ADVANCES IN MATHEMATICS 37, 272–299 (1980)

# The Rigidity of Certain Cabled Frameworks and the Second-Order Rigidity of Arbitrarily Triangulated Convex Surfaces

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## 1. INTRODUCTION

When is a triangulated polyhedral surface rigid? In 1813 Cauchy [4] showed the following:

THEOREM C. *If each natural face of a convex polyhedral surface is held rigid, then the whole surface is rigid.*

It had been suspected that any (connected) polyhedral surface, convex or not, (with its triangular faces, say, held rigid) was rigid, but this has turned out to be false (see Connelly [7]). We extend Cauchy's theorem to show that any convex polyhedral surface, no matter how it is triangulated, is rigid. Note that vertices are allowed in the relative interior of the natural faces and edges.

Here we say a triangulated surface in three-space is rigid if any continuous deformation of the surface that keeps the distance fixed between any pair of points in each triangle (a part of the triangulation of the surface), keeps the distance fixed between any pair of points on the surface (and thus extends to an isometry of three-space). A natural face of a convex polyhedral surface is the two dimensional intersection of a support plane with the surface (see Fig. 3). Similarly natural edges and vertices are one- and zero-dimensional intersections, respectively. Thus in Cauchy's theorem it is insisted that under a deformation of the surface the distance is fixed between any pair of points of a natural face and not just some triangle of a triangulation. Alexandrov [1] showed that if a convex polyhedral surface is triangulated with no vertices in the relative interior of a natural face (but they are allowed in natural edges), then the surface is rigid. This theorem is the basic result for the results of this paper. The author is grateful for a description of Alexandrov's proof by Asimow and Roth [5]. Whiteley [15] also has a somewhat

* Partially supported by an NSF grant.



0001-8708/80/090272–28$05.00/0

different proof of Alexandrov's theorem in which he derives our Theorem 5.1 as a corollary.

However, it does not obviously follow from Alexandrov's result that if vertices are allowed in the interior of natural faces then the surface is rigid. In fact Alexandrov's theorem precisely is the following:

THEOREM A. *If a convex polyhedral surface is triangulated with vertices only in the natural edges, then the surface is infinitesimally rigid, which implies it is rigid.*

(Infinitesimally rigidity will be defined later.)

This is a sharpened version of a theorem by Max Dehn on the infinitesimal rigidity of such surfaces [8]. (See also Gluck [10], for a nice proof of Dehn's result.)

If we call the "weak" Alexandrov theorem the case when in Theorem A there are no other vertices than the natural vertices in the triangulation, and the "strong" Alexandrov theorem as above, then it turns out by techniques in this paper we can show how the weak version implies the strong.

When vertices are in the relative interior of a natural face, then the surface is definitely not infinitesimally rigid. It was conceivable that there was a deformation (flex) of an appropriately triangulated convex surface that immediately moved the surface into a non-convex shape, and this would not contradict any known theorems.

We extend the notion of infinitesimal rigidity to second-order rigidity and show that second-order rigidity implies rigidity (as does first-order or infinitesimal rigidity). The basic theorem is the following:

Theorem 6.1. *Any triangulated convex surface is second-order rigid and thus rigid.*

Ironically the main ingredient here is Alexandrov's Theorem A applied twice.

As far as we know this is the first it is known that a polyhedral surface is rigid no matter how it is triangulated. Even for a tetrahedron it was not known whether for some clever triangulation it would be flexible, moving immediately into some non-convex shape.

As a consequence of the techniques developed here we also show that if one removes a convex polygonal hole from the interior of some of the natural faces of a convex polyhedral surface, where the boundary of each hole touches the boundary of each natural face in at most one point, which is a natural vertex, then any triangulation of this surface "with boundary" is still second-order rigid (Theorem 6.2).

Along the way we also develop techniques for showing the rigidity of certain cabled "frameworks." A framework is simply a collection of points in three-space (the vertices) together with two collections of unordered pairs of vertices called "rods" and "cables." A flex (or deformation) of a framework is just a motion (in time) of the vertices so that the vertices of any rod stay at a fixed distance, and the vertices of any cable do not increase in distance. Thus we show that if we take as vertices the natural vertices of a convex polyhedral surface, as rods the natural edges of this surface, and a certain system of cables in each natural face, then this framework is *rigid.* That is, any flex of this framework does not change any distance between any pair of vertices and thus is the restriction of a rigid motion of all of three-space. In fact we show a bit more (see Whiteley [15], also):

THEOREM 5.1. *Let $\mathscr{F}$ be a cabled framework obtained as follows: The vertices of $\mathscr{F}$ are the natural vertices of a convex surface. The rods of $\mathscr{F}$ are the natural edges of this surface. The cables of $\mathscr{F}$ are any system of cables in each natural face such that the vertices, rods, and cables of each natural face form an infinitesimally rigid framework in the plane of that face. Then $\mathscr{F}$ is infinitesimally rigid (see Fig.* 14). (*We will also define infinitesimally rigid cabled frameworks later.*)

There are many ways of cabling a convex polygon in the plane to make it infinitesimally rigid, so the theorem above applies to many cabled frameworks.

We must mention that we owe a great deal not only to the works of Alexandrov [1] and Gluck [10], but also to a set of notes "Lectures in Lost Mathematics" by Grünbaum [11]. These notes define the notion of cabled frameworks, conjecture the theorems about convex triangulated surfaces, and generally serve as the inspiration of this paper. These notes also have many interesting and illuminating examples and pictures which were very helpful.

Lastly we point out that there is a strong parallel between the results and ideas of this paper and the corresponding situation in the smooth (of class $C^3$ or $C^\infty$ say) category as discussed by Efimov in [9]. For instance Efimov defines second-order rigidity for smooth surfaces and shows that second-order rigidity implies rigidity but only for *analytic* deformations. (Note in the piecewise-linear case a deformation is essentially automatically analytic so the theory turns out to be a bit nicer.) Even our result about cabled frameworks is analogous to Efimov's result about sliced ovaloids where, in the holes sliced out, nearby points are not allowed to move apart. Also our results about the rigidity of convex polyhedral surfaces with holes removed seem to parallel results about the rigidity of similar smooth surfaces. Nevertheless, there does not seem to be any obvious way to deduce the results of one category from the other, despite their similarity.

We give a brief outline of the following sections. Section 2 defines infinitesimal rigidity and gives a brief discussion of different definitions of rigidity. Section 3 introduces second-order rigidity and shows that it or first-order rigidity imply rigidity. Section 4 shows that certain cabled frameworks in the plane are infinitesimally rigid and thus rigid. Section 5 shows that the cabled frameworks mentioned before, associated to a convex polyhedral surface, are infinitesimally rigid. Note that here Alexandrov's Theorem A is used the first time. Section 6 applies the result of Section 5 to show that convex surfaces are second-order rigid. Note this uses Alexandrov's theorem the second time. Also if the reader is willing to look at a bit more detail, we generalize Theorem 6.1 to show the result about convex surfaces with convex polygonal holes removed. (This is Theorem 6.2.)

## 2. Rigidity and Infinitesimal Rigidity

In order to discuss the rigidity of surfaces and other gadgets we define the notion of a framework in $m$-space. Let $p_1, p_2, \dots, p_V$ be $V$ points in $\mathbb{R}^m$, possibly not distinct. Let $\mathscr{E}$ be a collection of unordered pairs of these points. We call $\mathscr{E}$ the *rods* and $p = (p_1, \dots, p_V) \in \mathbb{R}^{mV}$ the *vertices* of a *framework*, $\mathscr{F}$. Note here we often regard the ordered collection of vertices simply as a single $mV$ vector in $\mathbb{R}^{mV}$. For a triangulated surface, the vertices of the triangulation are the vertices of the associated framework, and the edges of the triangulation are the rods of the associated framework. Often it is convenient to regard a rod as the unordered pair of indices $\{i, j\}$ rather than $\{p_i, p_j\}$. Figure 1 is a picture of one such framework.

As indicated in the introduction we say a *deformation* or *flex* of a framework $\mathscr{F}$ is simply a continuous path, for $0 \leqslant t \leqslant 1$, $p(t) =$

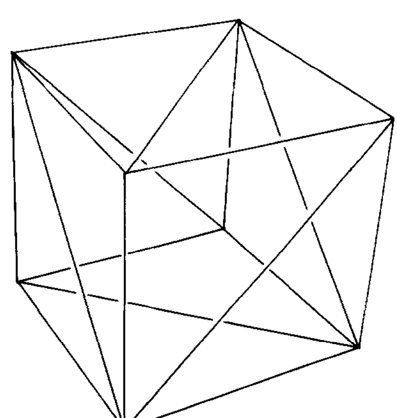

Figure 1

$(p_1(t),..., p_V(t))$ such that the length of any rod $|p_i(t) - p_j(t)|$ $\{i, j\} \in \mathscr{E}$, is constant in $t$, and $p(0) = p$.

