TENSEGRITY STRUCTURES: WHY ARE THEY STABLE?

R. CONNELLY

June 14, 1998

ABSTRACT. A particular definition of stability for tensegrity structures is presented, super stability. This is a stronger case of prestress stability that applies to many examples of tensegrities found in nature.

1. Introduction: The Basic Object.

A basic question in dealing with tensegrity structures is: What are they? Here several definitions are briefly described with some of their basic characteristics, but it is proposed that one particularly strong condition for stability, what has been called, prestress stability, is of central importance. Furthermore, it is claimed that a particular type of prestress stability, what we call super stability, is especially relevant.

There are also several different categories or types of models for tensegrities. The following, though, is a good starting point for the basic object. The question of the stability or rigidity of this object will be dealt with next, and is a basic part of the whole picture.

Start with a finite configuration of labeled points $p_i, i = 1, \ldots, n$ in R^d. This is denoted as

$$p = (p_1, p_2, \ldots, p_n).$$

Define the graph $G = (V, E)$ of the tensegrity as some graph on the set of vertices $V = \{1, 2, \ldots, n\}$, where E is a set of unordered edges (without loops or multiple edges) of G such that each edge is designated as either a *cable*, *strut*, or *bar*. An edge of G is often referred to as a *member*. The whole tensegrity is written as $G(p)$.

The basic general idea of these definitions is:

Cables can shrink in length, but not increase.

Struts can increase in length, but not decrease.

Bars stay the same length.

But the specific form that these conditions take can be quite varied. Figure 1 shows how the tensegrity is denoted graphically.

2. Stability.

The whole point of a tensegrity is what sort of stability is to be considered. There are many inequivalent, but related definitions of rigidity and/or stability. They usually
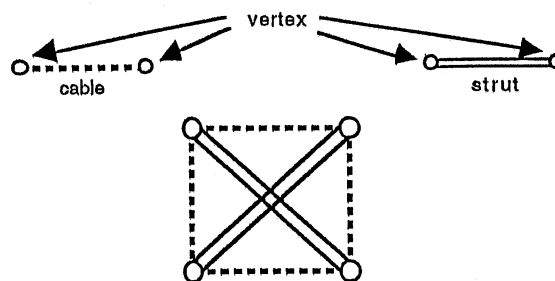
Snelson's X tensegrity. This is one of
the simplist tensegrities.

Figure 1

involve the following rigidity constraints. The configuration p is fixed and we consider the following constraints on other configurations q:

$$|q_i - q_j| \leq |p_i - p_j|$$

for $\{i, j\}$ a cable

$$|q_i - q_j| \geq |p_i - p_j|$$

for $\{i, j\}$ a strut.

-**Rigidity**: Any configuration q, sufficiently close to p and satisfying the constraints is congruent to p.

-**Infinitesimal Rigidity (= Static Rigidity)**: This uses a linearized form of the constraints given by a matrix inequality $R(p)p' \leq 0$. If this has only the minimal trivial solutions, given by derivatives of one parameter families of congruences, $G(p)$ is said to be *infinitesimally rigid*.

-**Generic Rigidity**: This is just a property of the underlying graph, G. If there is some configuration p (with algebraically independent coordinates) such that $G(p)$ is infinitesimally rigid, then G is said to be *generically rigid*.

-**Second Order Rigidity**: This uses the first-order motion p' and asks for an extension p'' satisfying the appropriate equations.

-**Prestress Stability**: For $G(p)$ this means that there are potential functions for each strut and cable, such that the sum over all nearby configurations q has a quadratic local minimum only when q is congruent to p. (This will be discussed in more detail in the next section.)

-**Global Rigidity**: $G(p)$ is *globally rigid* if any configuration q in R^d satisfying the constraints is congruent to p.

-**Super Stability**: $G(p)$ is *super stable* if there is a particular positive semi-definite quadratic form defined on all configurations q, in all dimensions, that is minimized when q is an affine image of p, together with other conditions mentioned below.

