1 + t\Omega$ for small t must have a negative eigenvalue.

$$(1-t)\Omega_1 + t\Omega = \Omega_t$$

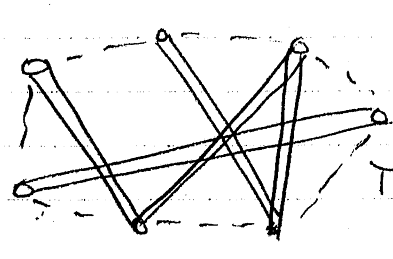
for $t=0$ Ω_t has negative eigenvalue
for $t=1$ it is positive semi-definite.

So there must be a critical t_0 where $\Omega_{t_0} \geq 0$ but $\dim \ker \Omega_{t_0} = 5$. (universal configuration at least 4 dimensions).

Problem: Find t_0. Describe configuration. for next week.

Grünbaum 1970's Lectures in Lost Mathematics

Convex polygon



~~Struts~~ cables on edges
Struts on inside.
There is a proper, non-zero, stress.

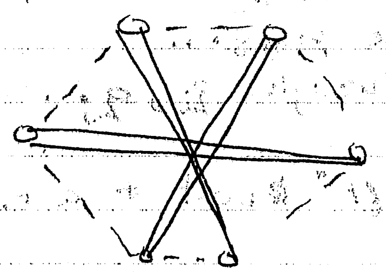
Theorem: These cable - strut (c-s) polygons are all super - stable.

next time : brief proof.

Feb 22

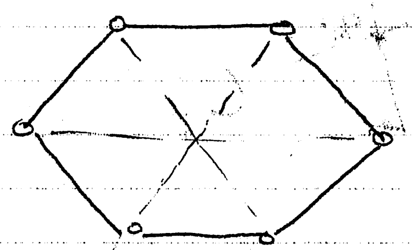
Def: A tensegrity $G(p)$ is a c-s (cable-strut) polygon if the vertices of $p = (p_1, \dots, p_n)$ form a strictly convex polygon in a plane, and the edges are cables, and the struts are the only other member.

Ex: Cauchy polygon.



globally rigid

regular b-c polygon



not globally rigid
(not even rigid)
See homework problem

Lemma: Let $G(p)$ be a c-s polygon tensegrity with proper non-zero equilibrium stress ω. Then p is universal for the stress ω. (i.e. Ω has kernel whose dimension is exactly 3).

Proof:

Let $q = (q_1, \dots, q_n)$ be any configuration in $\mathbb{E}^3$ that projects onto p and has ω as an equilibrium stress. Want to show $\dim \langle q_1, \dots, q_n \rangle = 2$.
affine span

Let Q = Convex hull of $q_1, \dots, q_n$.

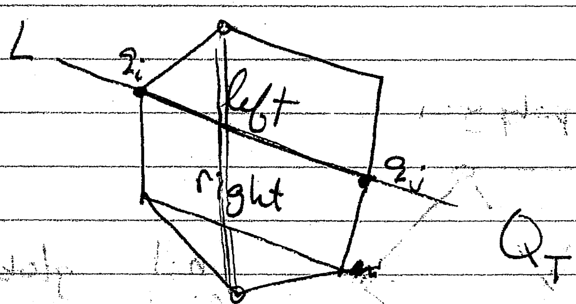
Let Q_T = top part of Q = facets of Q visible from $z = +\infty$
Q_B = bottom " " " " " " " $-\infty$

$Q_T \cap Q_B$ = lift of boundary of polygon.

If $\dim \langle q_1, \ldots, q_n \rangle = 3$ we will find a contradiction to q being in equilibrium with respect to ω.

So there must be an edge $\langle q_i, q_j \rangle$ on Q_T, q_i, q_j not an edge of the base polygon.
Let L be a line in $\mathbb{E}^3$ through q_i, q_j.

L seperates Q_T into a "left" part + a right part.



Let R_Θ be a rotation by Θ (radians) about the line L in $\mathbb{E}^3$.
Define $q_i(\Theta) = \begin{cases} q_i & \text{if } i \text{ is part of left half} \\ R_\Theta(q_i) & \text{if } i \text{ is part of right half} \end{cases}$

