# Limited choice and randomness in evolution of networks Lecture 1

Cornell Probability summer school
July 2012

Shankar Bhamidi

Department of Statistics and Operations Research
University of North Carolina

July, 2012

# Plan of the lectures

## Underlying theme

- Mathematical techniques for dynamic random graph models
- Effect of limited choice

# Plan of the lectures

## Underlying theme

- Mathematical techniques for dynamic random graph models
- Effect of limited choice

## Lecture content

- **Lecture 1:** Critical random graphs, Bounded size rules **[Scaling limits]**
- **Lecture 2:** Preferential attachment models, random trees **[Local Weak convergence]**

# Lecture 1

- General motivation
- Critical random graphs
- Bounded size rules and the emergence of the giant
- Method of proof (Joint work with Budhiraja and Wang)
- Extensions: “Explosive percolation”

# Lecture 2

- Preferential attachment models
- Convergence of random trees
- Implications: Convergence of the spectral measure (Arnab Sen, Steve Evans, SB)
- Power of choice in random trees
- Local weak convergence (Angel, Pemantle, SB)

# Power of two choices

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).

# Power of two choices

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).
- Centralized scheme (asking bins current load) computationally expensive and time consuming

# Power of two choices

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).
- Centralized scheme (asking bins current load) computationally expensive and time consuming
- Simplest scheme, each stage choose bin at random and place ball
- Each ball has $\sim Poi(1)$ # of balls at end

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).
- Centralized scheme (asking bins current load) computationally expensive and time consuming
- Simplest scheme, each stage choose bin at random and place ball
- Each ball has $\sim Poi(1)$ # of balls at end

Max load $\sim \Theta(\log n / \log(\log n))$

# Power of two choices

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).
- Centralized scheme (asking bins current load) computationally expensive and time consuming
- Simplest scheme, each stage choose bin at random and place ball
- Each ball has $\sim Poi(1)$ # of balls at end

$$\text{Max load} \sim \Theta(\log n / \log(\log n))$$

- **Limited choice** Choose $2$ bins u.a.r.

# Power of two choices

## Application setting

- Consider $n$ bins (servers) into which we are going to sequentially place $n$ balls (jobs).
- Centralized scheme (asking bins current load) computationally expensive and time consuming
- Simplest scheme, each stage choose bin at random and place ball
- Each ball has $\sim Poi(1)$ # of balls at end

$$\text{Max load} \sim \Theta(\log n / \log(\log n))$$

- **Limited choice** Choose $2$ bins u.a.r.
- Put ball in bin with minimal # of balls at that stage

$$\text{Max load} \sim \Theta(\log \log n)$$

## Motivation

- Last few years have seen an explosion in empirical data on real world networks.
- Has motivated an interdisciplinary study in understanding the emergence of properties of these network models.
- Formulation of many mathematical models of network formation.

# Network models

## Motivation

- Last few years have seen an explosion in empirical data on real world networks.
- Has motivated an interdisciplinary study in understanding the emergence of properties of these network models.
- Formulation of many mathematical models of network formation.

## Limited choice

- Incorporate effect of limited choice in network formation
- Mathematically understand explosive percolation
- Simple variants of standard models give much better fit but hard to mathematically analyze

# Erdos-Renyi random graph

## Setting

- $n$ vertices
- Edge probability $t/n$
- Phase transition at $t = 1$

# Erdos-Renyi random graph

## Setting

- $n$ vertices
- Edge probability $t/n$
- Phase transition at $t = 1$

$$\text{\# of edges} \sim n/2$$

- $t < 1$, $\mathcal{C}_1(t) \sim \log n$

# Erdos-Renyi random graph

## Setting

- $n$ vertices
- Edge probability $t/n$
- Phase transition at $t = 1$

$$\text{\# of edges} \sim n/2$$

- $t < 1$, $\mathcal{C}_1(t) \sim \log n$
- $t > 1$, $\mathcal{C}_1 \sim f(t)n$
-

$$t = 1$$

# Erdos-Renyi random graph

## Setting

- $n$ vertices
- Edge probability $t/n$
- Phase transition at $t = 1$

$$\# \text{ of edges} \sim n/2$$

- $t < 1$, $\mathcal{C}_1(t) \sim \log n$
- $t > 1$, $\mathcal{C}_1 \sim f(t)n$
- $$t = 1 + \frac{1}{n^{1/3}}$$

# Erdos-Renyi random graph

## Setting

- $n$ vertices
- Edge probability $t/n$
- Phase transition at $t = 1$

$$\#\text{ of edges} \sim n/2$$

- $t < 1$, $\mathcal{C}_1(t) \sim \log n$
- $t > 1$, $\mathcal{C}_1 \sim f(t)n$
- $$t = 1 + \frac{1}{n^{1/3}}$$

**Beautiful math theory**

# Bounded size rules

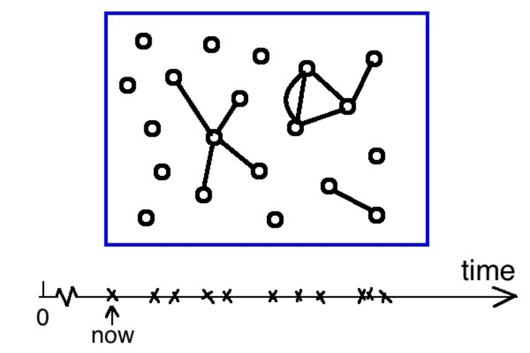
## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.