3. Prestress Stability.

I claim that the most important notions of stability are prestress stability and super stability, which will be described more carefully.

Start with the "ideal" configuration $p = (p_1, p_2, \ldots, p_n)$ and define real valued potential functions, whose domain is the positive reals

$$f_{i,j} : R^1_+ \to R^1$$

for each $\{i, j\}$ an edge of G.

48

The total potential for any configuration q is

$$E(q) = \sum_{\{i,j\}} f_{i,j}(|q_i - q_j|^2).$$

If E has a non-degenerate local minimum when $q = p$, which is unique up to rigid congruences, then we say $G(p)$ is *prestress stable*.

The $f_{i,j}$ determine the stress-strain characteristics of the cable or strut $\{i,j\}$, and we assume that they all are differentiable and have a strictly positive second derivative where they are defined.

This potential function defines an (equilibrium) stress as follows:

$$\omega_{i,j} = \frac{1}{2} f'_{i,j}(|p_i - p_j|^2).$$

These can be interpreted as tensions (when positive) for the cables and compressions (when negative) for the struts.

Lemma. *When p is a critical point for the potential function E, then the stresses above provide a vector equilibrium at each vertex. In other words the following vector equation holds at each vertex: For each i,*

$$\sum_j \omega_{i,j}(p_i - p_j) = 0.$$

Figure 2 shows some examples of prestress stable tensegrities.

See [1] for a more complete description of prestress stability.

4. The Hessian.

Suppose that $\omega = (\dots, \omega_{i,j}, \dots)$ is an equilibrium stress as above. Define the (symmetric) stress matrix Ω so that the $\{i,j\}$-th entry, when $i \neq j$ is $-\omega_{i,j}$, and so that the row and column sums are 0.

Define the rigidity matrix $R(p)$ as that e-by-dn matrix, where the row corresponding to the edge $\{i,j\}$ has all its entries 0, except for two blocks of entries, the row vector $p_i - p_j$ corresponding to vertex i, and $p_j - p_i$ corresponding to vertex j. The number of struts and cables of G is e, and the number of vertices is n. The ambient space is R^d.

For each cable or strut $\{i,j\}$, define the stiffness coefficient as

$$c_{i,j} = \frac{1}{4} f''_{i,j}(|p_i - p_j|^2),$$

and let D be the e-by-e diagonal matrix, where the $\{i,j\}, \{i,j\}$ entry is $c_{i,j} > 0$

Then a calculation shows that the Hessian of the potential function is the symmetric matrix:

$$H(E) = \Omega \otimes I^d + R(p)^t D R(p),$$

where $()^t$ is the transpose operation and I^d is the d-by-d identity matrix. The symbol $\otimes$ represents the tensor product of two matrices. In this case it repeats the action of Ω on each set of coordinates.

The second term is called the stiffness matrix and it is always positive semi-definite.