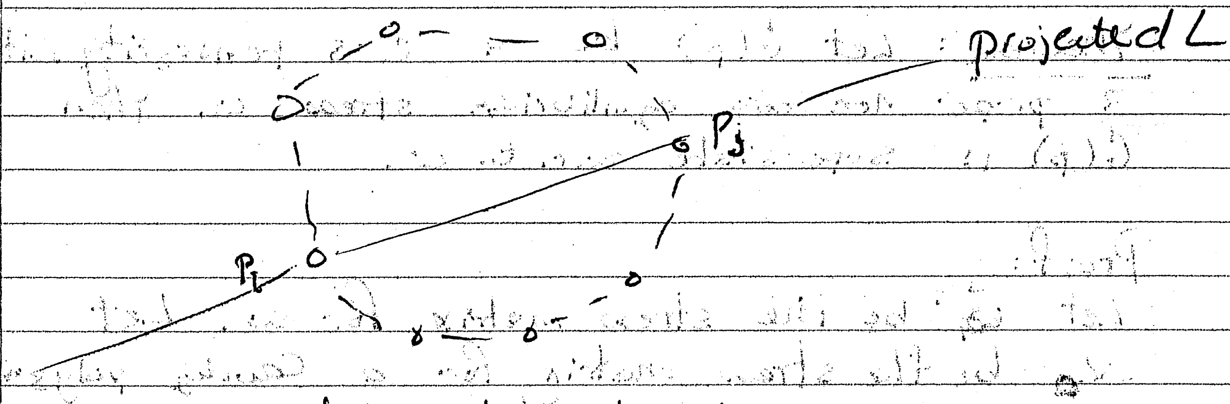
So $|q_i(\Theta) - q_j(\Theta)|^2$ is constant if $i + j$ are in the same half. But $\frac{d}{d\Theta}|q_i - q_j(\Theta)| > 0$ if i, j are in seperate halfs. at $\Theta = 0$.

So if i,j is a strut i,j in different parts, and $w_{i,j} < 0$ then $\frac{d}{d\Theta} E_\omega(q(\Theta)) < 0$.

So this contradicts that q is a critical point for E_ω and that q is in equilibrium with respect to ω.

But what if there is no struts going from one side to the other! ~~then we could just fold it over~~

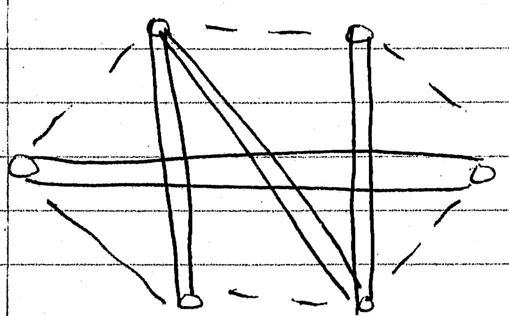
So we are left with the case where there are no struts crossing from the left to the right.



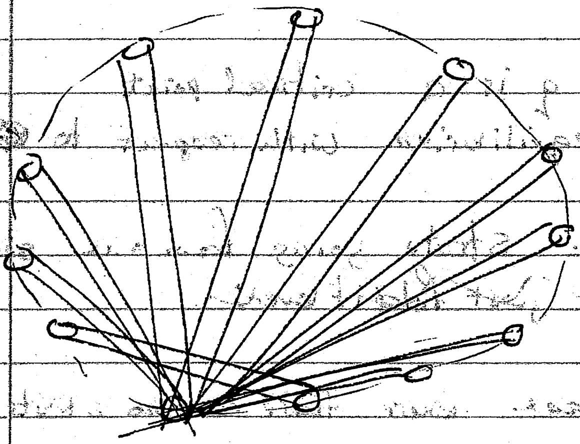
no struts intersecting L.

Even in $\mathbb{E}^2$, this cannot be in equilibrium. not even rigid!

Find a motion that opens each part as "arms," fixing all cable lengths and increasing all strut lengths in their first derivatives again $\frac{d}{dt} E_\omega(p(t)) < 0$. $\square$



(23)



always a proper stress

Grunbaum

<u>Theorem</u>: Let $G(p)$ be a c-s tensegrity with a proper non-zero equilibrium stress ω. Then $G(p)$ is superstable w.r.t. ω.

Proof:
Let Ω_1 be the stress matrix for ω. Let Ω_0 be the stress matrix for a Cauchy polygon on $p = (p_1, \ldots, p_n)$. Then consider
$\Omega_t = t\,\Omega_1 + (1-t)\,\Omega_0$.
By the lemma, the ~~kernel~~ dimension of the kernel of $\Omega_t = 3$, a constant. But Ω_0 is positive-semi-definite. $\dim\ker\Omega_0 = 3$. By continuity of eigenvalues $\Omega_1 \geq 0$ and the dimension of the kernel of $\Omega_1 = 3$.
So there are easily 3 stressed directions
$\Rightarrow G(p)$ is superstable. $\square$