Feb. 27

Read Graver, Seuvatias^2 (sp?)
Chapter 1 to 2.4

local / infinitesimal theory
mostly bar frameworks — what we were doing
first order theory — before: global theory
stress

Trivialities

Let $p = (p_1, \ldots, p_n)$ p_i in $\mathbb{E}^d$, a configuration

Def: A congruence of $\mathbb{E}^d$ is a transformation $T: \mathbb{E}^d \to \mathbb{E}^d$ where $T(p_i) = Ap_i + b$
where A = $n \times n$ orthogonal matrix
and b = vector in $\mathbb{E}^d$.

orthogonal means $A^T A = AA^T = I_{d\times d}$

$$|Tp_i - Tp_j|^2 = |Ap_i + b - Ap_i - b|^2 = |Ap_i - Ap_j|^2$$

$$= |A(p_i - p_j)|^2 = (p_i - p_j)^T A^T A (p_i - p_j)$$

$$= (p_i - p_j)^T (p_i - p_j)$$

$$= |p_i - p_j|^2$$

So distances are preserved.

Fact: If $T: \mathbb{E}^d \to \mathbb{E}^d$ preserves distances, then it is a congruence.
proof: an exercise.

Let $G(p)$ be a (tensegrity) framework. (can think of this as a homotopy.)

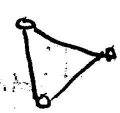
<u>Def</u>: A path $p(t) = (p_1(t), \dots, p_n(t))$, $0 \le t \le 1$ where $p(0) = p$ is called a <u>flex</u> of $G(p)$ if $G(p(t)) \le G(p)$.

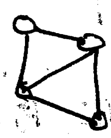
<u>Def</u>: If $p(t)$ is a flex, we say it is <u>trivial</u> if there is a family $T_t : \mathbb{E}^d \to \mathbb{E}^d$ where $T_0 = \text{identity}$, T_t is a congruence, and $T_t(p_i) = p_i(t)$ $\forall\ 0 \le t \le 1$. (normally think of T_t being continuous).

<u>Def</u>: We say $G(p)$ is <u>rigid</u> if every continuous flex $p(t)$ of $G(p)$ is trivial.

engineer would say this is not a finite mechanism

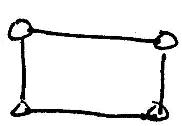
<u>Examples</u>:

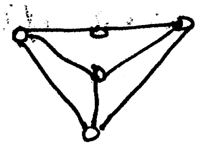

rigid


rigid in $\mathbb{R}^2$
(not rigid in $\mathbb{R}^3$)

should be using $\mathbb{E}^2$ or $\mathbb{E}^3$ (oh well...)


not rigid in $\mathbb{R}^2$


not rigid in $\mathbb{R}^2$


rigid in $\mathbb{R}^2$
(although engineer would not call it rigid).

Theorem: Let $G(p)$ be a tensegrity framework in $\mathbb{E}^d$. Then the following are equivalent.
(i) Every continuous flex $p(t)$ is trivial.
(ii) Every analytic flex of $p(t)$ is trivial.
(iii) There is a $\delta > 0$ such that if $|q-p| < \delta$ and $G(q) \leq G(p)$ then q is congruent to p. ($Tp_i = q_i$. T = congruence)

Proof (Hint): The member constraints

$$\underbrace{|q_i - q_j|^2}_{\text{variable}} \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} \underbrace{|p_i - p_j|^2}_{\text{fixed}}$$

polynomial inequalities ~~def~~ + equalities defined on the configuration space $\mathbb{E}^{nd}$ (the space of configurations.)

Such a finite set defined by polynomial inequalities + equalities is called a semi-algebraic set. (An algebraic set would be one defined by only equalities).

→ In Milner's Singular Points on Complex Hypersurfaces it is shown that in the neighborhood of any point p in X, a semi-algebraic set, there is an analytic fibration. (?) This provide the analytic parametrization.

End of hint for proof.

If $p(t)$ is an analytic flex of $G(p)$ then we can can take the derivative.

So $$|p_i(t) - p_j(t)|^2 \begin{Bmatrix} = \\ \le \\ \ge \end{Bmatrix} |p_i - p_j|^2 \qquad \{i,j\} = \begin{Bmatrix} \text{bar} \\ \text{cable} \\ \text{strut} \end{Bmatrix}.$$

Differentiate, at $t=0$

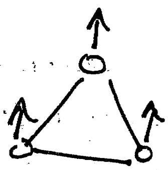
$$2\big(p_i(t) - p_j(t)\big)\cdot\big(p_i'(t) - p_j'(t)\big) \begin{Bmatrix} \le \\ = \\ \ge \end{Bmatrix} 0 \qquad \{i,j\} = \begin{Bmatrix} \text{cable} \\ \text{bar} \\ \text{strut} \end{Bmatrix}$$

where $p_i(t) = \frac{d}{dt} p_i(t)$ at $t=0$.

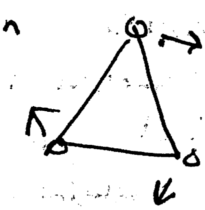
$$(p_i - p_j)\cdot(p_i' - p_j') \begin{Bmatrix} \le \\ = \\ \ge \end{Bmatrix} 0.$$

Any p' where holds + $p' = (p_1', \dots, p_n')$, is called a infinitesimal flex of $G(p)$.
So this is a little vector field associated with $G(p)$.

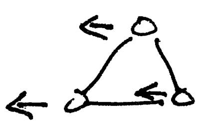
translation



rotation



or translate a different direction



 2 dimensional ~~flex~~ flexes

(2 dimensions of possible flexes)

Both are rigid, only trivial motions.

But  this object had 3 dimensions of flexes

(2 for translations + 1 for rotations.)

The reason these don't have the same number of dimensions is that the point $\bullet$ has too small a dimension (affine span not $\mathbb{E}^2$.)

More Trivialities:

Let $p = (p_1, \ldots, p_n)$ be any configuration.

Def: Then a <u>trivial</u> <u>infinitesimal</u> <u>flex</u> of $G(p)$ is the time 0 derivative of an analytic congruence T_t where $T_0 = I$.

Remember $T_t(p_i) = A_t p_i + b_t$ where $A_t^T A = I$.

Trivial flex

$$dA_t p_i + db_0 = 0$$

this is Chapter -1 of Lie theory.

If A_t is an analytic family of orthogonal matrices, $A_t^T A_t = I$.

$$0 = \frac{d}{dt}\left(A_t^T A_t\right) = \left(\frac{d}{dt} A_t^T\right) A_t + A_t^T \frac{dA_t}{dt}$$

Let $\left(\frac{d}{dt} A_t\right)\Big|_{t=0} = S$

~~Then~~

At $t=0$,

$$0 = S^T + S \qquad S^T = -S$$

So S is a skew-symmetric matrix.

So for p' to be a trivial infinitesimal flex, we have that

$$p_i' = S p_i + b'$$

b': d-dimensional space

Constant corresponds to infinitesimal translation

Skew symmetric corresponds to infinitesimal rotation

$\frac{d^2-d}{2} = \frac{d(d-1)}{2}$ dimensional space

Total dimension of trivial infinitesimal flexes is $\frac{d^2+d}{2} = \frac{d(d+1)}{2} = \binom{d+1}{2}$

(assuming our object does not have too low of dimension such as the point • in $\mathbb{E}^2$).

For $d=3$ we can write

$$S p_i = s \times p_i$$

s: vector

$\times$: ordinary cross product

Can use same formula in plane, if you like.

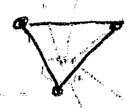
$$S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

$$s = \begin{pmatrix} -c \\ b \\ a \end{pmatrix}$$

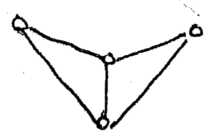
March 1

<u>Def</u>: A (tensegrity) framework $G(p)$ is <u>infinitesimally rigid</u> if every infinitesimal flex p' of $G(p)$ is trivial.

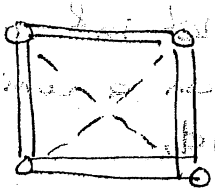
<u>Examples</u>:



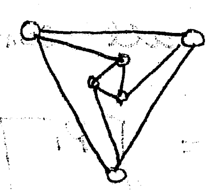
inf. rigid in $\mathbb{E}^2$



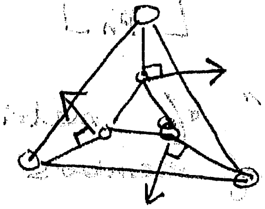
inf. rigid in $\mathbb{E}^2$



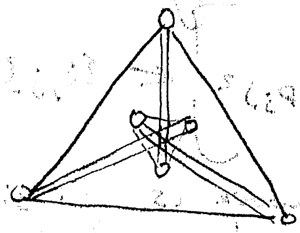
inf. rigid in $\mathbb{E}^2$
not globally rigid.



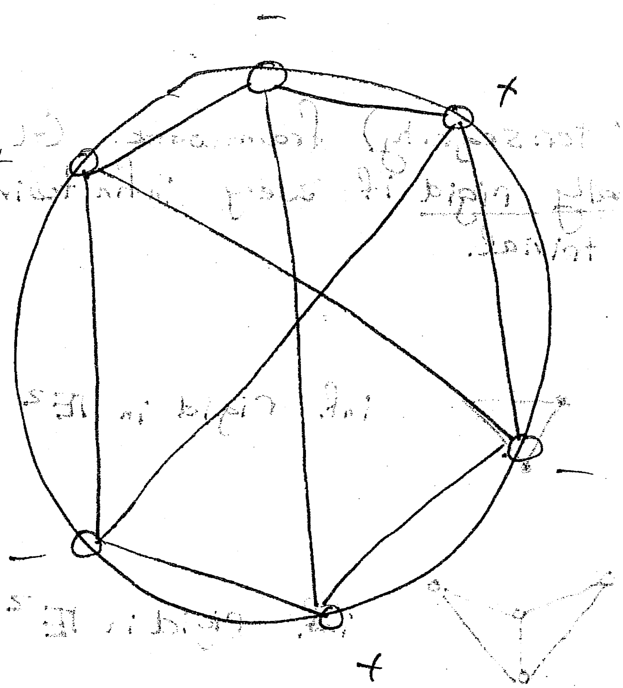
inf. rigid (we'll prove later)
no stress



~~inf. ri~~
not inf. rigid!
despite being rigid.



not inf rigid,
not globally rigid
just take ~~struts~~ inner
△ somewhere else
rigid.



Connect all +'s to all -'s

$K_{3,3}(p)$

not inf. rigid.

But if it doesn't lie on a conic, then it would be inf. rigid.

Let $\mathbb{E}^{nd}$ be the space of all configurations of n points in $\mathbb{E}^d$.

$$p = \cancel{(p_1, \dots, p_n)} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

Let $\mathbb{E}^e$ be the "space" of metrics for a graph G where e = # of members of G.

Define $f: \mathbb{E}^{nd} \to \mathbb{E}^e$ by

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \mapsto \begin{bmatrix} \vdots \\ (p_i - p_j)^2 \\ \vdots \end{bmatrix} \leftarrow \{i,j\} \text{ member of } G$$

So $G(p)$ a bar framework is rigid if

$$f^{-1}f(p) \cap \mathcal{U}_p = \{q \mid q \sim p\} \cap \mathcal{U}_p$$

where $\mathcal{U}_p$ = nbhd of p in $\mathbb{E}^{dn}$

We compute df_p the differential of f at p.