We say a flex $p(t)$ of $\mathscr{F}$ is *trivial* if for all pairs of vertices, the length $|p_i(t) - p_j(t)|$ is constant. We leave it to the reader to check that this is equivalent to finding a continuous path $T_t$, $0 \leqslant t \leqslant 1$, in the (Lie) group of rigid motions of $\mathbb{R}^n$ such that $p(t) = T_t p = (T_t p_1,..., T_t p_V)$.

We say a framework $\mathscr{F}$ is *rigid* if $\mathscr{F}$ admits only trivial flexes. The reader is referred to Gluck [10] for a discussion of these definitions as well as some examples. The following is a convenient alternate definition of rigidity following Gluck [10]:

Let $E$ be the number of rods. Define a map (the *rigidity map*) $f$: $\mathbb{R}^{mV} \to \mathbb{R}^E$ by $f(p_1,..., p_V) = (..., (p_i - p_j) \cdot (p_i - p_j),...)$. (Here $\cdot$ is the usual dot product).

THEOREM 2.1 (Gluck). *Let $m = 3$. Suppose the vertices of $p$ are not colinear. Then $\mathscr{F}$ is rigid if and only if the dimension of the component of $p$ in $f^{-1}f(p)$ is 6 (the dimension of the group of rigid motions of $\mathbb{R}^3$). (See Asimow and Roth [4] also for a discussion of this type of result.)*

By including a few more equations it is possible to define a new map $\bar{f}$ and say $\mathscr{F}$ is rigid if and only if dim $\bar{f}^{-1}\bar{f}$ is 0 in a neighborhood of $p$. This is an idea of Kuiper [12] and is outlined in the Appendix.

We now define the notion of infinitesimal rigidity. It is convenient notationally to define a map $R$: $\mathbb{R}^{mV} \times \mathbb{R}^{mV} \to \mathbb{R}^E$ by $R(p, q) = (..., (p_i - p_j) \cdot (q_i - q_j),...)$, where $p = (p_1,..., p_V)$, $q = (q_1,..., q_V)$, each $p_i, q_i$ is an $m$-vector in $\mathbb{R}^m$, and the dot product above appears in the $\{i, j\}$th slot corresponding to a typical rod in some framework $\mathscr{F}$ with $V$ vertices. Thus in the previous notation $f(p) = R(p, p)$, and the differential

$$df_p(q) = 2R(p, q).$$

We also regard $R(p)$ as the matrix defined by $R(p)(q) = R(p, q)$ (with respect to the standard basis say). Then $df_p = 2R(p)$. We call $R(p)$ the *rigidity matrix* (corresponding to a framework with vertices $p$, and rods $\mathscr{E}$).

Let $\mathscr{F}$ be a framework with vertices $p \in \mathbb{R}^{mV}$ and rods $\mathscr{E}$. We say a vector $p' \in \mathbb{R}^{mV}$ is an *infinitesimal deformation* (or *infinitesimal flex*) of $\mathscr{F}$ if $R(p, p') = 0$. Thus writing $p' = (p'_1,..., p'_i,..., p'_V)$

$$(p_i - p_j) \cdot (p'_i - p'_j) = 0$$

for each $\{i, j\}$ a rod of the framework. Intuitively we regard $p$ as a vector field defined on each vertex of $\mathscr{F}$, and the above condition simply says that the lengths of rods do not change infinitesimally, since $(p_i - p_j)^2$ is the length squared of the $\{i, j\}$th rod.

We say the infinitesimal flex $p'$ is *trivial* if $p' = Dp(0)$ where $T_t$, $0 \leqslant t \leqslant 1$, is a path in the group of rigid motions of $\mathbb{R}^m$, and $D$ is differentiation with respect to $t$. For instance, in the case $m = 3$, $p'$ takes the form $p'_i = \mathbf{t} + \mathbf{r} \times p_i$, where $\mathbf{t}$ and $\mathbf{r}$ are fixed vectors in $\mathbb{R}^3$. $\mathbf{t}$ can be regarded as an infinitesimal translation, and $\mathbf{r}$ an infinitesimal rotation, where $\times$ is the cross product. Note that this group of infinitesimal flexes is six dimensional.

We say a framework $\mathscr{F}$ with vertices $p$ is *infinitesimally rigid* if there are only trivial infinitesimal flexes of $\mathscr{F}$.

*Remark* 2.1. If a framework $\mathscr{F}$ is infinitesimally rigid, then it is rigid. The proof of this in Gluck [10], for instance (see also Asimow and Roth [5]), is to show that the rank of the rigidity matrix $R(p)$ is "too small," if there is a non-trivial flex, and thus there is a vector $p'$ in the kernel that is not a trivial infinitesimal flex. For instance when $m = 3$, the vertices of $p$ are not colinear, and $\mathscr{F}$ is flexible, by Theorem 2.1, the dimension of $f^{-1}f(p)$ near $p$ is greater than 6, and so the implicit function theorem says that the rank of $df_p = 2R(p)$ is less than $3V - 6$. (Here say $E \geqslant 3V - 6$.) Thus there is a vector $p'$ in the kernel of $R(p)$ that is not in the six-dimensional space of trivial infinitesimal flexes.

Intuitively, however, it seems to us to be more natural just to take $p' = Dp(0) = p'(0)$, where $p(t)$ is some non-trivial flex of $\mathscr{F}$. Unfortunately this does not always work. For instance we could replace a $p(t)$ that does "work" (so that $p'(0)$ is not trivial) by $\bar{p}(t) = p(t^2)$. So $\bar{p}'(0) = 0$. In the next section we see how to avoid this problem and use this second idea which is very useful for second-order rigidity.

The reader is referred to Asimow and Roth [5] for examples of infinitesimally rigid frameworks. We mention only a few here.

EXAMPLE 2.1. Any affine independent set of points in $\mathbb{R}^m$ with all possible rods is infinitesimally rigid. For example, a triangle in the plane or three-space, or a tetrahedron in three-space Fig. 2).

EXAMPLE 2.2. Recall from Theorem A of the introduction that any triangulated convex surface with no vertices in the interior of a natural face is infinitesimally rigid (see Fig. 3).

Next we define second-order rigidity. Let $\mathscr{F}$ be a framework with vertices

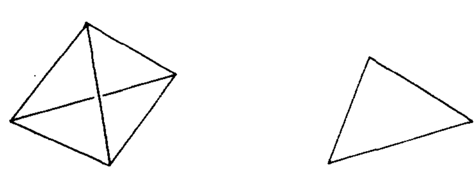

FIGURE 2

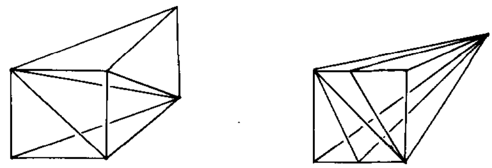

FIGURE 3

$p$. Let $p'$ be an infinitesimal flex of $\mathscr{F}$. Then for another vector $p'' \in \mathbb{R}^{3V}$ we say $(p', p'')$ is a *second-order deformation* (or a *second-order flex*) for $\mathscr{F}$ if

$$R(p', p'') + R(p, p'') = 0. \qquad (*)$$

We say $\mathscr{F}$ is *second-order rigid* if for every second-order flex $(p', p'')$, the first-order flex $p'$ is trivial.

*Remark* 2.2. If $\mathscr{F}$ is second-order rigid, then it is rigid. However we do not see how to use the idea of Gluck's proof (actually due more to Alexandrov). Instead we use the more natural idea indicated in Remark 2.1. One price we pay for this, perhaps, is that we do not see how to extend the notion to higher-order rigidity so that $n$th order rigidity implies rigidity. Note Efimov [9] has the same problem. But fortunately this is enough for Theorem 6.1.

*Remark* 2.3. The proof of the infinitesimal rigidity of triangulated convex surfaces in Gluck [10] unfortunately only applies to the case when each natural face is a triangle. This is enough for Gluck's main theorem, but it seems more work must be done to show the stronger result of Theorem A, which we need.

## 3. SECOND-ORDER RIGIDITY

Here we investigate the notion of higher-order rigidity. Our main purpose is to show that second-order rigidity implies rigidity. To do this we generalize the notion of a first-order flex to an $n$th-order flex. An $n$th-*order flex* of an $mV$ vector $p$ of a framework is a sequence of $n$, $mV$ vectors, $p', p'', ..., p^{(n)}$ such that

$$\sum_{i=0}^{k} \binom{k}{i} R(p^{(i)}, p^{(k-i)}) = 0 \qquad k = 1, ..., n. \qquad (**)$$

This is motivated by differentiating the equation $R(p, p) = \text{constant}$ $n$ times via Leibniz's rule, where we regard $p = p(t)$ as a function of the parameter $t$

and $p^{(i)}$ is the $i$th derivative of $p(t)$. However, formally the $p^{(i)}$'s are only some $mV$ vectors satisfying (**). We say the $n$th-order flex of $\mathscr{F}$, $p',\ldots, p^{(n)}$ is *trivial* if there is a path $T_t$, in the group of rigid motions of $\mathbb{R}^m$ such that if $T_t(p) = (T_t p_1,\ldots, T_t p_V)$ then

$$D^k T_t p(0) = p^{(k)}, \qquad k = 0, 1,\ldots, n.$$

We say the framework $\mathscr{F}$ is *nth-order rigid* if for every $n$th-order flex $(p',\ldots, p^{(n)})$ of $\mathscr{F}$, $p'$ is trivial (see Remark 2.1).