Figure 2 shows some examples of prestress matrices. Notice that the Figures 2a is not rigid in 3-space. Its stress matrix is positive semi-definite, but there are realizations of the framework in higher dimensional spaces, where affine transformations serve as flexes.
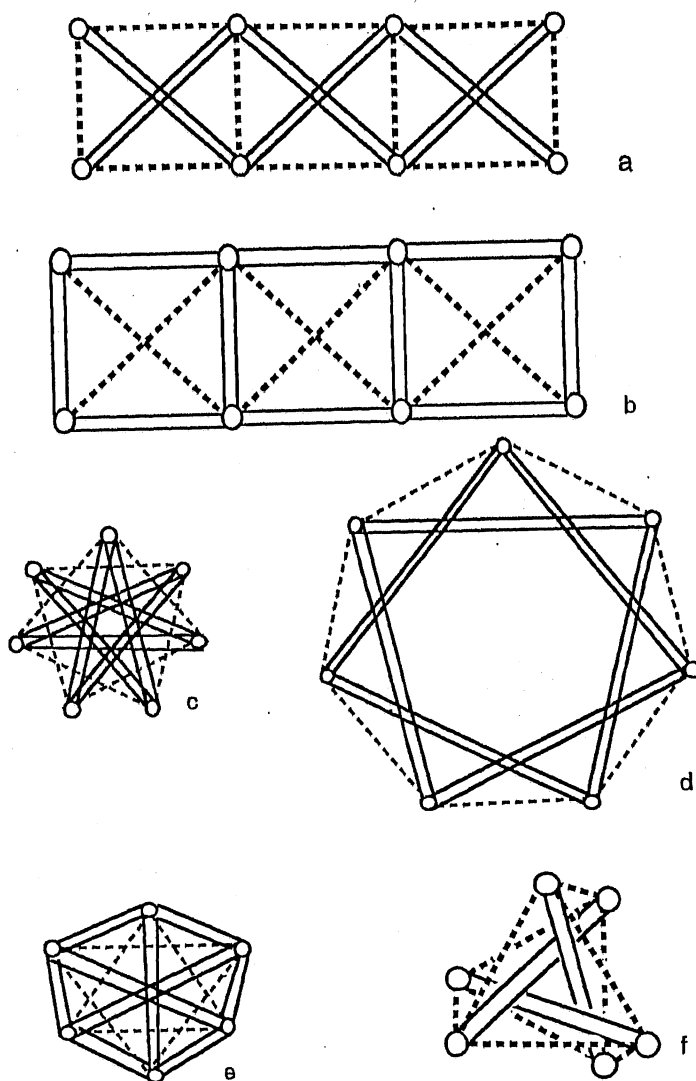
Figure 2

Figures 2a, 2b, 2c, 2d and 2e are all infinitesimally rigid in the plane, and so they are automatically prestress stable, since the stiffness matrix will dominate the quadratic form if the stresses are chosen small enough.

The Snelson octet truss, Figure 2f, has a positive semi-definite stress matrix, but is not infinitesimally rigid in three-space.

5. Super Stability.

Prestress stability has some disadvantages, especially when the stress matrix Ω has some negative eigenvalues. Here it is proposed that a particular sort of prestress stability be considered.

We say the tensegrity framework $G(p)$ with n vertices in R^d is *super stable* if there is an equilibrium stress $\omega = (\dots, \omega_{i,j}, \dots)$ with a stress matrix Ω such that

(1) The stresses $\omega_{i,j}$ are positive for cables and negative for struts.

(2) For $i \neq j$ the (i,j) entry of Ω is $-\omega_{i,j}$, and $(1,1,\dots,1)\Omega = 0$. (This determines the symmetric stress matrix Ω from the stress ω.)

50

(3) As a quadratic form Ω is positive semi-definite.
(4) The rank of Ω is $n - d - 1$ (the largest possible).
(5) There are no affine (infinitesimal) flexes of $G(p)$.

For example for the Snelson X tensegrity, the stress matrix becomes:

$$\Omega = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

which is positive semi-definite of rank 1. Note that if the cables and struts are reversed, the stress matrix is NOT positive definite.

6. Properties and Advantages of Super Stability.

If $G(p)$ is super stable with respect to the equilibrium stress ω in R^d, then the following properties hold:

(1) $G(p)$ is globally rigid in any R^k, for $k \geq d$. That is any other configuration q in R^k that satisfies the distance constraints is congruent to p.
(2) $G(p)$ is prestress stable, stabilized by any positive multiple of the stress w.
(3) If q is any non-singular affine image of p, $G(q)$ is super stable with respect to same equilibrium stress w.
(4) If q is any non-singular projective transformation of p, then there is a suitably altered equilibrium stress ω' (possibly with changes in sign) such that $G'(q)$ is super stable with respect to ω', where G' changes the assignment of struts and cables appropriately.
(5) The subgraph of G determined by the cables alone must be connected.
(6) The graph G determined by struts and cables is vertex $(d+1)$-connected.