Let $p' \in \mathbb{E}^{nd}$ (Think infinitesimal flex.)

Compute $\left.\frac{d}{dt} f(p+tp')\right|_{t=0} \equiv df_p(p')$

$$p(t) = p + tp'$$

$$(p_i(t) - p_j(t))^2 = (p_i + tp_i' - p_j - tp_j')^2$$

$$= [(p_i - p_j) + t(p_i' - p_j')]^2$$

$$= (p_i - p_j)^2 + 2t(p_i - p_j)\cdot(p_i' - p_j') + t^2(p_i' - p_j')^2$$

$$df_p(p') = 2\begin{bmatrix} \vdots \\ (p_i - p_j)\cdot(p_i' - p_j') \\ \vdots \end{bmatrix}$$

In terms of standard coordinates of $\mathbb{E}^{nd}$, the matrix of f $[\frac{1}{2} df_p]$ is:

$$R(p) = \begin{array}{c} \begin{bmatrix} \\ 0 \ \cdots \ (p_i - p_j)^T \ 0 \cdots 0 \ (p_j - p_i)^T \ 0 \ \cdots \ 0 \\ \\ \end{bmatrix} \\ \qquad\qquad i \qquad\qquad\qquad\qquad j \end{array}$$

$$2R(p)p' = \begin{bmatrix} \vdots \\ (p_i - p_j)\cdot(p_i' - p_j') \\ \vdots \end{bmatrix}$$

Define $R(p)$ $\equiv$ rigidity matrix

Note for a bar framework, $G(p)$ is infinitesimally rigid iff the kernel of $R(p)$ consists only of trivial infinitesimal flexes.

What is the dimension of trivial infinitesimal flexes of a configuration $(p_1, \ldots, p_n)$?

$\dim \left(S = \text{skew symmetric}, b' = \text{translations}\right) = \dfrac{d(d+1)}{2}$

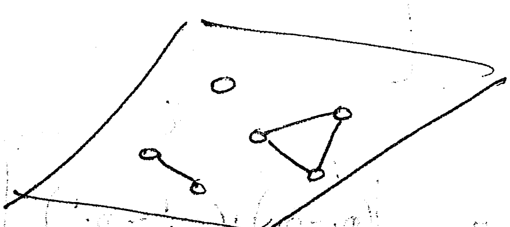
But problem if dimension is not same as space you are in. ex: • point.

Skew matrices $\times$ translations
$\rightarrow$ trivial flexes on a given configuration $\subseteq$ inf flexes of $G(p)$

Claim: This map is ~~onto~~ one-to-one (has kernel 0) when $\dim \langle p \rangle = d$ (or $d-1$)

What happens to inf. rigidity when $\langle p \rangle \neq \mathbb{E}^d$?

Ex: $d = 3$



have something in plane.

only examples inf. rigid in $\mathbb{E}^3$.

<u>Theorem</u>: If $G(p)$ is a bar framework, $\langle p \rangle = \mathbb{E}^d$, then $G(p)$ is infinitesimally rigid iff the dimension of the kernel of $R(p)$ is equal to $\frac{d \cdot (d+1)}{2}$.

<u>Corollary</u>: Rank $R(p) = nd - \frac{d(d+1)}{2}$

So if $G(p)$ is inf. rigid (not a simplex) then $e \geq nd - \frac{d(d+1)}{2}$

$d = 2 \qquad e \geq 2n-3$

$d = 3 \qquad e \geq 3n-6$

(inf. rigidity should feel rigid)

n vertices all Δ's
$3n-6$ edges
$e \geq 3n-6$

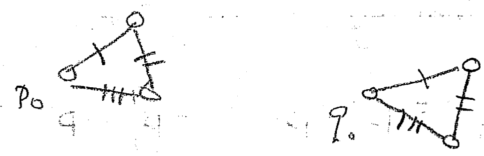
If you remove any edge, there is a nontrivial flex,

March 6

Hint for #2:

$f: \mathbb{E}^d \to \mathbb{E}^d$

$f(0)=0$



$p_0 = q_0 = 0$ $\quad$ $p_i \to q_i$

Infinitesimal rigidity

How do you detect it?

Examples: A bar simplex σ^k is infinitesimally rigid in all dimensions $\mathbb{E}^d$, $k \le d$.

Let $[p_0,\dots,p_k] = p$ be the vertices of the simplex σ^k.

Let $p' = (p_0',\dots,p_k')$ be an infinitesimal flex of $G(p)$ where G is the complete graph.

$p_0,\dots,p_k$ is affine independent, i.e.

$p_1-p_0,\dots,p_k-p_0$ is linearly independent in $\mathbb{E}^d$.

Trivial infinitesimal flexes of $G(p)$ are a linear space, so if we replace $p'=(p_0',\dots,p_k')$ with $(p_0'-p_0', p_1'-p_0', p_2'-p_0',\dots,p_k'-p_0')$ we have not changed whether p' is trivial. $(\ ,\ ) \to (-p_0',\dots,-p_0')$ is infinitesimal translation

Let us also assume $p_0=0$.

Then $(p_i-p_0)\cdot(p_i'-p_0') = p_i\cdot p_i' = 0$ $\quad i=1,\dots,k$

and $(p_i-p_j)\cdot(p_i'-p_j')=0$

$$-p_i\cdot p_j' - p_j\cdot p_i' = 0$$

$$p_i\cdot p_j' = -p_j\cdot p_i' \qquad i,j=1,\dots,k$$

$p_1, \ldots, p_b$ are linearly independent
Let $S: \langle p_1, \ldots, p_b \rangle \to \mathbb{E}^d$
be given by $Sp_i = p_i'$.

$$Sp_i \cdot p_j = p_i' \cdot p_j = -p_j' \cdot p_i = -p_i \cdot Sp_j$$

S = linear transformation.

Claim: S can be extended to all of $\mathbb{E}^d$ so then
$Sv \cdot w = -v \cdot Sw$, for all v, w in $\mathbb{E}^d$.
Then S is skew symmetric.
(Exercise.)

Let $A: \mathbb{E}^d \to \mathbb{E}^d$ be linear.
$G(p)$ is infinitesimally rigid in $\mathbb{E}^d$.
$G(Ap) = G(Ap_1, Ap_2, \ldots, Ap_n)$

Let p' be an inf. flex of $G(p)$.

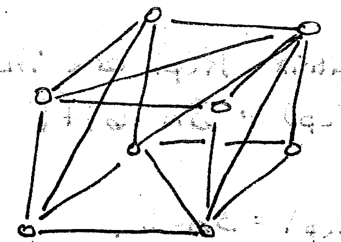
$$0 = (p_i - p_j) \cdot (p_i' - p_j')$$
$$0 = (Ap_i - Ap_j) \cdot ((A^{-1})^T p_i' - (A^{-1})^T p_j')$$
$$= (p_i - p_j)(p_i' - p_j')$$

Assume A invertible

Test for triviality of an infinitesimal flex:
If $G(p)$ is not a k-dimensional ~~bar~~ simplex, $k < d$,
then p' is a non-trivial flex of p iff
$(p_i - p_j) \cdot (p_i' - p_j') \neq 0$ for some i, j.
If $(p_i - p_j) \cdot (p_i' - p_j') = 0 \quad \forall i, j$ then p' is trivial.
i.e. $K_n(p)$ is infinitesimally rigid in $\mathbb{E}^d$,
which is true if affine span of p is d-dimensional.

This can be shown by starting with a simplex + building.

March 8



$n = \#$ vertices $= 8$
$e = \#$ edges $= 12+6-1 = 17$

cube with one extra bar on each face except one face

$3n-6 = 18 > 17$ so not inf. rigid.

Note: with one extra bar inf. rigid. (?)

Cauchy (1813) Convex polyhedra are "unique" in the class of convex polyhedra.

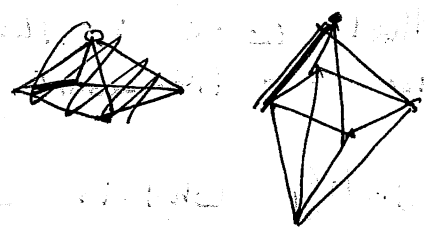
M. Dehn student of Hilbert
solved Hilbert's 3rd problem before it was stated

note: facets of above framework are squares not $\triangle$'s.

(1916) Theorem: Let P be a convex polyhedron with all its facets (faces) triangles and let $G(p)$ be the bar framework formed using the vertices and edges of P. Then $G(p)$ is infinitesimally rigid in $\mathbb{E}^3$.

Examples:

First nontrivial examples:



octahedron

Proof: (of Dehn's theorem)

Note since P has all triangulated facets, $3n-6 = e$ (from Euler formula).

matrix multiplication — air plane flying into building

So the rows of the rigidity matrix $R(p)$ are independent if and only if the rank of $R(p) = 3n-6$ iff $G(p)$ is inf. rigid.

So we want to show rank $R(p) = 3n-6$.

we're looking at cokernel

So we wish to show that the only solutions to $\omega R(p) = 0$ are $\omega = 0$ where $\omega = (\ldots, \omega_{ij}, \ldots)$ ← edge between i & j.

$$\omega R(p) = (\ldots, \omega_{ij}, \ldots) \begin{array}{c} \begin{bmatrix} (p_i - p_j)^T & 0, \ldots & (p_j - p_i)^T \end{bmatrix} \\ \underbrace{\quad}_{\substack{i \\ 3 \text{ columns}}} \qquad \underbrace{\quad}_{\substack{j \\ 3 \text{ columns}}} \end{array}$$

$$= \Big[\ldots, \overbrace{\underbrace{\sum_j \omega_{ij} (p_i - p_j)^T}_{\substack{3 \text{ entries} \\ (\text{columns})}}}^{i}, \ldots \Big] = 0$$

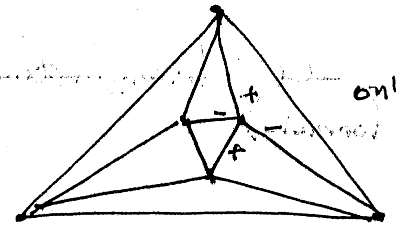
iff ω is an equilibrium stress for $G(p)$.

We need to show that $\omega = 0$ is the only equilibrium stress for our framework.

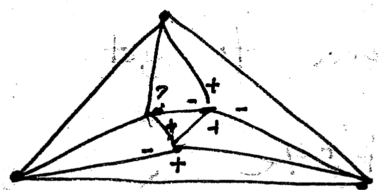
---

pause to justify what is coming next

Let's look at octahedron again (this time we look down on octahedron.)



only possibility for signs of stresses (or negative of this) have to have this same sign pattern everywhere.



So octahedron must be inf. rigid.

Combinatorial Lemma:

Lemma (Cauchy): Label each edge of a planar graph G (without loops or multiple edges) plus or minus. ($+$ or $-$). Then: G has a vertex i_0 that has no changes in sign or just 2 changes in sign in the cyclic order around a vertex (or stated: as the edges are seen cyclically.)

Proof:

Suppose that every vertex of G has at least 4 changes in sign. Look for a contradiction.