The following are some examples of $n$th-order rigid frameworks for $n = 1, 2,\ldots$. They are typical of the kind of phenomenon to come.

EXAMPLE 3.1. In $\mathbb{R}^2$, an arc, with one vertex in the middle, is not first-order rigid, where the ends are held fixed (Fig. 4).

We can push the center vertex perpendicular to the line on which it sits keeping the end points fixed. However if there is to be a second-order flex we are stuck, since from (*) $(p_2')^2 + (p_2'' - p_1'') \cdot (p_2 - p_1) = 0$ hence $p_2'' \cdot (p_2 - p_1) < 0$ and similarly $p_2'' \cdot (p_2 - p_3) < 0$, but $(p_2 - p_1) = -\lambda(p_2 - p_3)$, $\lambda > 0$ and we have a contradiction. Thus this is second-order rigid.

EXAMPLE 3.2. See Fig. 5. Here there are non-trivial first and second-order flexes, but no third order. Namely, we can push $p_2$ by $p_2'$ to the first-order perpendicular to $(p_1 - p_3)$, but all other $p_i'$'s are 0. Then by choosing $p_1''$ and $p_3''$, both pointing toward $p_2 (p_2'' = 0$, say), appropriately, we get a second-order flex. This is third-order rigid since $p_3''$ and $p_1''$ are 0 if there is to be a third-order flex from the above. So $p_2'$ is then also 0 from the above. Thus all the $p_i' = 0$, and thus it is third-order rigid.

It is easy to see how to continue this construction for any $n$ to get a structure that is $n$th-order rigid, but not $n-1$-order rigid in the plane.

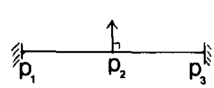


FIGURE 4

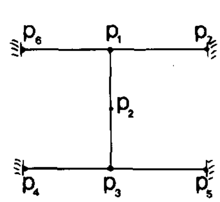


FIGURE 5

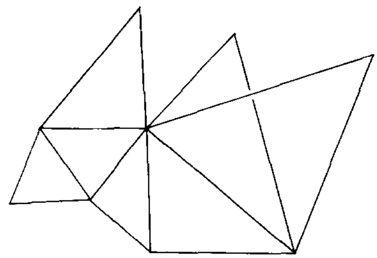

FIGURE 6

EXAMPLE 3.3. See Fig. 6. Any collection of non-degenerate triangles stuck together along common edges will be first-order rigid in $\mathbb{R}^2$. In fact such a "truss" can be used to keep the stationary vertices of Examples 3.1 and 3.2 fixed.

EXAMPLE 3.4. The edges of a tetrahedron in $\mathbb{R}^3$ when the vertices do not lie in a plane are easily seen to describe an infinitesimally rigid framework (Fig. 7). However, if we do allow them to lie in some plane then it becomes infinitesimally flexible as we can see by pushing one point perpendicular to the plane of the other three. However we shall see that this framework is still second-order rigid.

In the following we replace an $n$th-order flex, trivial up to some order $k$ by another $n$th-order flex 0 up to the order $k$. The approach suggested by N. H. Kuiper is possible also. See the Appendix.

*Remark* 3.1. If $D^k(p_i - p_j)^2(0) \neq 0$, for some $i, j$ then the derivatives of $p(t)$ cannot generate a *trivial* $k$th-order flex of any framework, because $(T_t p_i(0) - T_t p_i(0))^2 = (p_i(0) - p_j(0))^2$ for any rigid motion $T_t$.

THEOREM 3.1. *If a framework $\mathscr{F}$ is rigid of order* 1 *or* 2, *then $\mathscr{F}$ is rigid.*

*Proof.* The case $n = 1$ is well known (see Gluck [10]), but for completeness we prove it here as well from our point of view. Suppose $\mathscr{F}$ is not rigid, i.e., $\mathscr{F}$ is flexible. By Lemma 18.3 of Wallace [14], or by Lemma 3.1 of Milnor [13], there is an analytic path $p(t)$ with $p(0) = p$ such

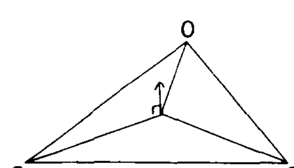


FIGURE 7

that $R(p(t), p(t)) = R(p, p)$ for all $t$, where $p = (p_1, \ldots, p_V)$ are the vertices of $\mathscr{F}$ as usual. If $p(t)$ is not a trivial flex there must be some first $k'$ for which the $k'$th derivative of the square of the length between some pair of points $p_i(t)$, $p_j(t)$ changes. Thus there is a path $T_t$ in the group of rigid motions of $R^n$ such that $D^j p(0) = D^j T_t p(0)$, $j = 1, \ldots, k-1$, for $k \leqslant k', k$ as large as possible. So by replacing $p(t)$ by $T_t^{-1} p(t)$ (or by using Kuiper's formula in the Appendix) we may assume $p^{(j)}(0) = 0$ for $j = 1, \ldots, k-1$. Then by (**) $p^{(k)}(0)$ satisfies $R(p, p^{(k)}(0)) = 0$ and by assumption $p^{(k)}(0)$ is not trivial as a $k$th-order flex. But it is also not trivial as a first-order flex, since if there is a $T_t$ such that $DT_t p(0) = p^{(k-1)}(0)$, then $D^j T_{(t^k/k!)}(0) = p^{(j)}(0)$, $j = 0, \ldots, k$. Thus $\mathscr{F}$ is not first-order rigid. (Note that $p$ may lie on a hyperplane, but if it does not, it turns out that $k = k'$.)

Next we must show $\mathscr{F}$ is not second-order rigid, continuing from above, assuming again we have a flex $p(t)$. Define $\bar{p}_n = p^{(n)}/n!$, for $n = 0, 1, \ldots$, so $p(t) = \sum_{n=0}^{\infty} \bar{p}_n t^n$. Here $p^{(k)}(0)$ and thus $\bar{p}_k$ is non-trivial as a first order flex. We set $q' = \bar{p}_k = p^{(k)}(0)/k!$ and we must find a vector to serve as $q''$ to satisfy the second equation of (**), i.e., $R(q', q') + R(p, q'') = 0$. Since $p'(0) = \cdots = p^{(k-1)} = \bar{p}_1 = \cdots = \bar{p}_{k-1} = 0$, looking at the $2k$th coefficient of the power series of $t$ we get

$$0 = \sum_{i=0}^{2k} R(\bar{p}_i, \bar{p}_{2k-i}) = R(\bar{p}_0, \bar{p}_{2k}) + R(\bar{p}_k, \bar{p}_k) + R(\bar{p}_{2k}, \bar{p}_0)$$

thus

$$0 = R(p, 2\bar{p}_{2k}) + R(\bar{p}_k, \bar{p}_k).$$

Set $q'' = 2p_{2k} = (2/(2k)!)\, p^{(2k)}(0)$. Thus $q', q''$ serves as a second-order flex and $q'$ is non-trivial. Thus second-order rigidity implies rigidity and we are done.

*Remark* 3.2. It would be interesting to know if, by any chance, third-order (or higher-order) rigidity implied rigidity also. The above proof breaks down, since, in the Leibnitz formula, beyond the $2k$ level, more cross terms enter and make the problem more difficult.

It might be thought that we can somehow reparametrize the flex $p(t)$ so that $p'(0)$ is not trivial. Perhaps this is possible for our situation but we should be reminded of the following example:

Consider the curve defined in the plane by $y^2 = x^3$. This has the "analytic" parametrization $y = t^3$ and $x = t^2$. So if $p(t) = (t^2, t^3)$, $p'(0) = 0$ and this is true for any such analytic parametrization. Thus perhaps it may be useful to work with some other definition of $n$th-order rigidity for $n \geqslant 3$.

We thank M. Gromov for pointing out to us that a reasonable converse is, however, true. Namely, Theorem 6.1 of Artin [3] (see also [2] for helpful comments) shows that if $\mathscr{F}$ is rigid then it is $n$th-order rigid for some $n$ (using the definitions above).

## 4. Cabled Frameworks and First Order Rigidity

In order to proceed with the discussion of rigidity of frameworks, it is necessary to generalize the notion slightly to "tensed cabled" frameworks or simply *cabled frameworks*, a notion from Branko Grünbaum's notes. Here we have a finite number of points $p_1, \ldots, p_V$ in $\mathbb{R}^n$ and a collection of unordered pairs of points $\mathscr{E}$ as before. However, we also give ourselves another collection of unordered pairs of points $\mathscr{E}_-$, which we call *cables*. The first collection $\mathscr{E}$ we still call *rods*. Here a flex $p(t)$ will not permit any of the rods to change length, but it must also not permit the cables to *increase* length.