# Bounded size rules

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

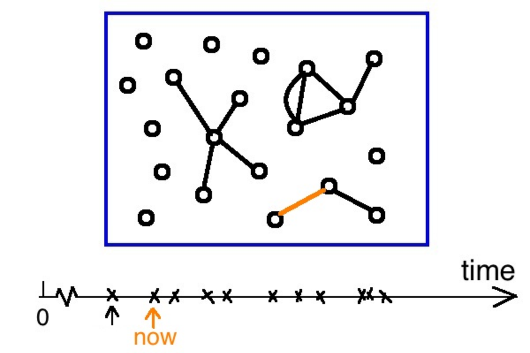
- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

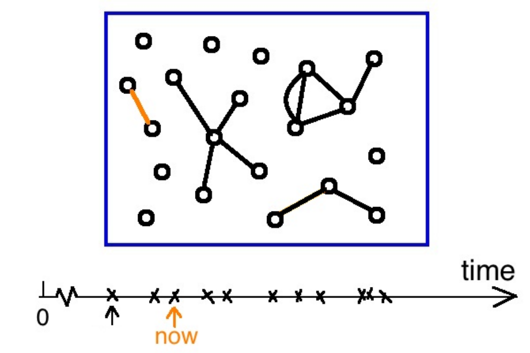
- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

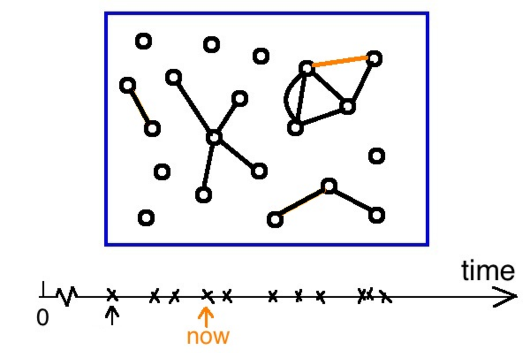
- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

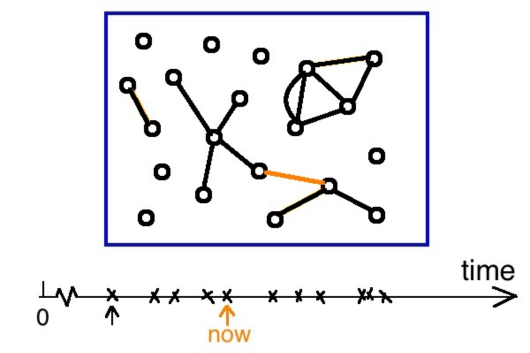
- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

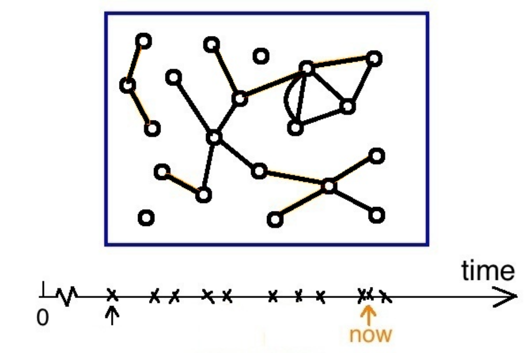
- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The Erdős-Rényi random graph of $\mathcal{G}_n^{ER}$

- $\mathcal{G}_n(0) = \mathbf{0}_n$ the graph with $n$ vertices but no edges
- Each step, choose one edge $e$ uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.
- $\mathcal{G}_n(t)$: add edges at rate $n/2$.



# The Erdős-Rényi random graph process

## The phase transition of $\mathcal{G}_n^{ER}(t)$

- The giant component: the component contains $\Theta(n)$ vertices.

# The Erdős-Rényi random graph process

## The phase transition of $\mathcal{G}_n^{ER}(t)$

- The giant component: the component contains $\Theta(n)$ vertices.
- Let $\mathcal{C}_n^{(k)}(t)$ be the size of the $k^{th}$ largest component
- $t_c = t_c^{ER} = 1$ is the critical time.

# The Erdős-Rényi random graph process

## The phase transition of $\mathcal{G}_n^{ER}(t)$

- The giant component: the component contains $\Theta(n)$ vertices.
- Let $\mathcal{C}_n^{(k)}(t)$ be the size of the $k^{th}$ largest component
- $t_c = t_c^{ER} = 1$ is the critical time.
- (super-critical) when $t > 1$, $\mathcal{C}_n^{(1)} = \Theta(n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (sub-critical) when $t < 1$, $\mathcal{C}_n^{(1)} = O(\log n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (critical) when $t = 1$, $\mathcal{C}_n^{(1)} \sim n^{2/3}$, $\mathcal{C}_n^{(2)} \sim n^{2/3}$.

# The Erdős-Rényi random graph process

## The phase transition of $\mathcal{G}_n^{ER}(t)$

- The giant component: the component contains $\Theta(n)$ vertices.
- Let $\mathcal{C}_n^{(k)}(t)$ be the size of the $k^{th}$ largest component
- $t_c = t_c^{ER} = 1$ is the critical time.
- (super-critical) when $t > 1$, $\mathcal{C}_n^{(1)} = \Theta(n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (sub-critical) when $t < 1$, $\mathcal{C}_n^{(1)} = O(\log n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (critical) when $t = 1$, $\mathcal{C}_n^{(1)} \sim n^{2/3}$, $\mathcal{C}_n^{(2)} \sim n^{2/3}$.

- after initial work by [ER1960], further work by [JKLP1994], finally proved by [Aldous1997].
- Merging dynamics through the scaling window of the components described by a Markov Process called the multiplicative coalescent.
- Formal existence of multiplicative coalescent.

# Bounded size rules: Effect of limited choice

## [Bohman, Frieze 2001]The Bohman-Frieze random graph

- Motivated by very interesting question of D. Achlioptas. Delay emergence of giant component using simple rules

# Bounded size rules: Effect of limited choice

## [Bohman, Frieze 2001]The Bohman-Frieze random graph

- Motivated by very interesting question of D. Achlioptas. Delay emergence of giant component using simple rules
- Each step, two candidate edges $(e_1, e_2)$ chosen uniformly among all $\binom{n}{2} \times \binom{n}{2}$ possible pairs of ordered edges. If $e_1$ connect two singletons (component of size 1), then add $e_1$ to the graph; otherwise, add $e_2$.

# Bounded size rules: Effect of limited choice

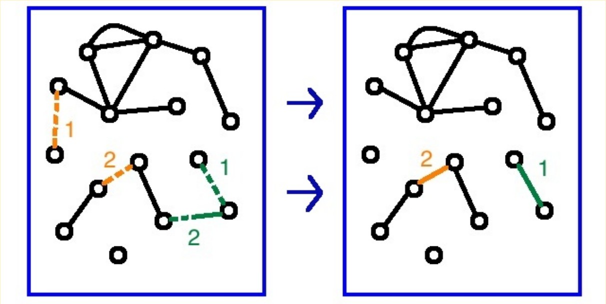
## [Bohman, Frieze 2001]The Bohman-Frieze random graph

- Motivated by very interesting question of D. Achlioptas. Delay emergence of giant component using simple rules
- Each step, two candidate edges $(e_1, e_2)$ chosen uniformly among all $\binom{n}{2} \times \binom{n}{2}$ possible pairs of ordered edges. If $e_1$ connect two singletons (component of size 1), then add $e_1$ to the graph; otherwise, add $e_2$.
- Shall consider continuous time version wherein between any ordered pair of edges, poisson process with rate $2/n^3$.