Let n = number of vertices of G

e = number of edges of G.

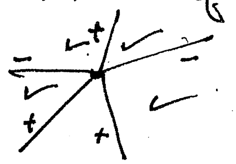
f_i = number of faces of G with i edges.

f = total number of faces of G.

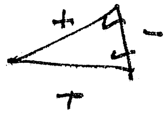
Euler: $n - e + f = 2$

$$f = f_3 + f_4 + \dots$$
$$2e = 3f_3 + 4f_4 + \dots$$

Put a ✓ between each $+$ & $-$ at a vertex of G:



Let c = total number of checks (✓)



can only have 2 checks in triangular face

$$4n \le c \le 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \dots$$

Now we use algebra to get

$$\begin{aligned} 4n &= 8 - 4f + 4e \\ &= 8 - 4(f_3 + f_4 + \dots) + 4(\tfrac{3}{2} f_3 + 2 f_4 + \dots) \\ &= 8 - 4(f_3 + f_4 + \dots) + 2(3f_3 + 4 f_4 + \dots +) \\ & \cancel{= 8 + 2f_3 + 8f_4} \quad \le 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \dots \end{aligned}$$

so $8 + 0f_3 + 0f_4 + 2f_5 + 2f_6 + \dots \le 0$

all coefficients ≥ 0.

contradiction.

(of lemma)

Finishing the proof of Dehn's ~~[illegible]~~ Theorem:

If $G(p)$ is not inf. rigid, there must be a non-zero ~~inf. flex~~ equilibrium stress ω where $\omega R(p) = 0$. Apply combinatorial lemma to the subgraph H of G with a nonzero stress. Then any vertex with at most 2 changes in sign is not in equilibrium. (this argument does not apply to triangulated polytypes with facets that are not Δ's.).

March 13

Projects?

- Bracing a square grid
- Bellows properties
- Cusp mechanisms & higher order rigidity
- Tension percolation
- Polyhedral combinatorics (relative inf. rigidity)

Plan

Near Future:
- Cauchy's original theorem (today)
- Generic rigidity
- Projective invariants
- Basic rigidity $\Leftarrow$ inf. rigidity
- Generic global rigidity

Theorem Cauchy (1813) Modern Version
Let $h : \partial P \to \partial Q$ be a homeomorphism where P, Q are compact convex polyhedra in $\mathbb{E}^3$ where ∂P, ∂Q are their boundaries, and h restricted to each face of P is a congruence, then h extends to a congruence $\tilde{h} : \mathbb{E}^3 \to \mathbb{E}^3$, i.e. $\tilde{h} / \partial P = h$.

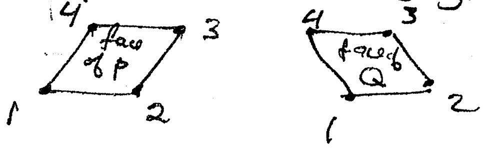
Proof:

Remark (Grünbaum): This result should be stated carefully.

False statement: Let there be a 1-1 incidence preserving correspondence between the faces of P & Q, two convex polytopes in $\mathbb{E}^3$ such that corresponding faces are congruent. Then P and Q are congruent.

Homework: Give false example.

- False because the correspondence & congruence may not match up.



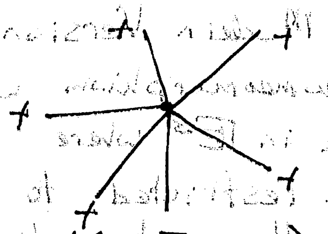
Proof: (of Cauchy's Theorem)

For each edge e of P, let Θ_e be the dihedral angle of P at e. (dihedral angle is internal angle between 2 planes along the edge e). Let $\Theta_{h(e)}$ be the corresponding dihedral angle of edge $h(e)$ in Q.

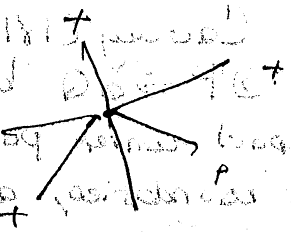
To each edge e associate a $+$ if $\Theta_e < \Theta_{h(e)}$ and a $-$ to e if $\Theta_{h(e)} < \Theta_e$.

We claim that at each vertex p_i of P, there are either no signed or there are at least 4 sign changes.

Let's look at a vertex p_i:



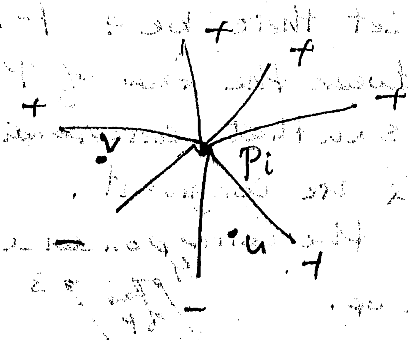
(2 sign changes)



(no sign changes)

Claim: these not possible.

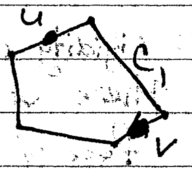
Suppose there are exactly 2 sign changes around p_i. Let u, v be two points $u, v \neq p_i$ in faces adjacent to p_i such that $\langle p_i, u\rangle, \langle p_i, v\rangle$ seperate the $+$'s from the $-$'s.



Choose a plane Π parallel to a support plane of P through h u and v such that p_i is on one side of Π and other vertices of P are on the other side.

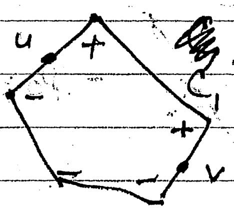
$C = \Pi \cap \partial P$ = planar convex polygon.

simple closed curve



$\Pi \cap P$

vertices of C correspond to the edges of P adjacent to p_i.



$\Pi \cap P$

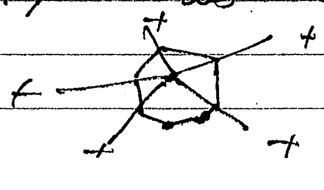
One path on C from u to v consists of only + or nothing vertices and under h the angles of C_1 must only increase or stay the same (and some increase).

So the Cauchy arm lemma implies $|h(u) - h(v)| > |u - v|$
(Note: Cauchy arm lemma works in any dimension.)

Now apply a similar argument in going from Q to P.
We get $|u-v| > |h(u) - h(v)|$. This is a contradiction.

This is why we need both P & Q to be convex.

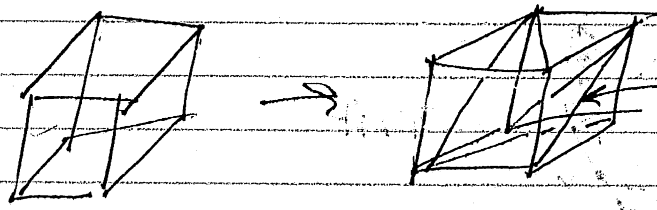
If all signs at p_i are + or all − use a similar argument, creates a contradiction.



So we can't have 2 changes in sign or no changes in sign. But the combinatorial lemma proved last time implies that all the edges have no sign attached. i.e. for all dihedral angles $\theta_e = \theta_{h(e)}$.
This implies that h extends to a congruence of $\mathbb{E}^3$. $\boxed{\because}$

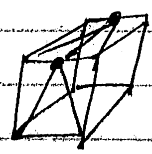
A. D. Alexandrov generalized Dehn's rigidity result.
Triangulate the surface ∂P, but there are no vertices in the interior of a 2-dimensional face of P.
Then, this bar framework is inf. rigid, + therefore rigid.

Classic example: cube



Weak Alexandrov theorem
no new vertices
Note: we can't just use proof of Dehn's theorem

Stronger version: subdivide in interior of edges


etc.

can use weak version to show the stronger version (if inf rigid w/out new vertices, then inf. rigid w/ new vertices).

Can generalize to more dimensions.
Just need to triangulate 2-skeleton.

<u>Theorem</u>: (Connelly)
Let $G(p)$ be a bar framework coming from any triangulation of the boundary of P, a convex polytope, then $G(p)$ is ~~second order~~ rigid + therefore rigid.
pre-stress stable

Next time: generic rigidity

March 27

Coming events:

- Finish Laman theorem today
- projective invariance of inf. rigidity
- Static Rigidity

Possible project: pebble game

the proof of this was started last time when I wasn't here

Laman's Theorem: G is generically rigid in $\mathbb{E}^2$ with $e = 2n - 3$ if and only if $e' \leq 2n' - 3$ for any subset of edges e' and their vertices n'.

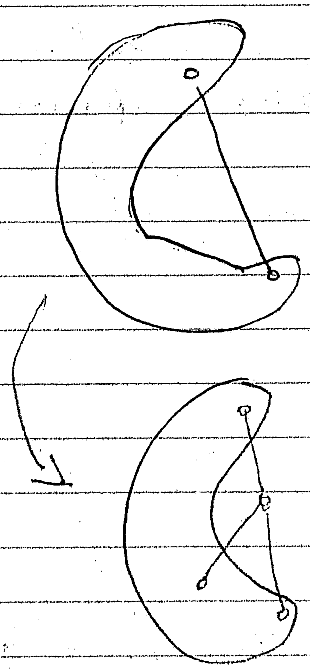
e: #edges, n: #vertices

$\Rightarrow$ clear

so we need to show $\Leftarrow$

Hinneberg operation

If



is generically rigid

edges $+2$

vertices $+1$

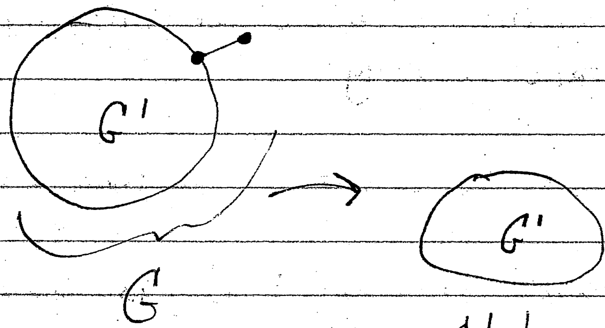
so still have $e = 2n-3$

is inf. rigid in this special position + therefore generically rigid

Noticed last time: average degree of G is < 4.

So G has a vertex of degree 1, 2, or 3.

Suppose has vertex with degree 1:



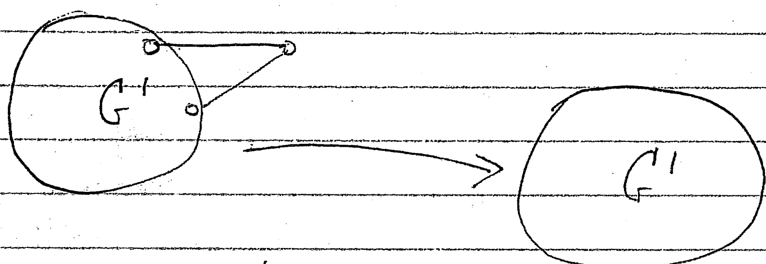
G' has $n' = n-1$ verties
$e' = e-1$ edges

So
$$2n'-3 = 2n-5$$
$$= (2n-3)-2$$
$$= e-2$$
$$= (e-1)-1$$
$$= e'-1 < e'$$

So $2n'-3 < e'$ $\Rightarrow\Leftarrow$

Thus we cannot have a vertex of degree 1.