Example 4.1. See Fig. 8. As before, we say $p(t)$ is trivial if there is a rigid motion $T_t$ of $\mathbb{R}^n$ such that $T_t p = p(t)$.

We shall mostly be concerned with flexes in the plane ($m = 2$), and, as before, we define the notion of an infinitesimal flex $p'$. $p'$ is an *infinitesimal flex* if

$$\begin{aligned} (p_i' - p_j') \cdot (p_i - p_j) &= 0 \qquad \text{if} \quad \{i, j\} \in \mathscr{E}, \\ (p_i' - p_j') \cdot (p_i - p_j) &\leqslant 0 \qquad \text{if} \quad \{i, j\} \in \mathscr{E}_-. \end{aligned} \tag{***}$$

$p'$ is *trivial* if it is trivial as a flex of $p$ as a rod framework. (Note the notion of triviality is independent of which pairs of vertices are rods.)

As before we say $\mathscr{F}$ is *rigid* if it only has trivial flexes, and $\mathscr{F}$ is *infinitesimally rigid* if it only has trivial infinitesimal flexes.

*Remark* 4.1. Note that the argument of Theorem 3.1 in the previous section implies that if $\mathscr{F}$, a cabled framework, is infinitesimally rigid, then it is rigid. The only problem that might arise is when we apply Wallace's lemma to obtain the analytic path. But the cabling conditions with inequalities still define a real variety (or algebraic set). This can be seen by

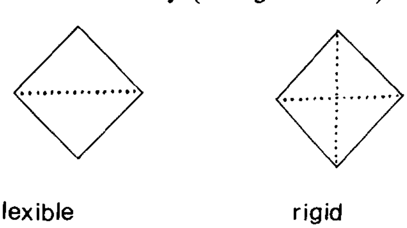


Fig. 8. The solid lines are rods, the dotted lines cables. $n = 2$.

introducing another variable $y_{ij}$ say for each inequality and replacing the cable conditions $|x_i - x_j|^2 \leqslant p_i - p_j|^2$ with $|x_i - x_j|^2 + y_{ij}^2 = |p_i - p_j|^2$. Thus Wallace's lemma still applies, and we obtain the analytic path $p(t)$ as before. The rest of the proof is the same. Alternatively, Milnor's statement in [13] applies.

Here we give an example of a tensed cable framework that is infinitesimally rigid. This is the infinitesimal analogue of an observation due to Grünbaum.

THEOREM 4.1. *Let $(p_1, ..., p_V) = p$ be the natural vertices of a framework $\mathscr{F}$ corresponding to a convex polygon in $\mathbb{R}^2$ with $\langle p_i, p_{i+1} \rangle$ $i = 1, 2, ..., V$ as rods and $\langle p_i, p_{i+2} \rangle$ $i = 1, 2, ..., V-2$ as cables, where $\langle p_i, p_{i+1} \rangle$ are the natural edges. Then $\mathscr{F}$ is infinitesimally rigid* (Fig. 9).

*Remark* 4.2. Grünbaum correctly points out that the fact that $\mathscr{F}$ is rigid follows from any of a number of standard sources, but we need the infinitesimal version here.

We copy the approach used by Cauchy for the infinitesimal version of this key result.

*Proof.* We proceed by induction on $V$, the number of vertices of $p$, starting with $V = 3$ where it is obvious. We make a slightly different statement that will imply the theorem and is more convenient.

Let $(p_1, ..., p_V) = p$ be a convex polygon with $\langle p_1, p_2 \rangle, ..., \langle p_{V-1}, p_V \rangle$ only as rods and $\langle p_1, p_V \rangle, ..., \langle p_{V-2}, p_V \rangle$ as cables (Fig. 10). Let $p' = (p'_1, ..., p'_V)$ be any infinitesimal flex. Then $\langle p_1, p_V \rangle$ is a cable, i.e., $(p'_1 - p'_V)(p_1 - p_V) \leqslant 0$. Furthermore if any cable is strictly bent, i.e., $(p'_i - p'_{i+2})(p_i - p_{i+2}) < 0$ $i = 1, ..., V-2$, then $\langle p_1, p_V \rangle$ is strictly bent. Obviously this implies the theorem (since if all the cables are rods $p$ is *infinitesimally* rigid) and is true for $V = 3$.

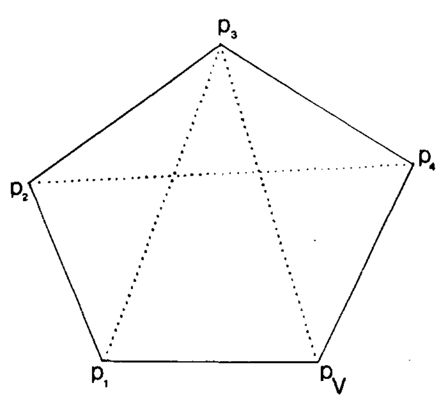


FIGURE 9

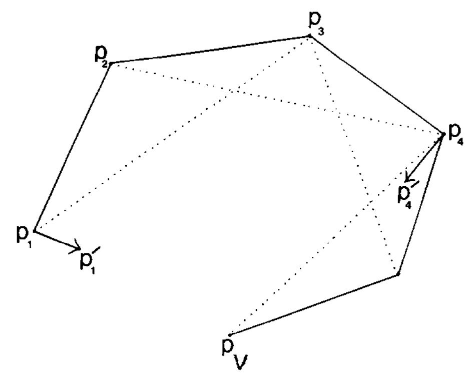


FIGURE 10

Let $V > 3$. By adding the appropriate trivial flex to $p$ we may assume $p_2' = p_3' = 0$ since $\langle p_2, p_3 \rangle$ is a rod.

Because $p$ is strictly convex at $p_2$ and $p_2' = p_3' = 0$, we get $p_i' \cdot (p_1 - p_i) \leqslant 0$ $i = 2, 3, 4, \ldots, V$ with strict inequality unless $p_1' = 0$ or $i = 2$. Similarly $p_4' \cdot (p_4 - p_1) \leqslant 0$. Consider the polygon $(p_1, p_3, p_4, \ldots, p_V) = \hat{p}$ with $V - 1$ vertices and the infinitesimal flex $\hat{p}' = (0, 0, p_4', \ldots, p_V')$. Since $p_4' \cdot (p_4 - p_1) \leqslant 0$, $\langle p_1, p_4 \rangle$ is a cable in $\hat{p}$. Since $\hat{p}_1' = \hat{p}_3' = 0$, $\langle p_1, p_3 \rangle$ is a rod. Thus $\hat{p}$ satisfies the induction hypothesis, and we get $p_V' \cdot (p_V - p_1) \leqslant 0$. Since $p_1' \cdot (p_1 - p_V) \leqslant 0$, $(p_V' - p_1')(p_V - p_1) \leqslant 0$. If any of the cables of $p$ are decreased strictly by $\hat{p}'$ then the $\langle p_1, p_V \rangle$ cable is decreased strictly by $\hat{p}'$ and the same is true for $p'$. Thus we are done if any but the $\langle p_1, p_3 \rangle$ and $\langle p_2, p_4 \rangle$ cables are decreased strictly. If the $\langle p_2, p_4 \rangle$ cable decreases strictly this means $p_4' \neq 0$ and thus the $\langle p_1, p_4 \rangle$ cable decreases strictly, and $\langle p_1, p_V \rangle$ decreases by induction. If the $\langle p_1, p_2 \rangle$ cable decreases strictly, then $p_1' \neq 0$ and thus the $\langle p_1, p_V \rangle$ cable decreases directly.

This completes the induction and the proof of the theorem.

*Remark* 4.3. If one of the vertices of $p$ is not a natural vertex, then the resulting polygon is in fact flexible as well as infinitesimally flexible (see Fig. 11).

DEFINITION. We call such a cabled polygon as in the theorem, a (cabled) *Cauchy polygon*.

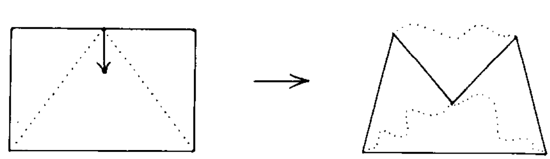

FIGURE 11

*Remark* 4.4. The proof above is very much in the spirit of Cauchy's original proof. The difficulty in the non-infinitesimal case comes in the induction process where an intermediate polygon may not be convex. In another paper we shall investigate other cabled polygons both for their infinitesimal rigidity and their uniqueness properties. For our purposes here it is sufficient to find some cabled polygon of the sort above.