Property (2) is the key for many physical situations. Suppose a tensegrity has some given equilibrium stress ω determined by the physics of the materials involved. If the tensegrity remains in a particular configuration for an extended period of time, we can reasonably expect that there is an equilibrium stress ω that forms the prestress in prestress stability. But what happens if ω is increased with no corresponding increase in the stiffness coefficients $\{c_{i,j}\}$? The stress matrix Ω dominates. If Ω has any negative eigenvalue, the structure will not remain at rest, even though it is an equilibrium configuration. The slightest perturbation will cause the structure to change its shape drastically. This can be what influences many physical tensegrities to be super stable. No matter how strong the equilibrium stress is relative to the stiffness coefficients, it will remain stable as long as the members themselves do not fail.

When the structure is super stable, increasing the prestress "stiffens" it. When there is a negative eigenvalue in the stress matrix increasing the prestress will cause a catastrophic failure.

For example, in Figure 2b, the array of squares with struts on the outside can only have an equilibrium stress with a negative eigenvalue. This is despite the fact that the tensegrity itself is infinitesimally rigid in the plane. If the prestress is increased sufficiently with no corresponding increase in the stiffness coefficients $c_{i,j}$, the structure will fail, even when it is constrained to stay in the plane.

On the other hand, the tensegrity in Figure 2a can only have a prestress that has positive eigenvalues. The rank of the stress matrix is low, however, and this prohibits it from being super stable strictly from the definition. It fails the condition (6) above. But it does retain many of the features of a super stable tensengrity. Each one of the squares is super stable and so they must retain their shape, even when they are allowed to move in three-space. Figures 2d and 2f are super stable.

51

But notice that Snelson octet truss in Figure 2f has only three struts and nine cables. In a sense, it is underbraced. For example, in order to be infinitesimally rigid, it is necessary that the total number of members in the graph G be at least one more than $3n - 6 = 12$ for three-space. See for example, Roth and Whiteley [5] for a good discussion of this. There are exactly 12 members, so this count alone implies that the stiffness matrix is of too small a rank to allow prestress stability by itself. But it turns out that the stress matrix has no negative eigenvalues, and so it takes over the stability, even for large stresses. Notice that the underlying graph is the same as the regular octahedron, and it is not infinitesimally rigid since it has an equilibrium stress and a triangulated sphere (such as the octahedron) is not infinitesimally rigid precisely when it does have an equilibrium stress.

One might be tempted to think that the underbraced nature of many tensegrities detracts from their stability, but that is not the case.

Property (5) is helpful in detecting many examples where the stress matrix must have a negative eigenvalue. For example, the tensegrity, Figure 2c, has only the two cable triangles. Imagine moving the two triangles separately, but individually rigidly quite far apart. The quadratic form associated to the stress matrix, must be negative in that configuration. So there must be a negative eigenvalue. When any equilibrium stress (with signs properly chosen for each member) is increased, the structure will become unstable even though it also is infinitesimally rigid.