Suppose ~~have~~ has vertex with degree 2:



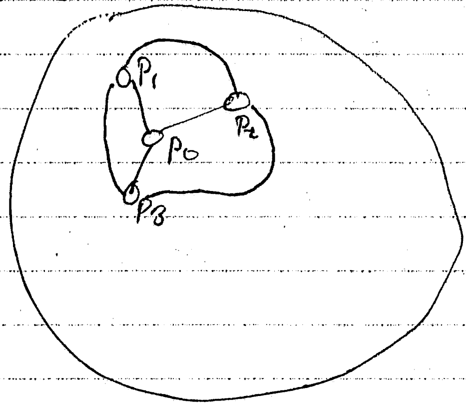
$n' = n-1$
$e' = e-2$

So $2n'-3 = e'$

So ~~and~~ by induction on n G' is 2-generically rigid. By problem 3 in HW G is inf. rigid.

(Base case $n=1$ • generically rigid.)

degree 3 (hard case)
Use induction $n = \#$ vertices



remove p_0 ~~to replace one of~~ P_1, P_2, P_3

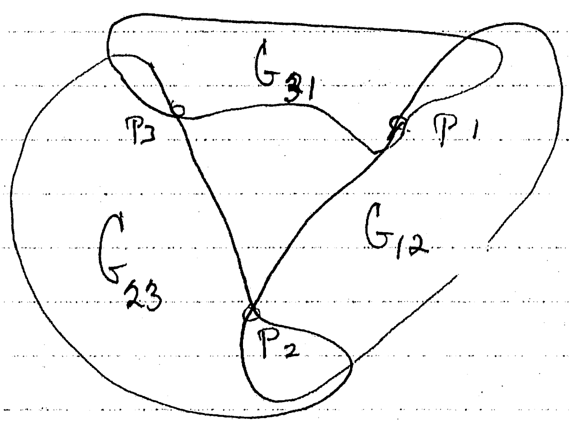
then can do Hinneberg operation.

We claim there ~~is a subgraph~~ for some $i \neq j \in \{1, 2, 3\}$ $G' \cup \{ij\}$ is generically 2-rigid.

If this happens for any i, j then the Hinneberg operation creates an inf. rigid G and therefore G is generically 2-rigid.

So assume all 3 cases yield a $G' \cup \{ij\}$ not generically 2-rigid.

So by induction we know that the "count" is wrong in each case. Let G_{ij} be the subgraph of $G' \cup \{ij\}$ where $e' > 2n' - 3$.



Each G_{ij} must be generically 2-rigid because when you add the edge $\{i j\}$ $e' > 2n' - 3$.
So in G_{ij} edges = 2vertices − 3.
So inductively generically 2-rigid.

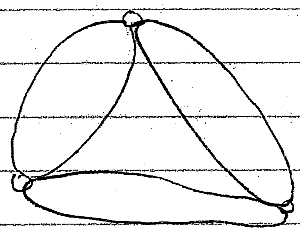
Consider $G_{ij} \cap G_{jk}$

Claim: $G_{ij} \cap G_{jk} = \{p_j\}$

If there is another vertex then $G_{ij} \cup G_{jk}$ is inf. rigid. Then adding p_0 + its three edges create a subgraph with $e = 2n-3+1$.

So $G_{ij} \cap G_{jk} = \{p_j\}$.

So we have



Now $G_{12} \cup G_{23} \cup G_{31}$ is inf. rigid.

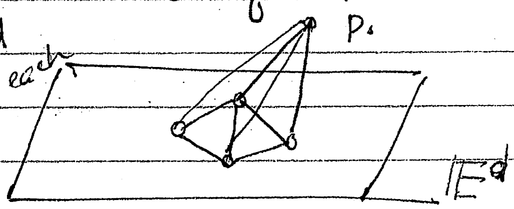
So $e' = 2n' - 3$ for this union.
Adding p_0 and its 3 edges create a count $e = 2n-3+1$ contradicting the hypothesis.

Thus some Henneberg can be done.

And we are done. $\square$

Projective invariance of infinitesimal rigidity.

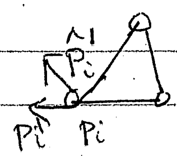
Choose $p_0 \notin \mathbb{E}^d$, $p_0 \in \mathbb{E}^{d+1} - \mathbb{E}^d$. Connect p_0 by a bar to each p_i in $\mathbb{E}^d$.



Let $G(p)$ be a framework. $\mathbb{E}^d \subset \mathbb{E}^{d+1}$

New framework $\hat{G}(\hat{p})$
$\hat{p} = (p_0, p_1, \dots)$

Let p' be an inf flex of $G(p)$ in $\mathbb{E}^d$.
Let $p_0' = 0$; we want to define $\hat{p}_i'$ as a corresponding inf. flex of $\hat{G}(\hat{p})$.



Let e_{d+1} be $\perp$ to $\mathbb{E}^d$. We want to define λ_i so that $(p_i - p_0)\cdot(p_i' + \lambda_i e_{d+1} - \underbrace{p_0'}_{=0}) = 0$

Define $\lambda_i = \dfrac{-p_i'\cdot(p_i - p_0)}{\underbrace{(p_i - p_0)\cdot e_{d+1}}_{\neq 0}}$

So bar constraints are satisfied and
$(p_i - p_j)\cdot(\hat{p}_i' - \hat{p}_j') = (p_i - p_j)\cdot(p_i' - p_j')$
so member contreinst elsewhere are satisfied.

? Prop: $G(p)$ is inf. rigid in $\mathbb{E}^d$ if and only if $\hat{G}(\hat{p})$ " " " " $\mathbb{E}^{d+1}$.

Would it be possible to have something inf. rigid in $\mathbb{E}^d$ but not ~~co~~ cone in $\mathbb{E}^{d+1}$?

Let $G(p)$ be a cone framework. ~~Then replace~~
Let $p_0 = 0$. Replace p_i with $\lambda_i p_i = \hat{p}_i$ where $\lambda_i \neq 0$.

Then $G(\hat{p})$ + $G(p)$ are both inf. rigid or both inf. flexible.

We can assume $p_0' = 0$.
Bar constraints $(p_i - p_0) \cdot (p_i' - p_0') = p_i \cdot p_i' = \frac{1}{\lambda^2} (\lambda p_i)(\lambda_i p_i')$

$$= \frac{1}{\lambda^2} \left(\hat{p}_i \cdot p_i' \right) = 0.$$

Other constraints $(p_i - p_j) \cdot (p_i' - p_j') = -p_i' \cdot p_j' - p_j' p_i'$

If cable, strut, remains cable or strut when $\lambda_i \lambda_j > 0$
Reverse roles when $\lambda_i \lambda_j < 0$

Projective invariance follows modulo this renaming members.

March 29

Question: Does there exist a generically rigid graph, embedded in the plane (no crossings) with no Δs that is generically rigid in the plane.

No. needs to have a Δ.

to be generically rigid need $e \geq 2v-3$ + use Euler characteristic.

$2e \geq 4f$

$e \geq 2f = 2(2-v+e) = 4-2v+2e$

$2v \geq 4+e$

$2v-3 \geq 1+e > e$

We continue the discussion of first order rigidity with

Static Rigidity

Consider all configurations

$p=(p_1,\dots,p_n)$, $p_i \in \mathbb{E}^d$

$p \in \mathbb{E}^{nd}$.

$\mathbb{E}^{nd}$ is configuration space

The rigid congruences act on $\mathbb{E}^{nd}$

$(p_1,\dots,p_n) \to (Tp_1,\dots,Tp_n)$

where $Tv = Av+b$, $A^TA=I$ orthogonal, b translation

$p \to Tp$

set of orthogonal matrices is Lie group.

The orbit of the configuration $p \in \mathbb{E}^{nd}$ is

$O_p = \{ q \mid q = Tp,\ T \text{ is congruence of } \mathbb{E}^{nd} \}$

affine span of p is all of $\mathbb{E}^d$.

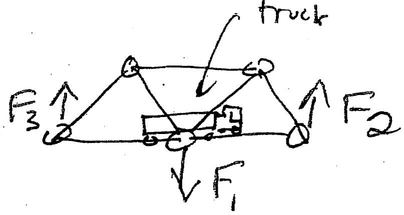
For "most" configurations p O_p is a manifold.

dimension of manifold is $\frac{d(d+1)}{2}$. when span is d or $d-1$.

orthogonal matrices are manifold.

We want to look at tangent space to p.

Lie groups?



Let T_p be the tangent space of O_p at p.
Tangent space is the vector space of trivial infinitesimal flexes. of p.

What does orthogonal complement look like?

We think now in terms of "forces" on the configuration p.
We say that a force $F = (F_1, \ldots, F_n)$, $F_i \in \mathbb{E}^{nd}$ is an equilibrium force if F is orthogonal to T_p the tangent space of O_p at p.

How do you calculate when a force F is an equilibrium force?

The definition says that $F \cdot p' = 0$ for all trivial

$$p' = \begin{bmatrix} p_1' \\ \vdots \\ p_n' \end{bmatrix}$$

So $\sum_{i=1}^{n} F_i \cdot p_i' = 0$ for all trivial inf. ~~metrics~~ motions.

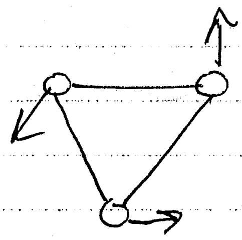
<u>Translational part</u>: In particular when $p_i' = b = \text{constant} \in \mathbb{E}^d$

$$\sum_{i=1}^{n} F_i \cdot b = 0 \quad \text{for all } b.$$

So we get that $\sum_{i=1}^{n} F_i = 0$ (linear momentum is 0).

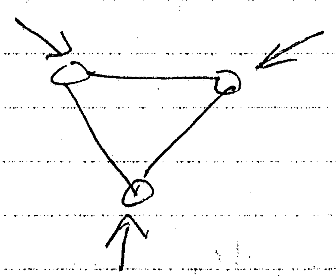
<u>Orthogonal part</u>: This says that for every skew symmetric matrix $S = -S^T$

$$\sum_{i=0}^{n} F_i \cdot S p_i = 0.$$



$\sum F_i = 0$

angular momentum $\neq 0$.



angular momentum $= 0$

push on it & nothing happens.

Side Bar: Exterior Algebra (homegrown version of exterior calculus)

Let $e_1, \ldots, e_d$ be a "standard" basis for $\mathbb{E}^d$.

Define for any $1 \leq k \leq d$, define symbols

$$\omega = e_{i_1} \wedge e_{i_2} \wedge \ldots e_{i_k}$$

And we look at equivalence classes where $e_i \wedge e_j = -e_j \wedge e_i$ and associativity holds.

The vector space generated by these equivalence classes ~~of~~ is called Λ^k.

that's a capital λ.

Note that a basis for Λ^k is the set $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots e_{i_k}$ where $i_1 < i_2 < \ldots < i_k$.

So the dimension of Λ^k is $\binom{d}{k}$

Note that $\cancel{\omega_1\wedge\omega_2}$ $\omega\wedge\omega = 0 \quad \forall\, \omega \in \Lambda^k$.

~~Abd~~ $\cancel{\omega_1\wedge\omega_2}$

$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1\wedge\omega_2 + \omega_1\wedge\omega_3$

$\Lambda^1 \Leftrightarrow \mathbb{E}^d$.