*Addendum.* Here we observe that a slightly different cabled framework from the Cauchy polygon is also infinitesimally rigid. Let $p_1,..., p_V$ be vertices of a strictly convex polygon (see Fig. 12). We have rods from $p_i$ to $p_{i+1}$, $i = 1,..., V$, $(p_{V+1} = p_1)$ as before, but instead of cables from $p_i$ to $p_{i+2}$, $i = 1,..., V-2$, we have a different cable system. Let $\bar{p}_i$, $i = 2,..., V-1$, be a point in the interior of the triangle $\langle p_{i-1}, p_i, p_{i+1}\rangle$. Then we put cables from $\bar{p}_i$ to $p_{i-1}, p_i$, and $p_{i+1}$. Call this new cabled framework an *altered Cauchy polygon*.

THEOREM 4.2. *An altered Cauchy polygon is infinitesimally rigid in the plane.*

*Proof.* Let $p'$ be an infinitesimal flex of the altered Cauchy polygon $\mathscr{F}$. If we can show $(p_i' - p_{i+2}')(p_i - p_{i+2}) \leqslant 0$, $i = 1,..., V-2$, then $p'$ will be a flex of the associated Cauchy polygon $\mathscr{F}$ and thus be trivial on the $p_i$'s. It then will be easy to see that the flex is trivial on all of the $\bar{p}_i$'s as well. So we

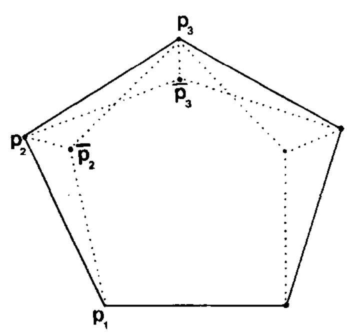


FIGURE 12

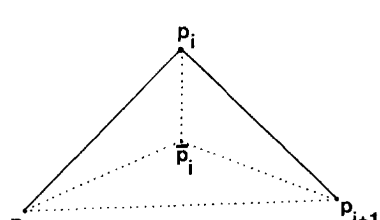


FIGURE 13

need only look at $p_{i-1}, p_i, p_{i+1}$ and $\bar{p}_i$ (see Fig. 13). We observe $\bar{p}_i = \lambda_1 p_{i-1} + \lambda_2 p_i + \lambda_3 p_{i+1}$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda_1, \lambda_2, \lambda_3 > 0$. For simplicity we may assume $p'_{i-1} = p'_i = 0$, without loss in generality. If we assume $-p'_{i+1} \cdot (p_{i-1} - p_{i+1}) > 0$ we wish to arrive at a contradiction. Thus $p'_{i+1} \cdot (\bar{p}_i - p_{i+1}) < 0$ since $\lambda_1 \neq 0$, and from the rod conditions. $\bar{p}'_i \cdot (\bar{p}_i - p_{i-1}) \leqslant 0$, $\bar{p}'_i \cdot (\bar{p}_i - p_i) \leqslant 0$ and $(\bar{p}'_i - p'_{i+1}) \cdot (\bar{p}_i - p_{i+1}) \leqslant 0$. But then

$$\begin{aligned}
\bar{p}_i \cdot \bar{p}'_i &= (\lambda_1 p_{i-1} + \lambda_2 p_i + \lambda_3 p_{i+1}) \cdot \bar{p}'_i \\
&= \lambda_1 p_{i-1} \cdot \bar{p}'_i + \lambda_2 p_i \cdot \bar{p}'_i + \lambda_3 p_{i+1} \cdot \bar{p}'_i \\
&\geqslant \lambda_1 \bar{p}_i \cdot \bar{p}'_i + \lambda_2 \bar{p}_i \cdot \bar{p}'_i + \lambda_3 (\bar{p}_i \cdot \bar{p}'_i - p'_{i+1} \cdot (\bar{p}_i - p_{i+1})) \\
&= \bar{p}_i \cdot \bar{p}'_i - \lambda_3 p'_{i+1} (\bar{p}_i - p_{i+1}) > \bar{p}_i \cdot \bar{p}'_i
\end{aligned}$$

a contradiction. Thus $-p'_{i+1}(p_{i-1} - p_{i+1}) \leqslant 0$ and $\langle p_{i-1}, p_{i+1} \rangle$ is a cable, i.e., $p'$ is an infinitesimal deformation of the Cauchy polygon $\mathscr{F}$. Thus $p'$ is trivial on the vertices $(p_1, \ldots, p_V)$. Thus we may assume each $p'_i = 0$. From the proof above, then each $\bar{p}'_i = 0$, and $\mathscr{F}$ is rigid. (We will see this idea also used later.)

*Note.* If $\bar{p}_i$ is on the interior of the rods $\langle p_{i-1}, p_i \rangle$ or $\langle p_i, p_{i+1} \rangle$ the altered Cauchy polygon is still rigid, but not infinitesimally rigid, which we need.

## 5. Infinitesimally Rigid Cabled Convex Surfaces

Here we show that if we take the Cauchy polygons of the previous section and put them together to form a convex polyhedral surface, then the resulting cabled framework in three-space is infinitesimally rigid. This is inspired from the discussion by Grünbaum in his "Lectures on Lost Mathematics" [9], where he has several examples of such frameworks which are experimentally seen to be rigid.

In the following we wish to use the information about rigidity in the plane to get information about rigidity in three-space. If $\mathscr{F}$ in the plane is a (cabled) framework that is rigid in $\mathbb{R}^2$, unless $\mathscr{F}$ is a triangle (or an edge or point), then $\mathscr{F}$ will not be infinitesimally rigid in three-space. For example we can hold three points fixed and choose any collection of $p'_i$'s (for the other vertices) that are perpendicular to the $\mathbb{R}^2 \subset \mathbb{R}^3$. We show first that this is essentially the only way we can infinitesimally flex $\mathscr{F}$ in $\mathbb{R}^3$.

Lemma 5.1. *Let $\mathscr{F}$ be a cabled framework in the plane, $\mathbb{R}^2$. Let $p'$ be an infinitesimal flex in three-space. Then $\pi p'$ is a flex of $p$ in $\mathbb{R}^2$ (and $\mathbb{R}^3$), where*

*$\pi p' = (\pi p'_1, \ldots, \pi p'_n)$ is the orthogonal projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ in each coordinate.*

*Proof.* We check, $(\pi p'_i - \pi p'_j) \cdot (p_i - p_j) = \pi(p'_i - p'_j) \cdot (p_i - p_j) = 0$, since $p'_i - p'_j = \pi(p'_i - p'_j) + \mathbf{n}$, where $\mathbf{n}$ is perpendicular to $\mathbb{R}^2$ and $p_i - p_j$, $\{i, j\} \in \mathscr{E}$.

*Remark* 5.1. Thus we see that if $\mathscr{F}$ is infinitesimally rigid in the plane, then an infinitesimal flex of $\mathscr{F}$, $p'$, is of the form $p' = \pi p' + \mathbf{n}$, where $\pi p'$ is a trivial flex in the plane and each $\mathbf{n}_i$ of $\mathbf{n}$ is perpendicular to the plane.

THEOREM 5.1. *Let $\mathscr{F}$ be a cabled framework obtained as follows: The vertices of $\mathscr{F}$ are the natural vertices of a convex surface. The rods of $\mathscr{F}$ are the natural edges of this surface. The cables of $\mathscr{F}$ are any system of cables in each natural face such that the vertices, rods, and cables of each natural face form an infinitesimally rigid framework in the plane that face. Then $\mathscr{F}$ is infinitesimally rigid.*

COROLLARY 5.1. *Let $\mathscr{F}$ be a framework defined as above, where a Cauchy polygon cables each natural face. Then $\mathscr{F}$ is infinitesimally rigid.*

*Remark* 5.2. In general the faces of $\mathscr{F}$ above will not even be rigid in three-space. We only insist that they be *infinitesimally* rigid in the plane. If

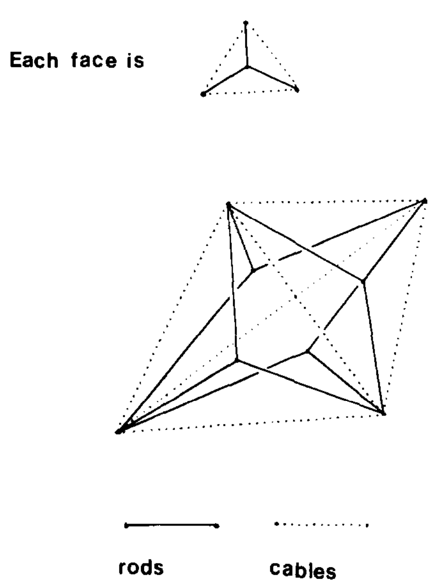


FIGURE 14

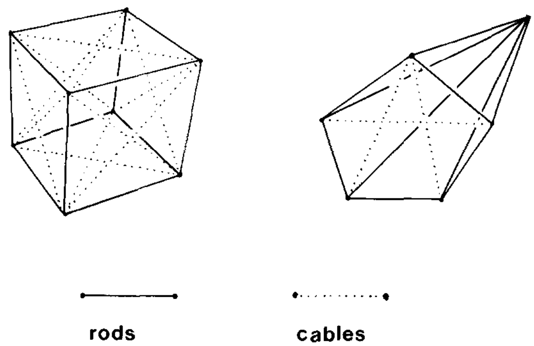


FIGURE 15

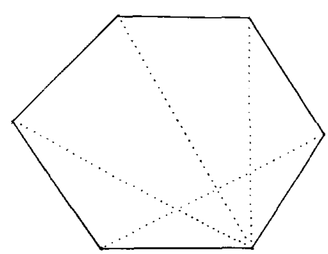

FIGURE 16

each face is just rigid in the plane this is not enough to make $\mathscr{F}$ rigid, as we shall see with later examples (Figs. 17 and 20).