# Bounded size rules: Effect of limited choice

## [Bohman, Frieze 2001]The Bohman-Frieze random graph

- Motivated by very interesting question of D. Achlioptas. Delay emergence of giant component using simple rules
- Each step, two candidate edges $(e_1, e_2)$ chosen uniformly among all $\binom{n}{2} \times \binom{n}{2}$ possible pairs of ordered edges. If $e_1$ connect two singletons (component of size 1), then add $e_1$ to the graph; otherwise, add $e_2$.
- Shall consider continuous time version wherein between any ordered pair of edges, poisson process with rate $2/n^3$.



# The Bohman-Frieze process

**[Bohman, Frieze 2001] The delay of phase transition**

Consider the continuous time version $\mathcal{G}_n^{BF}(t)$, then there exists $\epsilon > 0$ such that at time $t_c^{ER} + \epsilon$,

$$\mathcal{C}_n^{(1)}(t_c^{ER} + \epsilon) = o(n)$$

# The Bohman-Frieze process

## [Bohman, Frieze 2001] The delay of phase transition

Consider the continuous time version $\mathcal{G}_n^{BF}(t)$, then there exists $\epsilon > 0$ such that at time $t_c^{ER} + \epsilon$,

$$\mathcal{C}_n^{(1)}(t_c^{ER} + \epsilon) = o(n)$$

## [Spencer, Wormald 2004] The critical time

- $t_c^{BF} \approx 1.1763 > t_c^{ER} = 1$.
- (super-critical) when $t > t_c$, $\mathcal{C}_n^{(1)} = \Theta(n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (sub-critical) when $t < t_c$, $\mathcal{C}_n^{(1)} = O(\log n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.

# The Bohman-Frieze process

## [Bohman, Frieze 2001] The delay of phase transition

Consider the continuous time version $\mathcal{G}_n^{BF}(t)$, then there exists $\epsilon > 0$ such that at time $t_c^{ER} + \epsilon$,

$$\mathcal{C}_n^{(1)}(t_c^{ER} + \epsilon) = o(n)$$

## [Spencer, Wormald 2004] The critical time

- $t_c^{BF} \approx 1.1763 > t_c^{ER} = 1$.
- (super-critical) when $t > t_c$, $\mathcal{C}_n^{(1)} = \Theta(n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.
- (sub-critical) when $t < t_c$, $\mathcal{C}_n^{(1)} = O(\log n)$, $\mathcal{C}_n^{(2)} = O(\log n)$.

## Near Criticality

- Janson and Spencer (2011) analyzed how $s_2(\cdot), s_3(\cdot) \to \infty$ as $t \uparrow t_c$.
- Kang, Perkins and Spencer (2011) analyze the near subcritical $(t_c - \epsilon)$ regime.

# General bounded size rules

- Fix $K \geq 1$
- Let $\Omega_K = \{1, 2, \ldots, K, \omega\}$
- General bounded size rule: subset $F \subset \Omega_K^4$.
- Pick $4$ vertices uniformly at random. If $(c(v_1), c(v_2), c(v_3), c(v_4)) \in F$ then choose edge $e_1$ else $e_2$

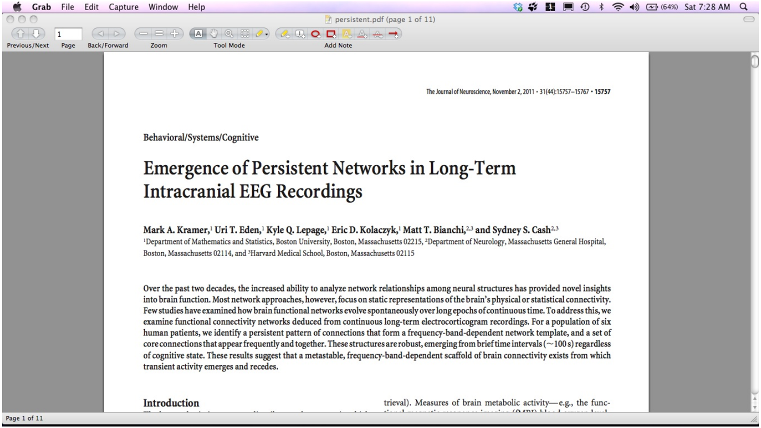
# General bounded size rules

- Fix $K \geq 1$
- Let $\Omega_K = \{1, 2, \ldots, K, \omega\}$
- General bounded size rule: subset $F \subset \Omega_K^4$.
- Pick $4$ vertices uniformly at random. If $(c(v_1), c(v_2), c(v_3), c(v_4)) \in F$ then choose edge $e_1$ else $e_2$

## BF model

$K = 1$, $F = \{(1, 1, \alpha, \beta)\}$.

# Applied context



The Journal of Neuroscience, November 2, 2011 • 31(44):15757–15767 • **15757**

Behavioral/Systems/Cognitive

## Emergence of Persistent Networks in Long-Term Intracranial EEG Recordings

**Mark A. Kramer,[1] Uri T. Eden,[1] Kyle Q. Lepage,[1] Eric D. Kolaczyk,[1] Matt T. Bianchi,[2,3] and Sydney S. Cash[2,3]**
[1]Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215, [2]Department of Neurology, Massachusetts General Hospital, Boston, Massachusetts 02114, and [3]Harvard Medical School, Boston, Massachusetts 02115

Over the past two decades, the increased ability to analyze network relationships among neural structures has provided novel insights into brain function. Most network approaches, however, focus on static representations of the brain's physical or statistical connectivity. Few studies have examined how brain functional networks evolve spontaneously over long epochs of continuous time. To address this, we examine functional connectivity networks deduced from continuous long-term electrocorticogram recordings. For a population of six human patients, we identify a persistent pattern of connections that form a frequency-band-dependent network template, and a set of core connections that appear frequently and together. These structures are robust, emerging from brief time intervals (~100 s) regardless of cognitive state. These results suggest that a metastable, frequency-band-dependent scaffold of brain connectivity exists from which transient activity emerges and recedes.