$\Lambda^{d-1} \Leftrightarrow \mathbb{E}^d$

for instance $e_1 \wedge \dots e_{k-1} \wedge e_{k+1} \wedge \dots e_d \to (-1)^{k+1} e_k$

$\Lambda^d \Leftrightarrow \mathbb{E}^1 \qquad \left(p_1 \wedge \dots \wedge p_d = \det[\, p_1, \dots, p_d\,]\right)$

When $d=3$ $\quad p_1 \wedge p_2 \Leftrightarrow p_1 \times p_2$ $\quad$ Check:

$e_1 \wedge e_2 = e_3$

$e_2 \wedge e_3 = e_1$

$e_1 \wedge e_3 = -e_3 \wedge e_1$

$= -e_2$

$e_3 \wedge e_1 = e_2$

For any skew symmetric matrix $S_{d\times d}$ we can associate an element $s \in \Lambda^{d-2}$

Let $p_1, p_2 \in \mathbb{E}^d$ $\quad (p_1, p_2) \to p_1^T S p_2 \in \mathbb{E}^1$ is a bilinear form

Let $s \in \Lambda^{d-2}$ $\quad (p_1, p_2) \to p_1 \wedge s \wedge p_2 \in \Lambda^d = \mathbb{E}^1$ is also a bilinear form.

Note both forms are skew symmetric

$p_1^T S p_2 = -p_2^T S p_1$

? on (-) signs $\quad p_1 \wedge s \wedge p_2 = (-1)^k p_1 \wedge p_2 \wedge s = p_2 \wedge p_1 \wedge s = p_2 \wedge p_1 \wedge s = \overset{(-1)^k}{-} p_2 \wedge s \wedge p_1 \in \Lambda^d = \mathbb{E}^1$

So we can associate $s \Leftrightarrow S$ = skew matrix

With this in mind,

$$\sum_{i=1}^{n} F_i S_{p_i} = 0$$

Now the condition for being in equilibrium is

$$\sum_{i=1}^{n} F_i \wedge s \wedge p_i = 0 = \pm \sum_{i=1}^{n} F_i \wedge p_i \wedge s$$

$$= I\left(\sum_{i=1}^{n} F_i \wedge p_i\right) \wedge s = 0$$

For all $s \in \Lambda^{d-2}$ Then we must have

$$\left.\begin{array}{l} \sum_{i=1}^{n} F_i \wedge p_i = 0 \\ \sum_{i=1}^{n} F_i = 0 \end{array}\right\} \text{condition for an equilibrium force.}$$

What is a form?

Here we have forms of the ~~po~~ that look like

$$p_1 \wedge \ldots \wedge p_k \in \Lambda^k$$

We claim Λ^k is a vector space.

How do we add? How do we multiply by scalar.

$$p_1 \wedge \ldots \wedge p_k + q_1 \wedge \ldots \wedge q_k = (p_1+q_1)\wedge \ldots \wedge (p_k+q_k)$$

and we have $e_i \wedge e_j = -e_j \wedge e_i$

So $e_i \wedge e_i = 0$

Suppose $d = 3$ $\quad k=2$

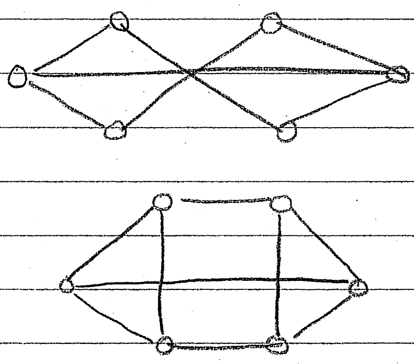
we have $e_i \wedge e_j$ where $i,j \in \{1,2,3\}$

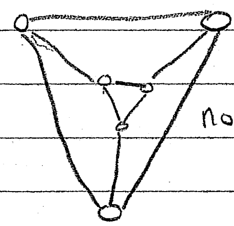
basis : $e_1 \wedge e_2$, $e_2 \wedge e_3$, $e_1 \wedge e_3$

April 3

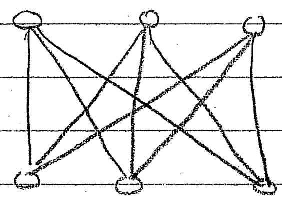
homework stuff

all same graph

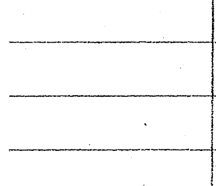



not inf. rigid

cases with 6 vertices + degree 3

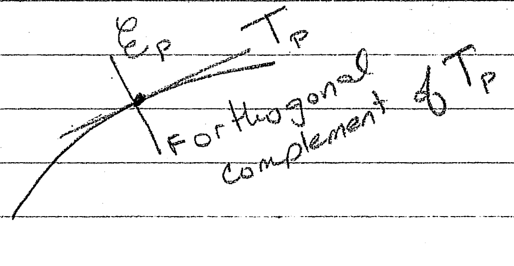


Question: Let P be a convex polytope in $\mathbb{E}^3$, where each facet is a triangle. Can you color the edges of P with 3 colors such that each facet has 3 different colors for its edges
(follows from the 4 color theorem)
4 color vertices from $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad$ (a,b) $\quad$ (c,d) at 2 vertices
$(a-c, b-d)$ on edge then at other vertex would need (a,b) to have same label at other edge so So would work.

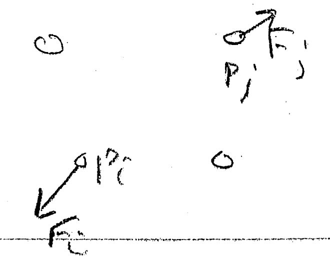


Equilibrium forces for a configuration
$F = [F_1, \dots, F_n]$ $\qquad$ $p = (p_1, \dots, p_n)$

$\sum_{i=1}^{n} F_i = 0$ and $\sum_{i=1}^{n} F_i \wedge p_i = 0$



all sits in $\mathbb{E}^{nd}$
(configuration space)



Example: For $p=(p_1,\dots,p_n)$

Define $F_{ij} = [\,0 \;\dots\; p_i - p_j \quad 0 \;\dots\; p_j - p_i \;\dots\,]$

$p_i - p_j + p_j - p_i = 0$

Let $\mathcal{E}$ be the equilibrium forces at p.

$\dim \mathcal{E} = nd - \dfrac{d(d+1)}{2}$

when the affine span of p is $\mathbb{E}^d$ (or $d-1$ dimensional

Notice also that the rigidity matrix

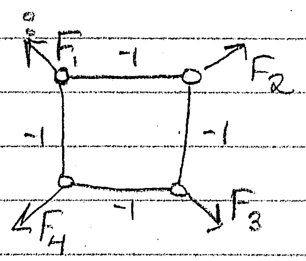
$$R(p) = \begin{bmatrix} \vdots \\ F_{ij} \\ \vdots \end{bmatrix}$$

$F = (F_1, \dots, F_n)$

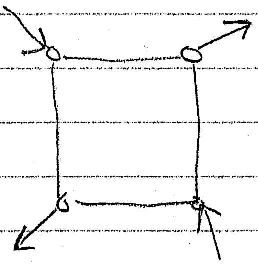
Let F be any equilibrium force for a configuration p.
We say F can be resolved by a stress ω for $G(p)$ if for each $i = 1, \dots, n$

(±) $F_i = \sum_j \omega_{ij}(p_j - p_i)$

Example:



But this cannot be resolved:



We say a (tensegrity) framework is statically rigid if every equilibrium force can be resolved by a proper stress ω.

Result that we will hopefull get to someday:
static rigidity & infinitesimal rigidity are equivalent.

Note that the only possible F's that can be resolved are equilibrium forces.

$$\omega R(p) = \begin{bmatrix} \sum \omega_{ij}(p_i - p_j) \\ \vdots \end{bmatrix} = \sum_{ij}' \omega_{ij} F_{ij}$$

Each F_{ij} is an equilibrium force so any linear combo is an equilibrium force as well.

In fact $\{\omega R(p) \mid \omega \text{ proper}\}$ is the set of resolvable forces.

Suppose $G(p)$ is an inf. rigid bar framework.
Let p' be an inf. flex of $G(p)$.
So $R(p)p' = 0$.
Let $\bar{p}'$ be the orthogonal projections of p' into $\mathcal{E}$ the space of equilibrium forces.
Claim: $\bar{p}'$ is also an inf. flex of $G(p)$ and $\bar{p}' = 0$ if and only if p' is trivial.

$\bar{p}' = p' - u$, u is trivial.
When $\{i,j\}$ = member of G, $(p_i - p_j)\cdot(\bar{p}_i' - \bar{p}_j')$
$= (p_i - p_j)\cdot(p_i' - p_j' - u_i + u_j)$
$= (p_i - p_j)\cdot(p_i' - p_j')$
$- (p_i - p_j)\cdot(u_i\ u_j)$

So $\bar{p}'$ is an inf. flex of $G(p)$ as well.

So if $G(p)$ is statically rigid, then there is a stress ω where ω resolves $\bar{p}' \in \mathcal{E}$. So $\omega R(p) = \bar{p}'$. So $\underbrace{\omega \underbrace{R(p)\bar{p}'}_{0}}_{0} = \bar{p}' \cdot \bar{p}'$ So $\bar{p}' = 0$.
So p' is trivial.

Thus if $G(p)$ is not inf. rigid, then it is not statically rigid.

Say affine span of p is all of $\mathbb{E}^d$.

So $G(p)$ as a bar framework is inf. rigid iff

$$\operatorname{rank} R(p) = nd - \frac{d(d+1)}{2}$$

and it is statically rigid iff the image of $\omega \mapsto \omega R(p)$ is all of $\mathcal{E}'$

but $\dim \mathcal{E} = nd - \frac{(d+1)d}{2}$ iff $\operatorname{rank} R(p) = nd - \frac{d(d+1)}{2}$

So if $G(p)$ bar framework, then inf. rigidity $\Leftrightarrow$ static rigidity.

Would lik to extend this to tensegrity frameworks.

Static rigidity $\leftrightarrow$ Forces (Engineering viewpoint)
Infinitesimal rigidity $\leftrightarrow$ Velocities (Differential Geometry)
Herglotz $\leftrightarrow$ Cauchy theorem

Rogovelov?

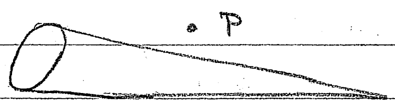
We want to show that static rigidity and infinitesimal rigidity are equivalent for tensegrities. This will use duality.

Static rigidity means every equilibrium force F can be resolved by a *proper* equilibrium stress. $\omega R(p) = F$, ω proper

Infinitesimal rigidity means all infinitesimal flexes of $G(p)$ are trivial. $R(p)p' \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} 0 \Rightarrow p'$ trivial.

Convexity Results:

1. Let X = closed convex set in $\mathbb{E}^d$ and $p \in \mathbb{E}^d - X$.



Then there ... a unique point q in X nearest to p.

2. There is a half space H containing X, but not p, with q, on ∂H.

3. If X is a cone from the origin, then $0 \in \partial H$.

We will prove this on next page.