Property (1) can also be used directly to detect negative eigenvalues. The tensegrity in Figure 2c has the same graph as the tensegrity in Figure 2d, and, indeed, the cables
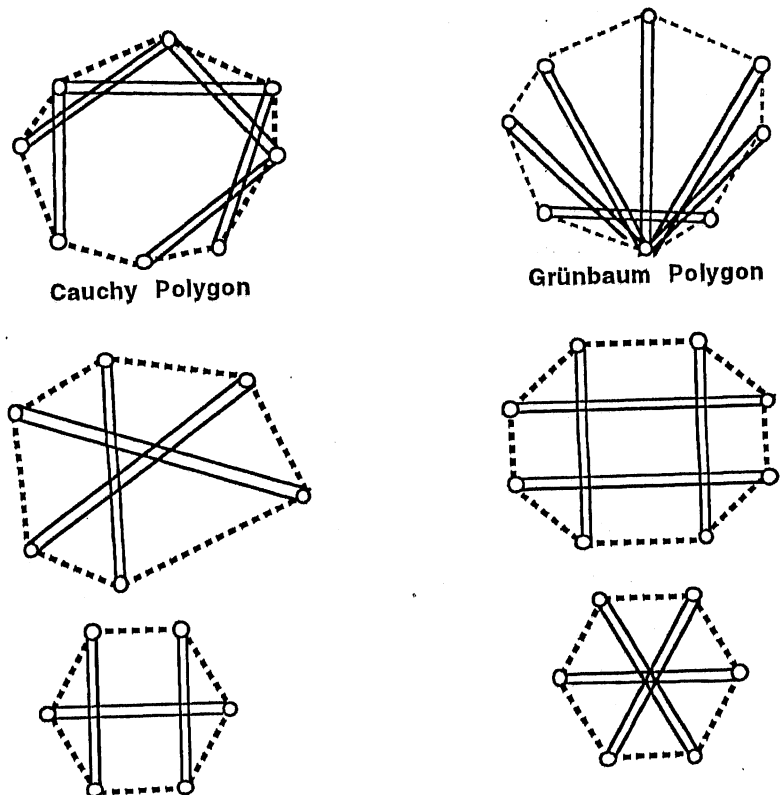

Cauchy Polygon   **Grünbaum Polygon**

Figure 3

in both graphs are the same lengths. But the struts in Figure 2d are longer than those in Figure 2c, so the stress matrix for Figure 2c must have a negative eigenvalue, while tensegrity in Figure 2d is super stable.

It can be expected that there are many instances in nature where super stable tensegrities appear. There are many examples and a strong case is made for the importance of tensegrities in a biological setting in the paper by Ingber [4].

7. Starting Examples for Super Stability.

The next question is: How does one construct super stable tensegrities? One category of super stable tensegrities to start further constructions is the following.

The Polygon Theorem. *Any convex planar polygon with an equilibrium stress (and corresponding stress matrix Ω), that is positive on the external edges and negative (or zero) otherwise, is super stable with respect to Ω.*

(For a proof see R. Connelly, [2])

Figure 3 shows some examples where the Polygon Theorem applies to guarantee super stability. The first two, the Cauchy polygon and Grünbaum polygon are part of a general class of polygons, and they happen to be infinitesimally rigid as well. As long as the vertices of the configuration p are part of a convex polygon, then the combinatorial structure, given by the graph G, will always give a super stable tensegrity $G(p)$. On the other hand, the other polygons in the Figure depend on there being a particular conditions on the configuration, just to insure that there is a non-zero equilibrium stress. For example, the polygon on the middle left must have the vertices lie on an a conic (in addition to being convex). The other three polygons are drawn as regular polygons.

Another source of examples is the catalogue of highly symmetric super stable tensegrities developed with Allen Back. See Connelly, Back [3]. See also our website at

http://math.lab.cit.cornell.edu/visualization/tenseg/tenseg.html

8. Combining by Superposition.

Start with two different tensegrities, each with its own equilibrium stress, and superimpose some of their vertices. The combined stress will be the sum of the stress matrices, regarded as quadratic forms. It can even turn out that some of the stresses may cancel, eliminating it as a cable or strut.
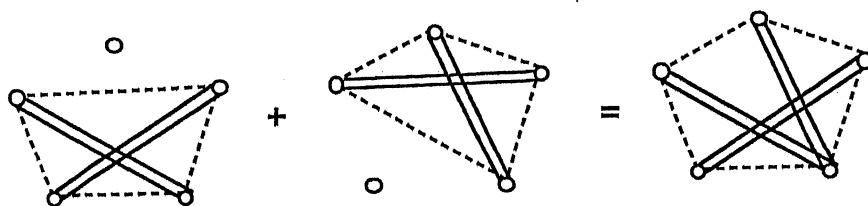
Figure 4 shows an example:

Figure 4

Note that when one uses this technique and adds the stress matrices, each subgraph, where each stress is non-zero, must overlap with the others in such a way that the whole configuration is forced to be in the appropriate dimensional Euclidean space. In the example of Figure 2a, each of the smaller squares is forced to lie in a two-dimensional space, but two successive squares have only two vertices in common. So there is a "hinge" where the whole configuration can rotate into higher dimensions.

Figure 5 shows some examples where the same unit is repeated with the effect that there is an extended truss. The top and bottom frameworks are super stable with the
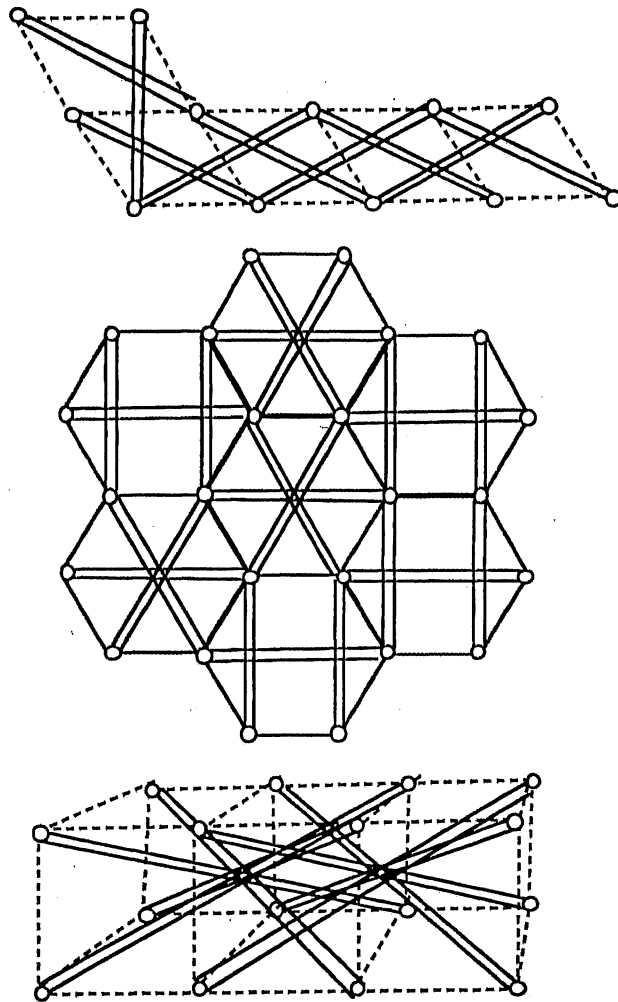
Figure 5

sum of stresses on the appropriate units as the stabilizing stress. For the middle example, if we take as the stabilizing stress the sum of the stresses on each hexagon, the rank of the stress matrix will be one less than what is required in condition (4) of Section 5. Nevertheless, in all the cases of Figure 5, the configuration is such that it is forced to stay rigid in the plane even considering it as being in three-space.

REFERENCES

1. R. Connelly and W. Whiteley, Second-Order Rigidity and Prestress Stability for Tensegrity Frameworks 9 (1996), no. 3, 453–491.
2. R. Connelly, *Rigidity and Energy*, Invent. Math. 66 (1982), 11–33.
3. R. Connelly and A. Back, *Mathematics and Tensegrity*, American Scientist, March-April (1998), 142–151.
4. D. Ingber, *The Architecture of Life*, Scientific American, January (1998), 48–57.
5. B. Roth and W. Whiteley, *Tensegrity Frameworks*, Trans. Amer. Math. Soc. 265 (1981), 419–446.