**Introduction**

trieval). Measures of brain metabolic activity—e.g., the func-

## Main questions

- Question: when $t = t_c$, do we have $\mathcal{C}_n^{(1)} \sim n^{2/3}$? How do components merge?

## Main questions

- Question: when $t = t_c$, do we have $\mathcal{C}_n^{(1)} \sim n^{2/3}$? How do components merge? scaling window?
- What about the surplus of the largest components in the scaling window?

# Notation

- $\mathcal{C}_n^{(i)}(t)$ size of $i$-th largest component at time $t$

# Notation

- $\mathcal{C}_n^{(i)}(t)$ size of $i$-th largest component at time $t$
- Surplus (Complexity) of a component

$$\xi_n^{(i)}(t) = E(\mathcal{C}_n^{(i)}(t)) - (\mathcal{C}_n^{(i)}(t) - 1)$$

# Notation

- $\mathcal{C}_n^{(i)}(t)$ size of $i$-th largest component at time $t$
- Surplus (Complexity) of a component

$$\xi_n^{(i)}(t) = E(\mathcal{C}_n^{(i)}(t)) - (\mathcal{C}_n^{(i)}(t) - 1)$$

- $l_\downarrow^2 = \{(x_i)_{i\geq 1} : x_1 \geq x_2 \geq \cdots \geq 0, \sum_i x_i^2 < \infty\}$
- $l_\downarrow^{2,*} = \left\{(x_i, y_i)_{i\geq 1} : (x_i) \in l_\downarrow^2, y_i \in \mathbb{Z}_+, \sum_i x_i y_i < \infty\right\}$
- $d((x,y),(x',y')) = \sqrt{\sum_i (x_i - x_i')^2} + \sum_i |x_i y_i - x_i' y_i'| + \sum_{i=1}^\infty \frac{|y_i - y_i'|}{2^i}$

# The Erdős-Rényi random graph

## Theorem (Aldous 1997)

*Let* $(\mathcal{C}_n^{(1)}(t), \mathcal{C}_n^{(2)}(t), \ldots)$ *be the component sizes of* $\mathcal{G}_n^{ER}(t)$ *in decreasing order and* $\xi_i(t)$ *the corresponding complexity (surplus). Define rescaled size vector* $\mathbf{C}_n^*(\lambda)$*,* $-\infty < \lambda < +\infty$ *as*

$$\Big( (\frac{1}{n^{2/3}} \mathcal{C}_n^{(i)}(t_c + \frac{\lambda}{n^{1/3}}),$$

# The Erdős-Rényi random graph

## Theorem (Aldous 1997)

*Let* $(\mathcal{C}_n^{(1)}(t), \mathcal{C}_n^{(2)}(t), \ldots)$ *be the component sizes of* $\mathcal{G}_n^{ER}(t)$ *in decreasing order and* $\xi_i(t)$ *the corresponding complexity (surplus). Define rescaled size vector* $\mathbf{C}_n^*(\lambda)$, $-\infty < \lambda < +\infty$ *as*

$$\left( \left(\frac{1}{n^{2/3}}\mathcal{C}_n^{(i)}(t_c + \frac{\lambda}{n^{1/3}}), \xi_n^{(i)}(t_c + \frac{\lambda}{n^{1/3}})\right) : i \geq 1 \right)$$

*Then* $\mathbf{C}_n(\lambda) \xrightarrow{d} \mathbf{X}(\lambda) = (X(\lambda), \xi(\lambda))$. *Here* $(X(\lambda), -\infty < \lambda < +\infty)$ *is the standard multiplicative coalescent, a continuous time Markov process on the state space* $l^2_\downarrow$.

# The Erdős-Rényi random graph

## Theorem (Aldous 1997)

*Let* $(\mathcal{C}_n^{(1)}(t), \mathcal{C}_n^{(2)}(t), \ldots)$ *be the component sizes of* $\mathcal{G}_n^{ER}(t)$ *in decreasing order and* $\xi_i(t)$ *the corresponding complexity (surplus). Define rescaled size vector* $\mathbf{C}_n^*(\lambda)$*,* $-\infty < \lambda < +\infty$ *as*

$$\left( \left(\frac{1}{n^{2/3}} \mathcal{C}_n^{(i)}\left(t_c + \frac{\lambda}{n^{1/3}}\right), \xi_n^{(i)}\left(t_c + \frac{\lambda}{n^{1/3}}\right)\right) : i \geq 1 \right)$$

*Then* $\mathbf{C}_n(\lambda) \xrightarrow{d} \mathbf{X}(\lambda) = (X(\lambda), \xi(\lambda))$*. Here* $(X(\lambda), -\infty < \lambda < +\infty)$ *is the standard multiplicative coalescent, a continuous time Markov process on the state space* $l^2_\downarrow$*.*

## Distribution for fixed $\lambda$

- For fixed $\lambda \in \mathbb{R}$, let
$$W_\lambda(t) = W(t) + \lambda t - \frac{t^2}{2},$$
- $\bar{W}_\lambda(\cdot)$ is the above process reflected at $0$.
- $X(\lambda)$ has same distribution as lengths of excursions away from $0$ of $\bar{W}(\cdot)$ arranged in decreasing order

## The standard multiplicative coalescent $\mathbf{X}(\lambda)$

- Think of each edge having a Poisson rate $1/n$ clock
- $\bar{\mathcal{C}}_i(\lambda) = n^{-2/3}\mathcal{C}_i(1+\lambda/n^{1/3})$
- At some time $\lambda$, rate at which two components $i, j$ merge:

$$\mathcal{C}_i(1+\frac{\lambda}{n^{1/3}})\mathcal{C}_j(1+\frac{\lambda}{n^{1/3}})$$

# The standard multiplicative coalescent $\mathbf{X}(\lambda)$

- Think of each edge having a Poisson rate $1/n$ clock
- $\bar{\mathcal{C}}_i(\lambda) = n^{-2/3}\mathcal{C}_i(1+\lambda/n^{1/3})$
- At some time $\lambda$, rate at which two components $i, j$ merge:

$$\mathcal{C}_i(1+\frac{\lambda}{n^{1/3}})\mathcal{C}_j(1+\frac{\lambda}{n^{1/3}}) \cdot \frac{1}{n} \cdot$$