Let $\mathcal{E}$ be a linear subspace of $\mathbb{R}^d$. Let $a_1, \dots, a_n \in \mathcal{E}$.
Define $H_{a_i} = \{ p \in \mathcal{E} \mid p \cdot a_i \leq 0 \}$
Then $\bigcap_{i=1}^{n} H_{a_i} = \{0\}$ if and only if for all $b \in \mathcal{E}$,
there are scalars $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i a_i = b$. (*)

Proof:

(⇐) Suppose for all b there are scalars such that (*) holds. Let $p \in \bigcap_{i=1}^{n} H_{a_i}$. Write $p = \sum_{i=1}^{n} \lambda_i a_i$ each $\lambda_i \geq 0$. Then

$$p \cdot p = p\cdot\Big(\sum_i \lambda_i a_i\Big) = \sum_{i=1}^{n} \lambda_i \, p\cdot a_i \leq 0$$

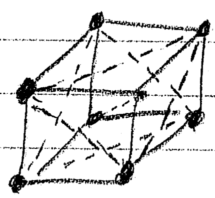
$$\Rightarrow p = 0$$

(⇒) Suppose $\bigcap_{i=1}^{n} H_{a_i} = \{0\}$.

Let $b \in \mathcal{E}$. Let $\underline{X} = \{p \mid p = \sum_{i=1}^{n} \lambda_i a_i\}$ convex cone including 0.
And suppose $b \notin \underline{X}$.
Then there exists a hyperplane H through 0 that separates $\underline{X}$ from b. Let $H_v = \{p \mid p \cdot v \leq 0\}$ $b \notin H_v$.
$H_v \supset \underline{X}$. $a_i \in \underline{X}$ So $a_i \cdot v \leq 0$. So $v \in \bigcap H_{a_i}$.
So $v = 0$. $\Rightarrow\Leftarrow$ Thus $b \in \underline{X}$. $\boxed{☺}$

off topic

?: Can you make cube inf. rigid with 7 cables?



bars.

(need to put some cables inside)

Theorem: Infinitesimal rigidity + static rigidity are equivalent.
Proof: Let $G(p)$ be infinitesimally rigid. And let F be an equilibrium force on the configuration $p = (p_1, \ldots, p_n)$.
Let $\mathcal{E}$ = equilibrium forces for p. So we are looking for a stress ω such that $\omega R(p) = F$ and each ω is proper.
Let $R_+(p)$ be $R(p)$ with the strut rows multiplied by (-1). So there is a proper stress ω if + only if there is a stress $\bar{\omega} \geq 0$ such that $\bar{\omega} R_+(p) = F$.

The condition for infinitesimal rigidity of $G(p)$ is $R(p)\,p' \leq 0$ implies $p' = 0$ if $p' \in \mathcal{E}$. Let a_i = i^{th} row of $R_+(p)$

This says $\bigcap H_{a_i} = \{0\}$

So our duality result implies that inf. rigidity $\Leftrightarrow$ static rigidity.

April 10

Theorem (1980 ±, Roth - Whiteley)
Let $G(p)$ be a tensegrity framework in $\mathbb{E}^d$. Let $\bar{G}(p)$ be the underlying bar framework.
Then $G(p)$ is infinitesimally rigid ($\Leftrightarrow$ statically rigid)
$\Leftrightarrow$ the following two conditions hold:
i) $\bar{G}(p)$ is infinitesimally rigid and
ii) $G(p)$ has a proper stress such that $\omega_{ij} \neq 0$ for each cable + strut $\{i,j\}$.
(proper means ω_{ij} has right sign depending on whether $\{i,j\}$ is cable or strut)

Proof:
($\Leftarrow$) We will begin by showing (i) + (ii) $\Rightarrow$ $G(p)$ inf. rigid.
(supposedly this is the "easy" way).
Suppose (i) + (ii) hold. We want to show $G(p)$ inf. rigid. Let p' be an inf. flex of $G(p)$.
We will show that p' is trivial.
Let ω be the proper equilibrium stress given by (ii).
Since ω is an equilibrium stress, $\omega R(p) = 0$ where $R(p)$ = rigidity matrix.
So $0 = \omega R(p)\, p' = \sum \omega_{ij} (p_i - p_j)\cdot(p_i' - p_j')$
For $\{i,j\}$ = cable, $\omega_{ij} > 0$, $(p_i - p_j)\cdot(p_i' - p_j') \leq 0$
~~with strict inequality unless~~ $\cancel{(p_i - p_j)\cdot(p_i' - p_j') = 0}$.
~~Similarly for a strut,~~ $\cancel{\omega_{ij} \leq 0, (p_i - p_j)\cdot(p_i' - p_j') \geq 0}$.
~~So~~ $\cancel{\omega_{ij} (p_i - p_j}$
So $\omega_{ij} (p_i - p_j)\cdot(p_i' - p_j') \leq 0$ with strict inequality unless $(p_i - p_j)\cdot(p_i' - p_j') = 0$.

For $\{i,j\}$ = strut, $\omega_{ij} < 0$, $(p_i - p_j)\cdot(p_i' - p_j') \geq 0$.
So $\omega_{ij} (p_i - p_j)\cdot(p_i' - p_j') \leq 0$ with strict inequality unless $(p_i - p_j)\cdot(p_i' - p_j') = 0$.

For a bar $\{i,j\}$, $\omega_{ij} (p_i - p_j)\cdot(p_i' - p_j') = 0$.

So $\omega R(p) p' \leq 0$ with strict inequality unless $(p_i - p_j)\cdot(p_i' - p_j') = 0$ for all members $\{i,j\}$. So $(p_i - p_j)\cdot(p_i' - p_j') = 0$ for all members $\{i,j\}$

So p' is an inf. flex of $\bar{G}(p)$. So by (i) p' is trivial.

($\Rightarrow$) This is the harder + supposedly more interesting directio[n]. Suppose $G(p)$ is statically rigid in $\mathbb{E}^d$. We want to show (i) + (ii) hold. (Note that we're using the fact that inf. rigidity $\Leftrightarrow$ statically rigid.)

Clearly (i) holds. If $G(p)$ inf. rigid, then $\bar{G}(p)$ inf. rigid, So we just need to show (ii).

Let $F(\{i,j\}) = [0 \ldots, p_i - p_j, 0 \ldots, p_j - p_i, 0 \ldots]$

Claim: this is an equilibrium force.

Because $\sum F_k(\{i,j\}) = p_i - p_j + p_j - p_i = 0$.

$$\sum F_k(\{i,j\}) \wedge p_k = (p_i - p_j)\wedge p_i + (p_j - p_i)\wedge p_j = -p_j \wedge p_i - p_i \wedge p_j = 0$$

Let $\bar{\omega}(\{i,j\})$ be the resolving stress for this equilibrium force.

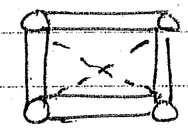
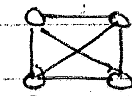
Now let ~~$\omega(\{i,j\}) = 1$ and ω~~ $\omega_{ij}\{i,j\} = 1 + \bar{\omega}_{ij}\{i,j\}$ and $\omega_{ij}\{k,l\} = \bar{\omega}_{kl}\{i,j\}$.

This is an equilibrium stress for $G(p)$, and it is proper + $\omega_{ij}(\{i,j\}) > 0$.

Similarly for a strut $\{i,j\}$ define $\omega(\{i,j\})$ so that it is a proper equilibrium stress for $G(p)$ and $\omega_{ij}(\{i,j\}) < 0$. Then

$\omega = \sum_{i,j} \omega(\{i,j\})$ is a proper equilibrium stress $\omega_{i,j} \neq 0$ for all cables + struts. $\square$

Note: Cannot ensure that bars have stress.
Example:  statically rigid, but no stress.

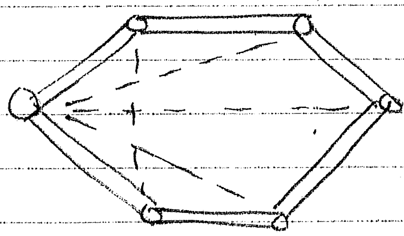
Examples: (1)  statically rigid because 

inf. rigid

& there is a stress

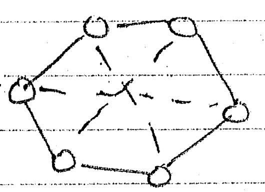
(2) In more generality, any Cauchy polygon is statically rigid.
(c-s, b-s, s-c, c-b, s-b, b-c, b-b)

(3) Grünbaum polygons



bar framework
inf. rigid
& there is a stress.

(4)



~~also~~ not statically rigid
no stress when hexagon not ~~rigid~~ regular
not inf. rigid when hexagon regular

[That's what mathematics is about – handwaving.

April 17

Pre-stress rigidity

energy

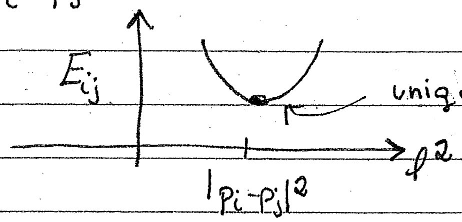
start with $G(p)$ a tensegrity members {cables, bars, struts} : G

configuration : p

We define an energy functional for each member $\{i,j\}$

$l = |p_i - p_j|$

bars:

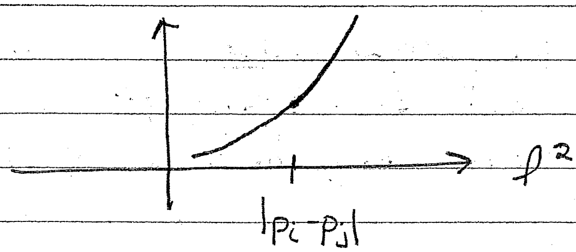


unique min at $l^2 = |p_i - p_j|^2$

want $E_{ij}'' > 0$

It costs us energy to move away from $l^2 = |p_i - p_j|^2$.

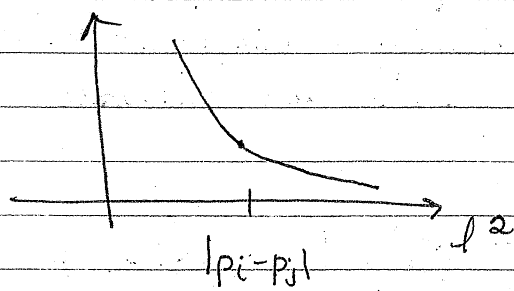
cable:



must be monotone increasing

want $E_{ij}'' > 0$

strut:



must be monotone decreasing

want: $E_{ij}'' > 0$

for any other configuration q (that is, a configuration other than

$$E(q) = \sum_{i<j} E_{ij}(|q_i - q_j|^2)$$

What we had previously was a special case.

Proposition: Principle of least work applies: If E is such that $E(p)$ is a local unique minimum up to congruences of p, then $G(p)$ is rigid in $\mathbb{E}^d$.

Proof: ~~Suppose~~ Let $\mathcal{U}_p$ be the neighborhood of p in $\mathbb{E}^{nd}$ such that $E(p)$ is a minimum up to congruences of p. Suppose $q \in \mathcal{U}_p$ + $G(q) \le G(p)$. (i.e. q satisfies the member constraints).

for cable $|q_i - q_j| \le |p_i - p_j|$, then $E_{ij}(|q_i - q_j|^2) \le E_{ij}(|p_i - p_j|^2)$.
same for bars + struts.