We cannot allow the boundary of the faces to be cables either, even if the framework in the plane of each face is inf. rigid. The framework of Fig. 14 is flexible, whereas the cabled framework in the plane of each face is infinitesimally rigid.

EXAMPLE 5.1. Figure 15 shows some pictures of the frameworks described by Corollary 5.1.

*Remark* 5.3. There are other ways of cabling a convex polygon to make it infinitesimally rigid in the plane and thus generate examples for Theorem 5.1. Figure 16 is an example conjectured by Grünbaum [11].

*Proof of Theorem* 5.1. Call the framework defined by the theorem $\mathscr{F}$. Let $p'$ be an infinitesimal flex of $\mathscr{F}$. We next consider another *rod* framework $\mathscr{F}$ with the same vertices $p$, but with a different collection of rods. Namely the rods of $\mathscr{F}$ will be the edges of some triangulation of the convex surface spanned by $p$, where the vertices of $p$ are the only vertices in the triangulation. (For instance we can triangulate each convex natural face separately and arbitrarily without adding any new vertices).

We claim that $p'$ is an infinitesimal flex for $\bar{\mathscr{F}}$ as well as $\mathscr{F}$. To see this let $\langle p_i, p_j \rangle$ be an edge of $\bar{\mathscr{F}}$ in some natural face $F$ of the convex surface. In the framework $\mathscr{F}$, the face $F$ is infinitesimally rigid in the plane, so if $q = (q_1, ..., q_k)$ are the vertices of $F$, and $q'$ is the corresponding restriction of $p'$, then $q' = \pi q' + \mathbf{n}$ where each $\mathbf{n}_i$ is perpendicular to $F$ and $\pi q$ is a trivial flex of the vertices of $F$. Thus $(p'_i - p'_j) \cdot (p_i - p_j) = (\pi p'_i + \mathbf{n}_i - \pi p'_j - \mathbf{n}_j) \cdot (p_i - p_j) = (\pi p'_i - \pi p'_j) \cdot (p_i - p_j) + (n_i - n_j) \cdot (p_i - p_j) = 0$; since $\langle p_i, p_j \rangle$ lies in $F$. This shows $p'$ is a flex of $\bar{\mathscr{F}}$, the rod polygon.

We now appeal to the theorem of Alexandrov referred to in the beginning. It says that since $p'$ is an honest flex of $\bar{\mathscr{F}}$, $p'$ is trivial. But if $p'$ is trivial for $\bar{\mathscr{F}}$, it is trivial for $\mathscr{F}$. Thus the cabled framework $\mathscr{F}$ is infinitesimally rigid, and we are done.

*Remark* 5.4. It is not possible to extend the above theorem to allow vertices in the interior of natural edges. If one takes a cube and puts vertices in the interior of the lateral edges (as in Fig. 17) there is no way of *cabling* the lateral faces to prevent the resulting framework from flexing even if the top and bottom faces are kept rigid.

Notice with this example, however, that the infinitesimal flex this flex generates is trivial on the vertices not in the interior of the natural edges, and the infinitesimal flex on these vertices in the interior of the natural edges points "in" to the center of the cube. Alexandrov's theorem is true for vertices in the interior of the natural edges (but not faces of course), but it is the infinitesimal rigidity of the appropriate polygons in the plane that fails us.

We also need to consider the case when a convex polyhedral surface is triangulated with vertices anywhere. We apply the technique of this section to see what we can say.

THEOREM 5.2. *Let $\mathscr{F}$ be the framework obtained from an arbitrary triangulation of a convex surface where all the edges are rods. Let $p = (p_1, ..., p_V)$ be the vertices of $\mathscr{F}$, and $q = (q_1, ..., q_{V'})$ be the vertices of $\bar{\mathscr{F}}$*

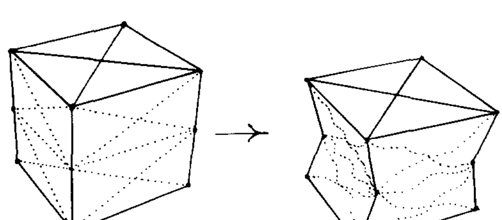

FIGURE 17

*that lie in the natural edges or vertices. Let $p'$ be an infinitesimal flex of $\mathscr{F}$. Then $q'$, the corresponding flex on the subset of vertices, is trivial.*

*Proof.* We proceed as before. Since the triangulation restricted to each face is infinitesimally rigid in the plane, the argument of the previous theorem shows that if we find any new triangulation of the convex surface with just $q$ as the vertices, then for this new rod framework $\bar{\mathscr{F}}$, $q'$ will be a legitimate flex. By the full strength of Alexandrov's theorem $\bar{\mathscr{F}}$ is infinitesimally rigid, and so $q'$ is trivial for $\bar{\mathscr{F}}$. Thus $q'$ is trivial as desired.

*Remark* 5.5. Note that this does not imply that $p'$ is trivial, or that even $\mathscr{F}$ is rigid, *yet*. If we add an appropriate trivial flex to $p'$, we may assume $q' = 0$. In this case the argument also implies that if $p_i$ is in the interior of a natural face, then $p_i'$ is perpendicular to that face, and as mentioned earlier any collection of such $p_i'$'s will be a valid infinitesimal flex of $\mathscr{F}$. Thus we have a complete picture of what the infinitesimal flexes of $\mathscr{F}$ look like, and to complete our study we must look at the possible second-order flexes. If $\mathscr{F}$ were to flex, this theorem says intuitively that the vertices on the natural edges or vertices do not start to move until the second order.

## 6. The Second-Order Rigidity of Triangulated Convex Surface

We know from the last theorem of the preceding section what a first-order flex of a triangulated convex surface looks like, and here we wish to get information about the second-order flex. Since the infinitesimal flex is trivial on the natural edges and vertices of the surface, any second-order flex will satisfy the equations of a first-order flex there, if the first-order flex is made 0. But this alone is not enough to make it rigid, since the framework, consisting of the natural edges and vertices only, will be flexible unless all the natural faces are triangles. See Asimow and Roth [4, part I]. In a sense, the natural faces keep the natural vertices and edges from moving in or out at the first order, but the important observation is that the natural vertices in a natural face still cannot move apart at the *second* order. This allows us to use the theorem in the previous section about cabled frameworks to conclude that the second-order flex on the natural vertices is trivial. But then it is impossible for there to have been a non-trivial first-order flex to start with, since it forces the vertices in the interior of natural faces to move toward each of the natural vertices on its face at the second order. This is our plan for showing second-order rigidity.

Suppose $p = (p_1, \ldots, p_V)$ describes the vertices of some triangulation of a disk in the plane. If $p'$ is an infinitesimal flex of an associated rod framework, then we regard that "vector field" as being defined at every point in the disk by using the triangulation and extending linearly. That is, if $x$ is in the triangle formed by $(p_1, p_2, p_3)$ and $x = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$, then $x' = \lambda_1 p_1' + \lambda_2 p_2' + \lambda_3 p_3'$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda_i \geqslant 0$, $i = 1, 2, 3$. It is easy to see that this infinitesimal deformation is trivial on each triangle. In particular if we consider $p_i$, $p_j$ two vertices such that the arc form $p_i$ to $p_j$ lies in the disk and $x_1, x_2, \ldots, x_n$ are the points between $p_i$ and $p_j$ that lie on edges (Fig. 18), then $p_1', x_1', \ldots, x_n', p_j'$ is an infinitesimal flex of the arc $\langle p_i, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_n, p_j\rangle$ though of as a framework. Clearly this also extend to any order.

We apply this idea to the following two lemmas.