# The standard multiplicative coalescent $\mathbf{X}(\lambda)$

- Think of each edge having a Poisson rate $1/n$ clock
- $\bar{\mathcal{C}}_i(\lambda) = n^{-2/3}\mathcal{C}_i(1 + \lambda/n^{1/3})$
- At some time $\lambda$, rate at which two components $i, j$ merge:

$$\mathcal{C}_i(1 + \frac{\lambda}{n^{1/3}})\mathcal{C}_j(1 + \frac{\lambda}{n^{1/3}}) \cdot \frac{1}{n} \cdot \frac{1}{n^{1/3}}$$

# The standard multiplicative coalescent $\mathbf{X}(\lambda)$

- Think of each edge having a Poisson rate $1/n$ clock
- $\bar{\mathcal{C}}_i(\lambda) = n^{-2/3}\mathcal{C}_i(1+\lambda/n^{1/3})$
- At some time $\lambda$, rate at which two components $i, j$ merge:

$$\mathcal{C}_i(1+\frac{\lambda}{n^{1/3}})\mathcal{C}_j(1+\frac{\lambda}{n^{1/3}})\cdot\frac{1}{n}\cdot\frac{1}{n^{1/3}} = \bar{\mathcal{C}}_i(\lambda)\bar{\mathcal{C}}_j(\lambda)$$

# The standard multiplicative coalescent $\mathbf{X}(\lambda)$

- Think of each edge having a Poisson rate $1/n$ clock
- $\bar{\mathcal{C}}_i(\lambda) = n^{-2/3}\mathcal{C}_i(1+\lambda/n^{1/3})$
- At some time $\lambda$, rate at which two components $i, j$ merge:

$$\mathcal{C}_i(1+\frac{\lambda}{n^{1/3}})\mathcal{C}_j(1+\frac{\lambda}{n^{1/3}}) \cdot \frac{1}{n} \cdot \frac{1}{n^{1/3}} = \bar{\mathcal{C}}_i(\lambda)\bar{\mathcal{C}}_j(\lambda)$$

## Dynamics of $\mathbf{X}(\lambda)$

- suppose $\mathbf{X}(\lambda) = (x_1, x_2, x_3, ...)$, each $x_l$ is viewed as the size of a cluster.
- each pair of clusters of sizes $(x_i, x_j)$ merges at rate $x_i x_j$ into a cluster of size $x_i + x_j$ .
- if $x_i, x_j$ is merging, then $(x_1, x_2, x_3, ...) \rightsquigarrow (x'_1, x'_2, x'_3, ...)$ where the latter is the re-ordering of $\{x_i + x_j, x_l : l \neq i, j\}$.

# Bounded size rules

## Theorem (Bhamidi, Budhiraja, Wang, 2012)

*Let* $(\mathcal{C}_n^{(1)}(t), \mathcal{C}_n^{(2)}(t), \ldots)$ *be the component sizes of* $\mathcal{G}_n^{BSR}(t)$ *in decreasing order and* $\xi_i(t)$ *the corresponding surplus. Define the rescaled size vector* $\mathbf{C}_n(\lambda)$, $-\infty < \lambda < +\infty$ *as the vector*

$$((\bar{\mathcal{C}}_i(\lambda), \xi_i(\lambda) : i \geq 1) = \left( \frac{\beta^{1/3}}{n^{2/3}} \mathcal{C}_n^{(i)}(t_c + \frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}), \xi_i(t_c + \frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}) : i \geq 1 \right)$$

*where* $\alpha, \beta$ *are constants determined by the BSR process. Then*

$$\mathbf{C}_n(\lambda) \xrightarrow{d} \mathbf{X}(\lambda)$$

*where* $(\mathbf{X}(\lambda), -\infty < \lambda < +\infty)$ *is the standard augmented multiplicative coalescent and convergence happens in* $l^2_\downarrow$ *with metric* $d$.

# Techniques

- **Branching process methods:** Great tool above and below criticality.

# Techniques

- **Branching process methods:** Great tool above and below criticality.
- **Exploration walks:** Very refined results

# Techniques

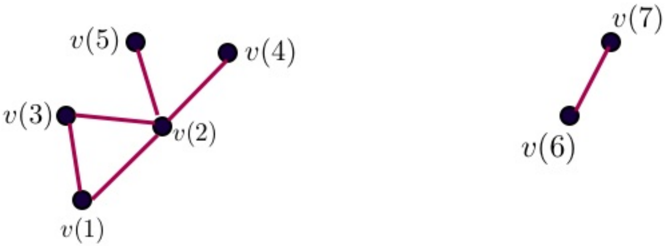
- **Branching process methods:** Great tool above and below criticality.
- **Exploration walks:** Very refined results **presence of lots of independence**,

# Techniques

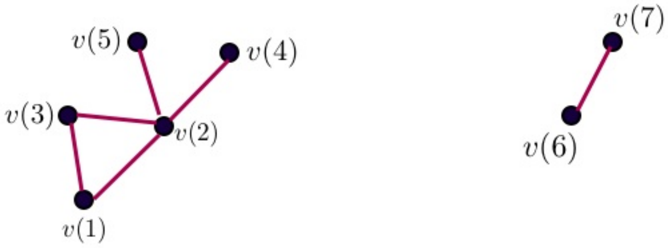
- **Branching process methods:** Great tool above and below criticality.
- **Exploration walks:** Very refined results **presence of lots of independence**, including structure of components
- **Differential equation method:** Technical, standard workhorse for such models. Can be pushed all the way to the critical window.

# Typical method of proof: Exploration

# Typical method of proof: Exploration



$c(1) = 2$
$c(2) = 2$
$c(3) = 0$
$\cdots$



# Typical method of proof

## Exploration of the graph

- Explore the components of the graph one by one
- choose a vertex. Let $c(1)$ be the number of children of this vertex
- choose one of the children of this vertex, let $c(2)$ be number of children of this vertex
- continue, when one component completed move onto another component
- Define $Z(0) = 0, \quad Z(i) = Z(i-1) + c(i) - 1$
- $Z(\cdot) = -1$ for the first time when we finish exploring component $1$, then hits $-2$ for first time when exploring component $2$ and so on.