So $E(q) \le E(p)$. So q congruent to p.
$G(p)$ is rigid. $\boxed{☺}$

Def: $G(p)$ is pre-stress stable if there are energy functions of this sort such that E is a relative min at p modulo congruences, because of "the second derivative test".

First derivative test (for minima):
Let us take the directional minima in the direction p'.
We consider $E(p+tp')$ and calculate $\frac{d}{dt}$ of $E(p+tp')$ at $t=0$.

$$\frac{d}{dt} E(p+tp') = \sum_{i<j} \frac{d}{dt} E_{ij}(|p_i - p_j + t(p_i' - p_j')|^2)$$

$$\frac{d}{dt} E_{ij}(|\quad|^2) = E_{ij}'(|p_i - p_j + t(p_i' - p_j')|^2) \cdot \left(2(p_i - p_j)\cdot(p_i' - p_j') + 2t(p_i' - p_j')^2\right)$$

At $t=0$, $\frac{d}{dt}E(p+tp')\Big|_{t=0} = 2\sum_{i<j} E_{ij}'(|p_i-p_j|^2)(p_i-p_j)\cdot(p_i'-p_j')$

Define $\omega_{ij} = E_{ij}'(|p_i-p_j|^2)$

Call $\omega = (\ldots, \omega_{ij}, \ldots)$ the "stress".

If p is a critical point for E, then
$2\sum_{i<j} \omega_{ij}(p_i-p_j)\cdot(p_i'-p_j') = 0$ for all directions p'.

In other words, $\omega R(p)p' = 0$ for all p'.
So $\omega R(p) = 0$. So ω is an equilibrium stress for $G(p)$.
Note also, ω is proper.

The second-derivative test.
p will be a local min if p is a critical point & $\frac{d^2}{dt^2}E(p+tp')\Big|_{t=0} \geq 0$ for all p', and
equality only when p' is a trivial infinitesimal flex.

$$\frac{d^2E(p+tp')}{dt^2}$$

$$\cancel{\frac{d^2}{dt^2}E_{ij}'(|(p_i-p_j)+t(p_i'-p_j')|^2)}$$

$$4E_{ij}''(|(p_i-p_j)+t(p_i'-p_j')|^2)\big((p_i-p_j)\cdot(p_i'-p_j') + 2t(p_i'-p_j')\big)^2$$
$$+4E_{ij}'(|p_i-p_j+t(p_i'-p_j')|^2)(|p_i'-p_j'|^2)$$

At $t=0$, we have

$$4E_{ij}''(|p_i-p_j|^2)\big((p_i-p_j)\cdot(p_i'-p_j')\big)^2 + 4\omega_{ij}|p_i'-p_j'|^2$$

$$\frac{d^2}{dt^2} E(p)\Big|_{t=0} = 4\sum_{i<j} w_{ij} |p_i - p_j|^2 + 4\sum_{i<j} c_{ij}\left((p_i-p_j)\cdot(p_i'-p_j')\right)^2$$

where $c_{ij} = E_{ij}''(|p_i - p_j|^2) > 0$

Call c_{ij}'s stiffness coefficients

Recall $(p')^T \Omega \otimes I^d p' = \sum_{i,j} w_{ij} (p_i' - p_j')^2$

Let $C = \begin{bmatrix} \ddots & & 0 \\ & c_{ij} & \\ 0 & & \ddots \end{bmatrix}$ = stiffness diagonal matrix

Calculate $[R(p)p']^T C R(p) p' = R(p)p' \cdot \begin{bmatrix} \vdots \\ c_{ij}(p_i-p_j)\cdot(p_i'-p_j') \\ \vdots \end{bmatrix}$

$$= \sum_{i<j} c_{ij} \left[(p_i - p_j)\cdot(p_i' - p_j')\right]^2$$

$$= (p')^T \underbrace{R(p)^T C R(p)}_{\substack{\text{symmetric} \\ \text{positive definite} \\ \text{matrix}}} p'$$

Second derivative is $\Omega \otimes I^d + R(p)^T C R(p)$

April 19

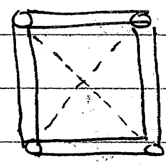
Today: Finish "prestressability"
Start Maxwell-Cremona correspondence

Reference for prestressability – paper of Connelly + Whiteley — this is the proof we will use
for Maxwell-Cremona – papers by Crapo + Whiteley, Gluck (1973) "Almost all..."

Recall that $G(p)$ is pre-stress rigid or (pre stressable) if there is a proper equilibrium stress for $G(p)$ such that $\Omega \otimes I^d + (R(p))^T R(p)$ is positive semi-definite with kernel exactly the trivial infinitesimal flexes of p, where Ω is the stress matrix associated to ω, and $R(p)$ is the rigidity matrix for the corresponding bar framework $\overline{G(p)}$.

Choose materials so that stiffness matrix is I.

Example:

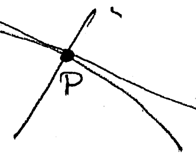


Ω has one negative eigenvalue but $R(p) + R(p)^T R(p)$ is positive semi-definite of max rank

This is unstable for large proper stress but stable for a small proper stress, since in that case the $R(p)^T R(p)$ will dominate.

Test for pre-stress stability.
Let $K_p = \{ p' \in \mathbb{E}^{nd} \mid R(p) p' = 0 \}$

Consider the quadratic form associated to the stress energy $\Omega \otimes I^d$ restricted to K_p.
$(p')^T \Omega \otimes I^d p'$



T_p (trivial)

<u>Proposition</u>: $G(p)$ is pre-stress stable if $(\Omega \otimes I^d)$ is positive definite on $K_p \cap \mathcal{E}$ where $\mathcal{E}$ = equilibrium forces.

<u>Proposition</u>: If $G(p)$ is inf. rigid, it is prestress stable.

<u>Proof</u>:

If $G(p)$ is a bar framework, choose $\omega = 0$, then $R(p)^T R(p) \geq 0$ + has only a trivial kernel.

If $G(p)$ is a tensegrity framework, choose a proper equilibrium stress ω that is non zero on all the cables + struts, by Roth–Whiteley (theorem that inf. rigid tensegrity has nonzero stress). Then choose $\varepsilon > 0$ s.t. $\varepsilon \Omega \otimes I^d + R(p)^T R(p)$ is still positive definite on $\mathcal{E} = (\text{trivials})^\perp$ and so $\varepsilon\omega$ serves as a stabilizing stress for pre-stress stability.

<u>Proposition</u>: If $G(p)$ is pre-stress stable, then it is rigid.

<u>Proof</u>:

So in $\mathcal{E}_p$, $E(q) = \sum_{i<j} \varepsilon_{ij} (|q_i - q_j|)^2$

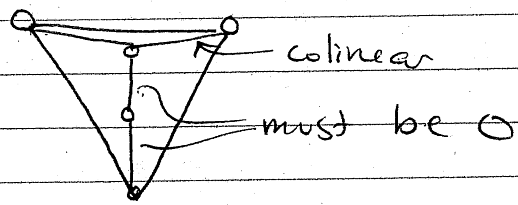
has a strict local minimum.

being a little sloppy here.

Principle of least work applies. So if $G(q) < G(p)$, then $E(q) < E(p)$, $(q-p) < \varepsilon$, then this contradicts local minimum. So $p \cong q$. $q = q$.

So this is an alternate way to show inf. rigid $\Rightarrow$ rigid.

<u>This</u> is the basic underlying geometry to engineering. Can you think of something that is rigid but not pre-stress stable? —

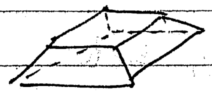


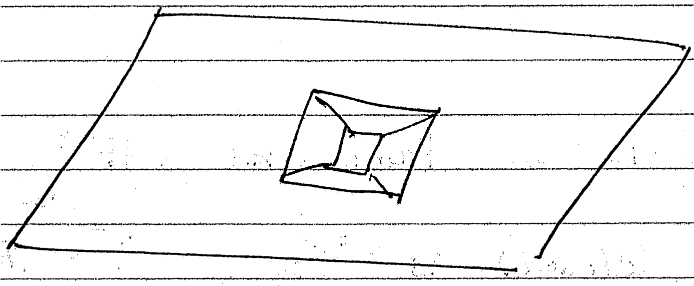
rigid but not prestress stable

April 24

Maxwell Cromona Correspondence

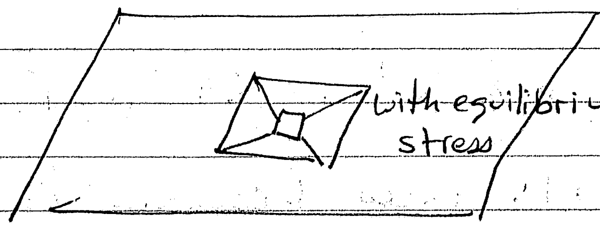
A special case in $\mathbb{E}^3$





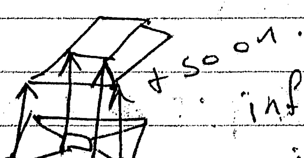
Lift

Correspondence



The correspondence from lift to eq. stress is natural.
Under ~~speci~~ certain assumptions, you can go from eq. stress to a lift.

What are the objects that are concerned here?
What is the correspondence?



inf. motion of the planar surface in $\mathbb{E}^2$

We'll consider triangulated surfaces.
And we'll allow self-intersections.

Infinitesimal motions $\leftrightarrow$ Equilibrium Stresses

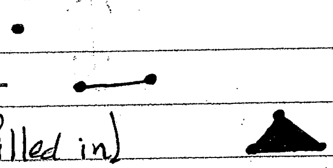
We have already seen duality between inf. motions + stresses.

The Objects

We eventually come to bar frameworks in $\mathbb{E}^3$.

Definitions: A simplex (closed) is $\sigma = \langle p_0, \ldots, p_n \rangle \subset \mathbb{E}^N$ (closed convex hull)
$\dim \sigma = n$, $p_0, \ldots, p_n$ are affine independent.

For us, σ^0 = point.
σ^1 = line segment
σ^2 = triangle (filled in)



A face of a simplex σ^n is $\langle p_{n_1}, \ldots, p_{n_i} \rangle$ (vertices)

A simplex has $2^n + 1$ faces (counting the empty set)

A simplicial complex K = collection of simplices such that:
(1) If $\sigma \in K$, all faces of σ are in K
(2) If $\sigma, \tau \in K$, $\sigma \cap \tau$ is a face of σ + τ.
$\rightarrow$ (3) $\emptyset \in K$

don't need this one necessarily (this just makes sure that K is nonempty)

For a simplicial complex K,
$K^{(i)} = \{ \sigma \in K \mid \dim \sigma \leq i \}$.
For $\sigma \in K$, star of σ
$= \mathrm{star}(\sigma, K) = \{ \tau \in K \mid \tau \supset \sigma \}$
This is not a complex.

~~Define~~ The closed star of σ in K is

$$\overline{st}(\sigma,K) = \{\tau \mid \tau \text{ is a face of } st(\sigma,K)\}$$

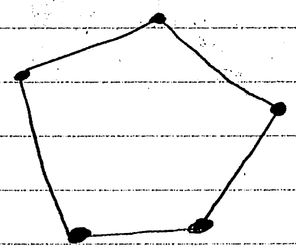
The link of σ in K is

$$lk(\sigma,K) = \overline{st}(\sigma,k) - st(\sigma,K)$$

A 0-sphere is a simplicial complex with just 2 vertices + $\emptyset$.