LEMMA 6.1. *Let $p = (p_1, \ldots, p_V)$ be the vertices of some triangulation of a convex disk in the plane. Let $p'$ and $p''$ be first- and second-order flexes of the associated rod framework in three-space such that $q' = 0$, where $q = (q_1, \ldots)$ represent the vertices of the boundary. Let $r_i, r_j$ be two of the natural vertices of $p$ on the boundary. Then $(r_i - r_j) \cdot (r_i'' - r_j'') \leqslant 0$, and if the arc from $r_i$ to $r_j$ is in the boundary, then $(r_i - r_j) \cdot (r_i'' - r_j'') = 0$.*

*Proof.* Let $x_1, \ldots, x_n$, be the intersection points along a straight line segment from $r_i$ to $r_j$. (If an interior edge is in the line segment just include its endpoints in the $x_k$'s.) From the above remarks we know that $p'$, and $p''$ induce a first- and second-order flex on this line segment with vertices $(r_i, x_1, \ldots, x_n, r_j)$. So if we define $x_0 = r_i$, $x_{n+1} = r_j$,

$$(x_k' - x_{k+1}')^2 + (x_k - x_{k+1}) \cdot (x_k'' - x_{k+1}'') = 0, \qquad k = 0, \ldots, n.$$

Since $r_i - r_j$ is a positive multiple of $(x_k - x_{k+1})$,

$$(r_i - r_j) \cdot (x_k'' - x_{k+1}'') \leqslant 0, \qquad k = 0, \ldots, n,$$

and adding $(r_i - r_j) \cdot (r_j'' - r_j'') \leqslant 0$, as desired.

If $r_i$ and $r_j$ are on a natural edge then $x_k' = 0$, for $k = 1, \ldots, n$, so $(r_i - r_j) \cdot (r_i' - r_j') = 0$.

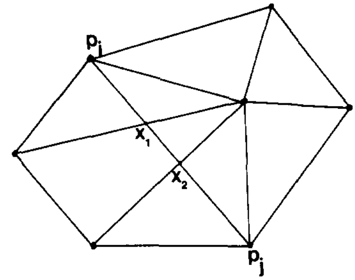


FIGURE 18

LEMMA 6.2. *Let $p$, $p'$, $p''$ be as above where we assume in addition that $r''=0$, where $r=(r_1,...)$ are the natural vertices. (Also $r'=0$ of course.) Then $p'=0$.*

*Proof.* Let $p_i$ be any vertex and $r_j$ a natural vertex on the boundary. Suppose $p_i' \neq 0$. From the calculation of the previous lemma we get

$$(r_j - p_i) \cdot (r_j'' - p_i'') < 0$$

since some $x_k' \neq x_{k+1}'$, since $r_j' = 0$ and $p_i' \neq 0$. But $r_j'' = 0$ so $p_i'' \cdot (r_j - p_i) > 0$, for all $r_j$. But this is impossible since $p_i$ is in the convex linear span of the $r_j$'s. (For example, let $p_i = \lambda_1 r_1 + \cdots \lambda_n r_n$, $\sum \lambda_i = 1$, $\lambda_i \geqslant 0$. Then $r_j p_i'' > \lambda_1 r_1 \cdot p_i'' + \cdots \lambda_n r_n \cdot p_i'' = p_i \cdot p_i''$, $j = 1,..., n$. But a weighted mean cannot be less than each of its terms.) Thus each $p_i' = 0$ as desired.

THEOREM 6.1. *Let $\mathscr{F}$ be rod framework associated to any triangulation of a convex surface. Then $\mathscr{F}$ is second-order rigid.*

*Proof.* Let $p=(p_1,..., p_V)$ be the vertices of a triangulation, $q=(q_1,...)$ those vertices of $p$ that lie in the natural edges and natural vertices, and $r=(r_1,...)$ the natural vertices of $p$. Suppose $p'$ and $p''$ is a second-order flex of $\mathscr{F}$. By Theorem 5.2 $q'$ is trivial. By adding a trivial flex to all of $p'$ we may assume that $q'=0$. Let $\bar{\mathscr{F}}$ be the cabled framework obtained with $r$ as vertices and a Cauchy polygon on each natural face.

By Lemma 6.1 and $r'=0$, $r''$ is a first-order flex of $\bar{\mathscr{F}}$ by definition.

By Theorem 5.1 $\bar{\mathscr{F}}$ is infinitesimally rigid. So we may assume $r''=0$. By Lemma 6.2 of this section $p'=0$. Thus there is no non-trivial second-order flex, and $\mathscr{F}$ is second-order rigid.

*Remark* 6.1. Instead of adding a trivial flex at each stage as in the proof above, it is possible to apply Kuiper's Eqs. (i) and (ii) of the Appendix. Here it seems best to only include the natural vertices in the collection of those points appearing in (i) and (ii). Theorem 5.2 implies that $q'$ is trivial in the above argument so $r'$ is trivial and thus 0 by (ii). Thus $q'=0$, or just knowing $r'=0$ by Remark 5.5 each $q_i'$ is perpendicular to each face in which it sits and thus is 0 also. Then Lemma 6.1, applied as above, shows $r''$ is a first-order flex of $\bar{\mathscr{F}}$ and Kuiper's equations imply $r''=0$, so $p'=0$ by Lemma 6.2 as above.

Interestingly the above argument can also show that the weak Alexandrov theorem implies the strong as mentioned in the Introduction.

*Remark* 6.2. We can improve the above theorem somewhat. Namely, if we remove convex holes from the interior some of the natural faces, triangulate this surface with boundary, and take the associated rod framework, it will still be second-order rigid. The idea here is the same as above except we must work a bit harder to create the cabled framework and to show that the vertices on the interior of natural faces do not move to the first order (assuming of course there were a non-trivial second-order flex).

As a simple example of the above consider some convex surface where some of the natural faces are pentigons as pictured in Fig. 19 (e.g., a regular dodecahedron).

If we remove the shaded portion and triangulate the rest, then the machinery we have already shows the resulting framework is second-order rigid. The disk minus the hole is clearly infinitesimally rigid in the plane, so when we retriangulate the whole convex surface and apply Alexandrov's theorem we get that $q'$ is trivial as before. So we assume $q' = 0$. We still have the cables for the Cauchy polygon so the same argument as above implies $r''$ is trivial (so we assume $r'' = 0$). To finally get $p' = 0$ we see that each vertex in the relative interior of a natural face is in the convex linear span of some three of the natural $r$ vertices of its natural face. The argument of Lemma 6.2 above applies and makes $p' = 0$, as defined.

*Remark* 6.3. With more work we can allow the holes to expand arbitrarily close to the natural edges, but if just one hole touches one edge in its relative interior, the resulting framework may be flexible as seen in

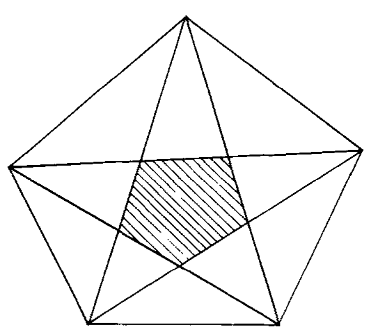

FIGURE 19

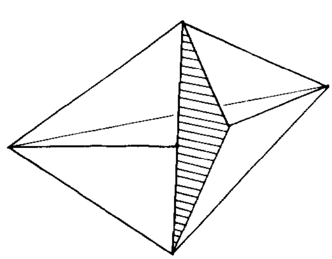

FIGURE 20

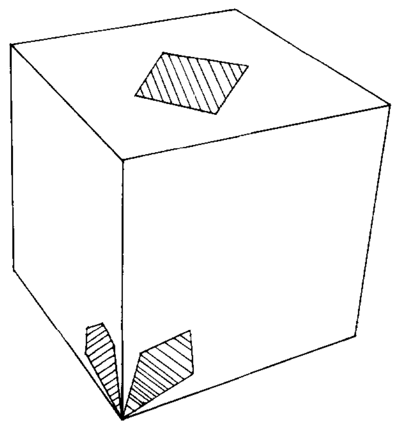

FIGURE 21

Fig. 20, a triangulation of a tetrahedron with the shaded triangle (really a 4-gon) removed.

Incidentally this is also a counterexample to Conjecture 4 of Grünbaum in his "Lectures on Lost Mathematics" [9].

*Addendum.* Here we show how to improve the above techniques to strengthen the results to allow "holes" in the natural faces of the convex surfaces.