# Typical method of proof

## Exploration of the graph

- Explore the components of the graph one by one
- choose a vertex. Let $c(1)$ be the number of children of this vertex
- choose one of the children of this vertex, let $c(2)$ be number of children of this vertex
- continue, when one component completed move onto another component
- Define $Z(0) = 0, \quad Z(i) = Z(i-1) + c(i) - 1$
- $Z(\cdot) = -1$ for the first time when we finish exploring component $1$, then hits $-2$ for first time when exploring component $2$ and so on.
- Try to use Martingale functional limit theorem to show $\frac{1}{n^{1/3}} Z(n^{2/3}t) \to_d W^{\lambda}(t)$

# Typical method of proof

## Exploration of the graph

- Explore the components of the graph one by one
- choose a vertex. Let $c(1)$ be the number of children of this vertex
- choose one of the children of this vertex, let $c(2)$ be number of children of this vertex
- continue, when one component completed move onto another component
- Define $Z(0) = 0, \quad Z(i) = Z(i-1) + c(i) - 1$
- $Z(\cdot) = -1$ for the first time when we finish exploring component $1$, then hits $-2$ for first time when exploring component $2$ and so on.
- Try to use Martingale functional limit theorem to show $\frac{1}{n^{1/3}} Z(n^{2/3}t) \to_d W^{\lambda}(t)$

# Bounded size rules

- Hard to think about exploration process especially at criticality

# Bounded size rules

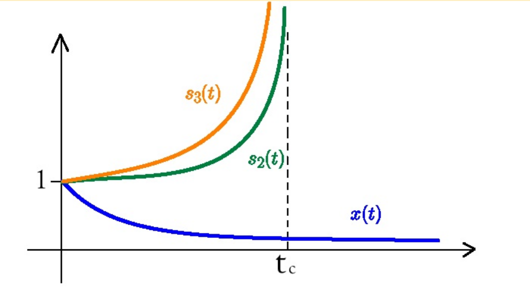
- Hard to think about exploration process especially at criticality
- Turns out: Easier to analyze the entire process

# Proof idea: The Bohman-Frieze process

## Where does $t_c$ come from ?

Define $X_n(t) = \#$ of singletons, $S_2(t) = \sum_i (\mathcal{C}_n^{(i)}(t))^2$, $S_3(t) = \sum_i (\mathcal{C}_n^{(i)})^3$.
and $\bar{x}_n(t) = X_n(t)/n$, $\bar{s}_2(t) = S_2/n$, $\bar{s}_3(t) = S_3/n$.
Then [Spencer, Wormald 2004] for any fix $t > 0$,

$$\bar{x}_n(t) \xrightarrow{\mathbb{P}} x(t), \qquad \bar{s}_2(t) \xrightarrow{\mathbb{P}} s_2(t), \qquad \bar{s}_3(t) \xrightarrow{\mathbb{P}} s_3(t)$$



# Why?

## Behavior of $x_n(t)$

- In small time interval $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 1/n$ at rate
$$\frac{2}{n^3}\left(\binom{n}{2} - \binom{X_n(t)}{2}\right) X_n(t)(n - X_n(t)) \sim n(1 - x_n^2(t))x_n(t)(1 - x_n(t))$$

# Why?

## Behavior of $x_n(t)$

- In small time interval $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 1/n$ at rate
$$\frac{2}{n^3}\left(\binom{n}{2} - \binom{X_n(t)}{2}\right) X_n(t)(n - X_n(t)) \sim n(1 - x_n^2(t))x_n(t)(1 - x_n(t))$$
- $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 2/n$ at rate
$$\frac{2}{n^3}\left[\binom{X_n(t)}{2}\binom{n}{2} + \left(\binom{n}{2} - \binom{X_n(t)}{2}\right)\binom{X_n(t)}{2}\right] \sim \frac{1}{2}(x_n^2(t) + (1 - x_n^2(t)x_n^2(t)))$$

# Why?

## Behavior of $x_n(t)$

- In small time interval $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 1/n$ at rate

$$\frac{2}{n^3}\left(\binom{n}{2} - \binom{X_n(t)}{2}\right) X_n(t)(n - X_n(t)) \sim n(1 - x_n^2(t))x_n(t)(1 - x_n(t))$$

- $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 2/n$ at rate

$$\frac{2}{n^3}\left[\binom{X_n(t)}{2}\binom{n}{2} + \left(\binom{n}{2} - \binom{X_n(t)}{2}\right)\binom{X_n(t)}{2}\right] \sim \frac{1}{2}(x_n^2(t) + (1 - x_n^2(t)x_n^2(t)))$$

- Suggests that $x_n(t) \to x(t)$ where

$$x'(t) = -x^2(t) - (1 - x^2(t))x(t) \qquad \text{for } t \in [0, \infty, ) \qquad x(0) = 1$$

# Why?

## Behavior of $x_n(t)$

- In small time interval $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 1/n$ at rate
$$\frac{2}{n^3}\left(\binom{n}{2} - \binom{X_n(t)}{2}\right) X_n(t)(n - X_n(t)) \sim n(1 - x_n^2(t))x_n(t)(1 - x_n(t))$$
- $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 2/n$ at rate
$$\frac{2}{n^3}\left[\binom{X_n(t)}{2}\binom{n}{2} + \left(\binom{n}{2} - \binom{X_n(t)}{2}\right)\binom{X_n(t)}{2}\right] \sim \frac{1}{2}(x_n^2(t) + (1 - x_n^2(t)x_n^2(t)))$$
- Suggests that $x_n(t) \to x(t)$ where
$$x'(t) = -x^2(t) - (1 - x^2(t))x(t) \qquad \text{for } t \in [0, \infty, ) \qquad x(0) = 1$$
- Similar analysis suggests that for $\bar{s}_2(t), \bar{s}_3(t)$
$$s_2'(t) = x^2(t) + (1 - x^2(t))s_2^2(t) \qquad \text{for } t \in [0, t_c), \qquad s_2(0) = 1$$
$$s_3'(t) = 3x^2(t) + 3(1 - x^2(t))s_2(t)s_3(t) \qquad \text{for } t \in [0, t_c), \qquad s_3(0) = 1.$$

# The Bohman-Frieze process

**Scaling exponents of $s_2$ and $s_3$ (Janson, Spencer 11)**

- Functions $x(t), s_2(t), s_3(t)$ are determined by some differential equations
- Differential equations imply $\exists$ constants $\alpha, \beta$ such that $t \uparrow t_c$

$$s_2(t) \sim \frac{\alpha}{t_c - t}$$

$$s_3(t) \sim \beta (s_2(t))^3 \sim \beta \frac{\alpha^3}{(t_c - t)^3}$$

# I: Regularity conditions of the component sizes at "$-\infty$"

- Let $\bar{\mathbf{C}}(\lambda) = n^{-2/3}\mathbf{C}\left(t_c + \beta^{2/3}\alpha\lambda/n^{1/3}\right)$.