A 1-sphere is a cycle of 1-simplices + their faces.

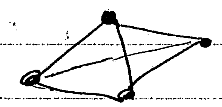
Ex:

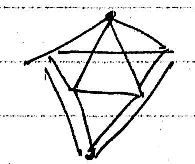


S^1

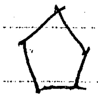
A combinatorial 2-manifold is a 2-dimensional simplicial complex K where for each vertex = 0-simplex v_{ij} $lk(v_{ij},K)$ = 1-sphere.

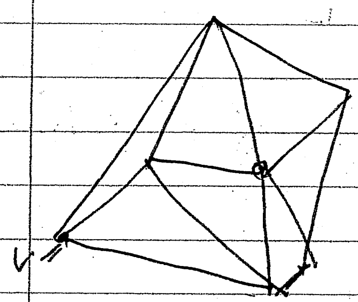
Examples:

①  boundary of 3-simplex

②  boundary of octahedron

③ boundary of regular icosahedron

$lk(v,K) =$ 



= manifold with boundary

$lk(v,K)$ = polygonal arc

2-manifolds are classified topologically by
(1) orientability
(2) Euler characteristic
$$\chi(K) = \sum_{i=0}^{2} (-1)^i f_i$$ where f_i = number of simplices of dimension i

A 2-sphere is an orientable 2-manifold with $\chi = 2$.

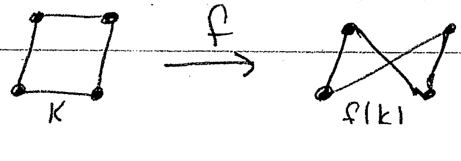
For a simplex $\sigma \subset \mathbb{E}^N$, an affine map $f: \sigma \to \mathbb{E}^d$ is the restriction of an affine linear map from $\mathbb{E}^N \to \mathbb{E}^d$.

Note: If is one-to-one, then $f(\sigma)$ = a simplex in $\mathbb{E}^d$.

Let the underlying space of K be
$$|K| = \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{E}^N.$$

A singular complex is a continuous map $f: K \to \mathbb{E}^d$ such that for every simplex $\sigma \in K$, $f|_\sigma$ is affine linear.

Example:



not a complex

A singular complex is non-degenerate if $f|_\sigma$ is one-to-one.

We consider singular, non-degenerate orientable 2-manifolds in $\mathbb{E}^3$.

For any such singular, non-degenerate 2-manifold, let $p = (p_1, \dots, p_n)$ be the image of the vertices of K. Let the 1-skeleton correspond to the bars of a framework $G(p)$ in $\mathbb{E}^3$.

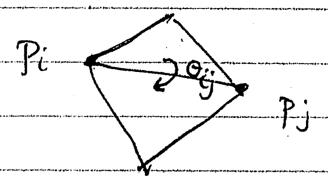
Now let p' be an inf. flex of $G(p)$ in $\mathbb{E}^3$.

Call $IF = \{ p' = (p_1', \dots, p_n')^T \mid p_i' \text{ is an inf. flex of } G(p) \}$

Let $\mathcal{S} = \{ \omega = (\dots, \omega_{ij}, \dots) : \omega R(p) = 0 \}$

The Maxwell-Cremona correspondence is a linear function from $IF \xrightarrow{mc} \mathcal{S}$ such that $\ker MC$ = trivial inf. flexes of the singular complex.

For each 1-simplex $\sigma^1 \leftrightarrow \{i,j\}$ of $G(p)$, let θ_{ij} = oriented dihedral angle for the edge $\{i,j\}$.



Corresponding to any p' is a θ_{ij}'.

Then the stress on $\{i,j\}$ is

$$\omega_{ij} = \theta_{ij}' / |p_i - p_j|$$

Notice that $\omega_{ij} > 0$ if $\theta'_{ij} > 0$

$\omega_{ij} < 0$ if $\theta'_{ij} < 0$

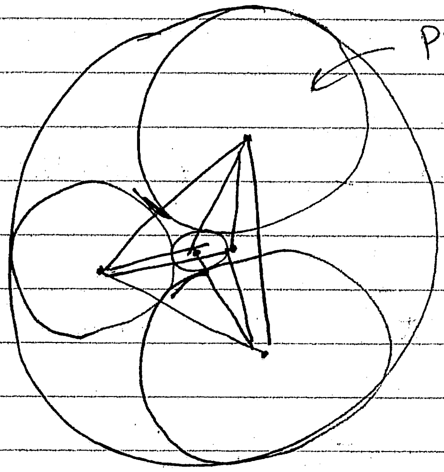
$\omega_{ij} = 0$ if $\theta'_{ij} = 0$.

April 26

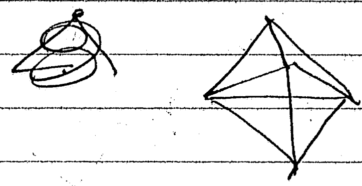
stuff on homework:

M^2 in $\mathbb{E}^3$

$f: K^2 \to \mathbb{E}^3$

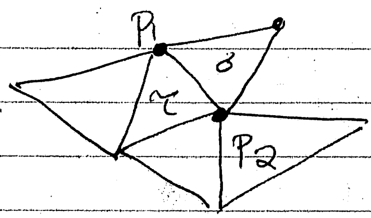


Draw line between all the circles that are tangent.

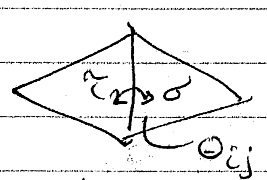


In this case, M^2 is a singular 2-sphere.

We have $p = (p_1, \dots, p_n)$ is the 0-skeleton of our possibly singular, non-degenerate combinatorial oriented 2-manifold M^2.



Let p' be an inf. flex of $G(p)$, the 1-skeleton of M.



orientation determines which θ we take.

Let θ_{ij} be the dihedral angle associated to σ and τ. The inf. motion p' defines a corresponding θ_{ij}'

$p \mapsto p + tp'$

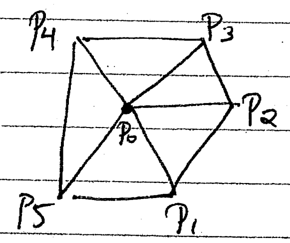
defines $\theta_{ij}(t)$, then $\theta_{ij}' = \frac{d\theta_{ij}(t)}{dt}\Big|_{t=0}$

So we define $w_{ij} = \dfrac{\Theta_{ij}'}{|p_i - p_j|}$

Claim: $w = (\ldots, w_{ij}, \ldots)$ is an equilibrium stress for $G(p)$.

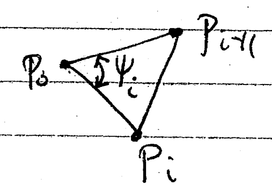
Proof: (of claim)
This proof has to do with beginning part of Lie Algebras.

Look at any vertex p_0 of G.
Let $p_1, \ldots, p_n$ be the adjacent vertices in cyclic order around p_0.



Say $p_0 = 0$
Let ψ_i = the angle between p_i & p_{i+1}

Note: $\psi_i' = 0$.

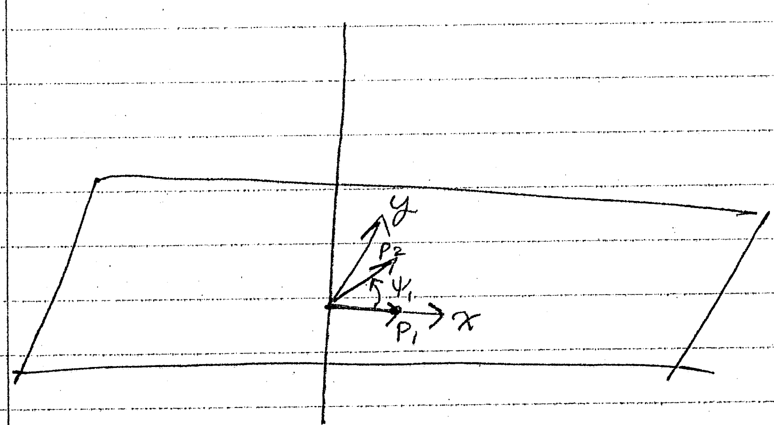


We also will assume that $|p_i| = 1$, $i = 1, \ldots, n$.

Define $B_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Assume by changing by appropriate rigid motion
$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} x_2 \\ y_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\psi_1 \\ \sin\psi_1 \\ 0 \end{bmatrix}$



$B^{-1}_{\psi_1} P_2 = P_1$

then $A^{-1}_{\theta_2} B^{-1}_{\psi_1}$ rotates the whole configuration into a position similar to the original. Now $A_{\theta_2} B_{\psi_1} P_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

and $A^{-1}_{\theta_2} B^{-1}_{\psi_1} P_3 = \begin{bmatrix}\cos\psi_2\\ \sin\psi_2\\ 0\end{bmatrix}$

Then, since the link of a vertex is a cycle, $A^{-1}_{\theta_1} B^{-1}_{\psi_n} \cdots B^{-1}_{\psi_2} A_{\theta_2} B^{-1}_{\psi_1} = I$, or equivalently $A_{\theta_1} B_{\psi_1} A_{\theta_2} B_{\psi_2} \cdots A_{\theta_n} B_{\psi_n} = I$ (or something similar)

We want to take the derivative of this.
The product rule of differentiation applies to matrices.
But remember $B'_{\psi_i} \equiv 0$

$$A'_\theta = \begin{pmatrix}0 & 0 & 0\\ 0 & -\sin\theta & -\cos\theta\\ 0 & \cos\theta & -\sin\theta\end{pmatrix}\theta'$$

So, taking the derivative, we get

$$0 = A'_{\theta_1} B_{\psi_1} A_{\theta_2} \cdots B_{\psi_n} + A_{\theta_1} B_{\psi_1} A'_{\theta_2} \cdots B_{\psi_n} + \cdots + A_{\theta_1} \cdots A_{\theta_n} B_{\psi_n}$$

Each term is of the form $A_{\theta_i} B_{\psi} \cdots \begin{bmatrix}0 & 0 & 0\\ 0 & -\sin\theta_i & -\cos\theta_i\\ 0 & \cos\theta_i & -\sin\theta_i\end{bmatrix} \cdots B_{\psi_n}\ \theta_i'$

after lots of work, we can conclude that

$$\sum_i \theta_i' p_i = 0$$

then $\theta_i' = \omega_{i0}$

That is equivalent to $\displaystyle\sum_{i=1}^{n} \frac{\theta_{i0}'(p_i - p_0)}{|p_i - p_0|} = 0$.

$$\omega_{ij} = \frac{\theta_{ij}'}{|p_i - p_j|}$$

$\square$ (End of proof of claim)

Inf. flexes of general oriented 2-manifold M

$\Updownarrow$

Equilibrium stress on M $\supset$ stresses on 2-sphere

$\cup$

Equilibrium stress on $M \subset \mathbb{E}^2$ $\leftrightarrow$ Reciprical diagrams in $\mathbb{E}^2$

$\cup$

Stresses on $M = S^2$ $\rightsquigarrow$ Lifts of M^2 to $\mathbb{E}^3$