THEOREM 6.2. *Let $\mathcal{F}$ be a framework obtained from a convex polyhedral surface by removing the interior of a convex polygonal hole (possibly empty) from each natural face and then triangulating the resulting surface with boundary (see Fig. 21). If the boundary of each hole intersects the boundary of the natural face at at most one point, a natural vertex, then $\mathcal{F}$ is second-order rigid.*

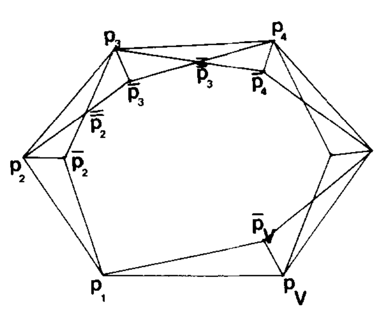


FIGURE 22

*Proof.* We first show a special case. Namely, if $p_1,..., p_V$ are the natural vertices (in order) of a natural face then we choose points $\bar{p}_2, \bar{p}_3,..., \bar{p}_V$ where $\bar{p}_i$ is near to $p_{1i}$ in the interior of the natural face Fig. 22). Let $\bar{\bar{p}}_2, \bar{\bar{p}}_3,..., \bar{\bar{p}}_{V-1}$ be defined by $\bar{\bar{p}}_i = \langle p_i, \bar{p}_{i+1}\rangle \cap \langle \bar{p}_i, p_{i+1}\rangle$. Then the hole removed will be the convex linear span of $p_1, \bar{p}_2, \bar{\bar{p}}_2, \bar{p}_3, \bar{\bar{p}}_3,..., \bar{\bar{p}}_{V-1}, \bar{p}_V$. We triangulate this surface arbitrarily. We now proceed as before. Let $p =$ vertices of $\mathscr{F}$, $q =$ vertices of $\mathscr{F}$ in the natural edges of the convex surface, $r =$ natural vertices of the convex surface. It is clear that the triangulation of each natural face, even though it has a hole in it possibly, is infinitesimally rigid in the plane of that face. Let $p', p''$ be the second-order flex of $\mathscr{F}$. Then by Alexandrov's theorem, and our argument about projections, $q'$ is trivial. Thus we assume $q' = 0$. Then $r''$ may be regarded as a first-order flex of the framework obtained by using rods for the natural edges. However, by the way the holes are placed, each natural face has an altered Cauchy polygon imbedded in the surface away from the hole. The argument of Lemma 5.1 still applies then to show that the projection of $r''$ is a first-order flex of this altered Cauchy polygon in the plane. Thus the projections are trivial in each plane, Alexandrov's theorem applies again, and we conclude that $r''$ is trivial. Thus we may assume $r'' = 0$. But then the argument of Lemma 6.2 applies to each $p_j$ in the interior of the natural face in the following way. If $p_j$ is on the line segment from $\bar{p}_i$ to $p_i$, $i = 2,..., V$, in the notation above, then the proof of the lemma implies that $p_j' = 0$, and in fact any vertex $p_k$ on the line segment from $p_j$ to $p_{i-1}, p_i$, on $p_{i+1}$ must have $p_k' = 0$. From the way the hole was chosen this includes all the vertices in the natural face. Thus $p' = 0$. Thus $\mathscr{F}$ is second-order rigid.

To get the general case we observe that a triangulation such as the ones considered above can be made a subcomplex of any triangulation for any admissible collection of holes. Let $\bar{\mathscr{F}}$ be the subframework described above, and $\bar{\bar{\mathscr{F}}}$ the given one subdivided, and $\mathscr{F}$ the given one. If $\bar{\bar{\mathscr{F}}}$ is second-order rigid, clearly $\mathscr{F}$ is. Since $\bar{\mathscr{F}}$ is known to be second-order rigid, if $p', p''$ is second-order flex of $\mathscr{F}$ we know that $p'$ is trivial for the vertices of $\bar{\mathscr{F}}$. So as before we may assume these $p' = 0$. But then $p''$ for $\bar{\mathscr{F}}$ is then a first-order flex. We know that for these flexes (which may be non-trivial) $q''$ is trivial. Thus we may assume $q'' = 0$ (the vertices of the natural edges). Then for any $p_j$ in $\mathscr{F}$ in the interior of a natural face, if we draw a line segment with $p_j$ in the interior with the endpoints in the natural edges (Fig. 23), since $q'' = 0$, we know $p_j' = 0$. Thus $p' = 0$ and $\mathscr{F}$ itself is second-order rigid.

*Remark* 6.4. The proof above fails for the example of the previous remark because the triangulation in the plane of the face with the hole is not infinitesimally rigid, even though it is rigid. We alter the example slightly as seen in Fig. 24.

We remove a smaller convex hole (or slit) now which touches the natural

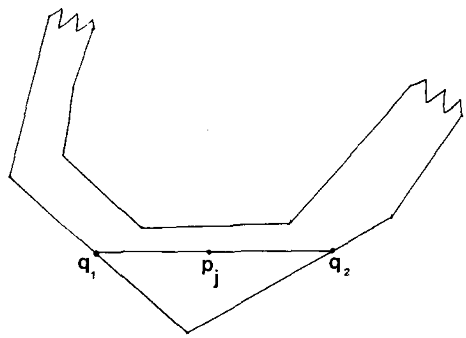


FIGURE 23

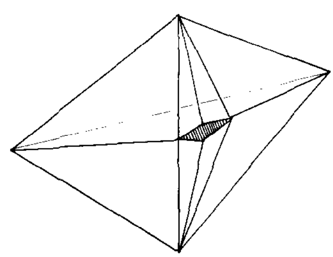

FIGURE 24

edge only in its interior. Note the framework in the plane of the face is infinitesimally rigid, but there is no way to show that the new vertices do not move at the $p'$ level, since there is no way to generate enough vertices, which are known not to move at the second level, which have the added vertices between them.

## APPENDIX: KUIPER'S DISCRETE ANGULAR MOMENTUM

In [12] Kuiper mentions a method which would restrict the motion of a finite collection of particles (in $\mathbb{R}^3$) such that if their motion is part of a rigid motion of all of $\mathbb{R}^3$, then they stay fixed.

Call the points $p(t) = (p_1(t),..., p_V(t))$, $0 \leqslant t \leqslant 1$, as before. First he assumes the baricenter or center of gravity is fixed at $0$.

$$\frac{1}{V} \sum_i p_i(t) = 0. \tag{i}$$

The second condition is

$$\sum_i p_i(0) \times p_i(t) = 0, \tag{ii}$$

which is a kind of "discrete angular momentum." (Note. $\sum_i p_i(t) \times p_i'(t)$ is the usual angular momentum.)

For our purposes we need to show two things:

A. If (i) and (ii) hold (and the $p_i(0)$'s do not lie on a line), and $p_i(t) = T_t p_i(0)$, $0 \leqslant t \leqslant 1$, for $T_t$ a continuous path in the group of rigid motions on $\mathbb{R}^3$, then $T_t = \text{identity}$.

B. If $p(t)$ is any continuous motion of the points, then there is a rigid euclidean motion $T_t$ such that (i) and (ii) hold for $T_t p(t)$.

We also observe that an infinitesimal version of A holds. Namely if $\sum_i p_i' = 0$ and $\sum_i p_i \times p_i' = 0$, where $p_i' = \mathbf{r} \times p_i + \mathbf{t}$ is a trivial infinitesimal motion, then $p_i' = 0$ for all $i$. This is a well-known fact from physics. If a rigid body moves with 0 linear and angular momentum it is not moving. Note that if the points $p_i$ move satisfying (i) and (ii), then at $t = 0$ the linear and angular momentum is automatically zero.

For A we observe that (i) implies that the center of gravity $0$ is a fixed point, so $T_t$ is a rotation. Suppose $T_t$ is a rotation about the vector $v_t$ in $\mathbb{R}^3$. Then each $p_i(0) \times p_i(t)$ is a scalar multiple of $v_t$, all of the same sign, not zero unless $T_t = \text{identity}$. So (ii) insures each $T_t = \text{identity}$ and A holds.

For B there is no problem in finding $T_t$ such that (i) holds for $T_t p_i(t)$. So we assume (i) holds to start with. Write $p_i(t) = (a_{i1}, a_{i2}, a_{i3})$. Then

$$\sum_{i=0}^{V} p_i(0) \times p_i(t) = \sum_{i=1}^{V} p_i \times (a_{i1} e_1 + a_{i2} e_2 + a_{i3} e_3)$$

$$= \sum_{j=1}^{3} \left( \sum_{i=1}^{V} a_{ij} p_i \right) \times e_j,$$

where $p_i = p_i(0)$, and $e_j$, $j = 1, 2, 3$, are the standard basis for $\mathbb{R}^3$. Let $P$ be the $3 \times 3$ matrix whose columns for $j = 1, 2, 3$ are

$$\sum_{i=1}^{V} a_{ij} p_i .$$

Let $\mathscr{O}$ be an orthogonal matrix so that

$$P = S\mathscr{O}^t,$$

where $S$ is a symmetric matrix and $(\ )^t$ denotes the transpose. Then $\mathcal{O}$ is the desired matrix because

$$\begin{aligned}\sum_i \mathcal{O}p_i(0) \times p_i(t) &= \sum_j \left(\sum_i a_{ij}\mathcal{O}p_i\right) \times e_j \\ &= \sum_j \mathcal{O}\left(\sum_i a_{ij}p_i\right) \times e_j \\ &= \sum_j \mathcal{O}P(e_j) \times e_j \\ &= \sum_j S(e_j) \times e_j \\ &= 0,\end{aligned}$$

since $S$ is symmetric.

Thus there is a $T_t$ such that (ii) holds, and this shows B. It is also clear that $T_t$ can be chosen as a continuous function of $t$.

## ACKNOWLEDGMENTS

The author would like to thank Peter Kahn, N. H. Kuiper, B. Roth, and J. Milnor for many helpful comments.

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