# I: Regularity conditions of the component sizes at "$-\infty$"

- Let $\bar{\mathbf{C}}(\lambda) = n^{-2/3}\mathbf{C}\left(t_c + \beta^{2/3}\alpha\lambda/n^{1/3}\right)$.
- For $\delta \in (1/6, 1/5)$ let $t_n = t_c - n^{-\delta} = t_c + \beta^{2/3}\alpha\frac{\lambda_n}{n^{1/3}}$, then $\lambda_n = -\beta^{2/3}\alpha n^{1/3-\delta}$.

# I: Regularity conditions of the component sizes at "$-\infty$"

- Let $\bar{\mathbf{C}}(\lambda) = n^{-2/3}\mathbf{C}\left(t_c + \beta^{2/3}\alpha\lambda/n^{1/3}\right)$.
- For $\delta \in (1/6, 1/5)$ let $t_n = t_c - n^{-\delta} = t_c + \beta^{2/3}\alpha\frac{\lambda_n}{n^{1/3}}$, then $\lambda_n = -\beta^{2/3}\alpha n^{1/3-\delta}$.
- Need to verify the three conditions

$$\frac{\sum_i \left(\bar{\mathcal{C}}_i(\lambda_n)\right)^3}{\left[\sum_i \left(\bar{\mathcal{C}}_i(\lambda_n)\right)^2\right]^3} \xrightarrow{\mathbb{P}} 1 \qquad \Leftrightarrow \qquad \frac{n^2 S_3(t_n)}{S_2^3(t_n)} \xrightarrow{\mathbb{P}} \beta$$

$$\frac{1}{\sum_i \left(\bar{\mathcal{C}}_i(\lambda_n)\right)^2} + \lambda_n \xrightarrow{\mathbb{P}} 0 \qquad \Leftrightarrow \qquad \frac{n^{4/3}}{S_2(t_n)} - \frac{n^{-\delta+1/3}}{\alpha} \xrightarrow{\mathbb{P}} 0$$

$$\frac{\bar{\mathcal{C}}_1(\lambda_n)}{\sum_i \left(\bar{\mathcal{C}}_i(\lambda_n)\right)^2} \xrightarrow{\mathbb{P}} 0 \qquad \Leftrightarrow \qquad \frac{n^{2/3}\mathcal{C}_n^{(1)}(t_n)}{S_2(t_n)} \xrightarrow{\mathbb{P}} 0$$

# II: Dynamics of merging in the critical window

## The dynamic of merging

- In any small time interval $[t, t+dt)$, two components $i$ and $j$ merge at rate

$$\frac{2}{n^3}\left[\binom{n}{2} - \binom{X_n(t)}{2}\right]\mathcal{C}_i(t)\mathcal{C}_j(t)$$
$$\sim \frac{1}{n}(1-\bar{x}^2(t))\mathcal{C}_i(t)\mathcal{C}_j(t)$$

  Let $\lambda = (t-t_c)n^{1/3}/\alpha\beta^{2/3}$ be rescaled time paramter, rate at which two components merge

$$\gamma_{ij}(\lambda) \sim \frac{(1-x^2(t_c+\beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}))}{n}\frac{\beta^{2/3}\alpha}{n^{1/3}}\mathcal{C}_i\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)\mathcal{C}_j\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)$$

# II: Dynamics of merging in the critical window

## The dynamic of merging

- In any small time interval $[t, t+dt)$, two components $i$ and $j$ merge at rate

$$\frac{2}{n^3}\left[\binom{n}{2} - \binom{X_n(t)}{2}\right]\mathcal{C}_i(t)\mathcal{C}_j(t)$$
$$\sim \frac{1}{n}(1-\bar{x}^2(t))\mathcal{C}_i(t)\mathcal{C}_j(t)$$

Let $\lambda = (t-t_c)n^{1/3}/\alpha\beta^{2/3}$ be rescaled time paramter, rate at which two components merge

$$\gamma_{ij}(\lambda) \sim \frac{(1-x^2(t_c+\beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}))}{n}\frac{\beta^{2/3}\alpha}{n^{1/3}}\mathcal{C}_i\left(t_c + \frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)\mathcal{C}_j\left(t_c + \frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)$$
$$= \alpha\left(1 - x^2\left(t_c + \beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}\right)\right)\bar{\mathcal{C}}_i(\lambda)\bar{\mathcal{C}}_j(\lambda)$$

# II: Dynamics of merging in the critical window

## The dynamic of merging

- In any small time interval $[t, t+dt)$, two components $i$ and $j$ merge at rate

$$\frac{2}{n^3}\left[\binom{n}{2} - \binom{X_n(t)}{2}\right]\mathcal{C}_i(t)\mathcal{C}_j(t)$$
$$\sim \frac{1}{n}(1-\bar{x}^2(t))\mathcal{C}_i(t)\mathcal{C}_j(t)$$

Let $\lambda = (t-t_c)n^{1/3}/\alpha\beta^{2/3}$ be rescaled time paramter, rate at which two components merge

$$\gamma_{ij}(\lambda) \sim \frac{(1-x^2(t_c+\beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}))}{n}\frac{\beta^{2/3}\alpha}{n^{1/3}}\mathcal{C}_i\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)\mathcal{C}_j\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)$$
$$= \alpha\left(1-x^2\left(t_c+\beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}\right)\right)\bar{\mathcal{C}}_i(\lambda)\bar{\mathcal{C}}_j(\lambda)$$
$$= \bar{\mathcal{C}}_i(\lambda)\bar{\mathcal{C}}_j(\lambda) \qquad \text{since } \alpha(1-x^2(t_c)) = 1$$

# How to check regularity conditions

## Analysis of $\mathcal{C}_n^{(1)}(t)$

- Key point: need to get refined bounds on maximal component in barely subcritical regime.
- known result: for fixed $t < t_c$, $\mathcal{C}_n^{(1)}(t) = O(\log n)$.
- not enough, since we want a sharp upper bound when $t \uparrow t_c$.

# How to check regularity conditions

## Analysis of $\mathcal{C}_n^{(1)}(t)$

- Key point: need to get refined bounds on maximal component in barely subcritical regime.
- known result: for fixed $t < t_c$, $\mathcal{C}_n^{(1)}(t) = O(\log n)$.
- not enough, since we want a sharp upper bound when $t \uparrow t_c$.

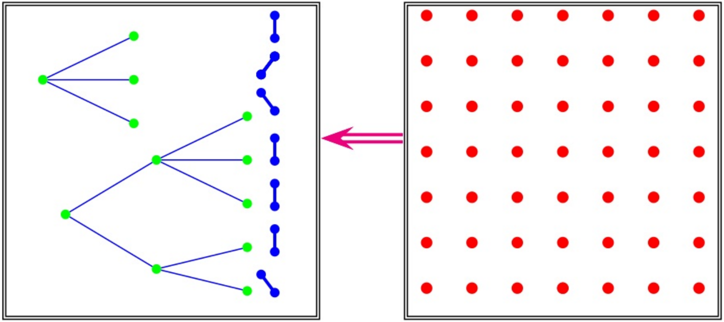
## Lemma (Bounds on the largest component)

*Let $\delta \in (0, 1/5)$, $t_c$ be the critical time for the BF process, $\mathcal{C}_n^{(1)}(t)$ be the size of the largest component. Then there exists a constant $B = B(\delta)$ such that as $n \to +\infty$,*

$$\mathbb{P}\{\mathcal{C}_n^{(1)}(t) \leq \frac{B \log^4 n}{(t_c - t)^2} \text{ for all } t < t_c - n^{-\delta}\} \to 1$$

*Proof strategy: Coupling with a near critical multi-type branching process on an infinite dimensional type space. delicate analysis of the maximal eigenvalue.*

# Random graph with Immigrating doubletons



# Sketch of the proof

## Regularity condition at time $\lambda = -\infty$

Check the following properties for the un-scaled component sizes. For $\delta \in (1/6, 1/5)$, and $t_n = t_c - n^{-\delta}$,

$$\frac{n^2 S_3(t_n)}{S_2^3(t_n)} \xrightarrow{\mathbb{P}} \beta \tag{1}$$

$$\frac{n^{4/3}}{S_2(t_n)} - \frac{n^{-\delta+1/3}}{\alpha} \xrightarrow{\mathbb{P}} 0 \tag{2}$$

$$\frac{n^{2/3}\mathcal{C}_n^{(1)}(t_n)}{S_2(t_n)} \xrightarrow{\mathbb{P}} 0 \tag{3}$$

# Sketch of the proof

## Regularity condition at time $\lambda = -\infty$

Check the following properties for the un-scaled component sizes. For $\delta \in (1/6, 1/5)$, and $t_n = t_c - n^{-\delta}$,

$$\frac{n^2 S_3(t_n)}{S_2^3(t_n)} \xrightarrow{\mathbb{P}} \beta \tag{1}$$

$$\frac{n^{4/3}}{S_2(t_n)} - \frac{n^{-\delta+1/3}}{\alpha} \xrightarrow{\mathbb{P}} 0 \tag{2}$$

$$\frac{n^{2/3}\mathcal{C}_n^{(1)}(t_n)}{S_2(t_n)} \xrightarrow{\mathbb{P}} 0 \tag{3}$$

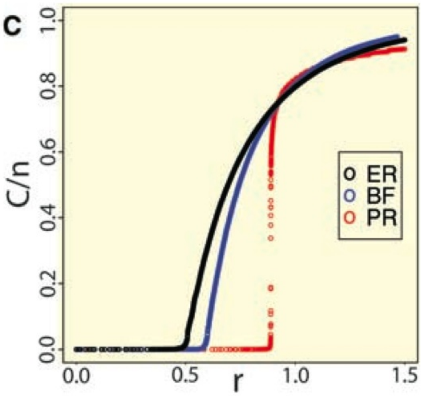
## Analysis of $S_2(t)$, $S_3(t)$

- above relations hold for limiting functions $s_2, s_3$ for $t_n$.
- Delicate stochastic analytic argument combined with result on $\mathcal{C}_n^{(1)}(t)$ to show this holds for $S_2, S_3$ near criticality.

# Work in progress

## Explosive percolation

In 2009, Achlioptas,D'Souza and Spencer considered "product rule". Conjectured that this process exhibits Explosive percolation



# Truncated product rule

## Fix $K$

- Choose 2 edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ at random
- If $\max\{C(v_1), C(v_2)C(v_3), C(v_4)\} \leq K$, then use the edge which minimizes of $\min\{C(v_1)C(v_2), C(v_3)C(v_4)\}$.
- Else use $e_2$.

# Truncated product rule

## Fix $K$

- Choose 2 edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ at random
- If $\max\{C(v_1), C(v_2)C(v_3), C(v_4)\} \leq K$, then use the edge which minimizes of $\min\{C(v_1)C(v_2), C(v_3)C(v_4)\}$.
- Else use $e_2$.

## Work in progress

- Consider the rescaled and re-centered component sizes

$$\mathbf{C}_K(\lambda) = \left(\frac{1}{n^{2/3}}\mathcal{C}_K^{(i)}\left(t_c(K) + \gamma(K)\frac{\lambda}{n^{1/3}}\right) : i \geq 1\right) \qquad \lambda \in \mathbb{R}$$

  Then we have $(\mathbf{C}_K(\lambda) : \lambda \in \mathbb{R}) \xrightarrow{d} (X(\lambda) : \lambda \in \mathbb{R})$ as $n \to \infty$.

- $t_c(K) \to t_c$;

# Truncated product rule

## Fix $K$

- Choose 2 edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ at random
- If $\max\{C(v_1), C(v_2)C(v_3), C(v_4)\} \leq K$, then use the edge which minimizes of $\min\{C(v_1)C(v_2), C(v_3)C(v_4)\}$.
- Else use $e_2$.

## Work in progress

- Consider the rescaled and re-centered component sizes

$$\mathbf{C}_K(\lambda) = \left(\frac{1}{n^{2/3}}\mathcal{C}_K^{(i)}\left(t_c(K) + \gamma(K)\frac{\lambda}{n^{1/3}}\right) : i \geq 1\right) \qquad \lambda \in \mathbb{R}$$

  Then we have $(\mathbf{C}_K(\lambda) : \lambda \in \mathbb{R}) \xrightarrow{d} (X(\lambda) : \lambda \in \mathbb{R})$ as $n \to \infty$.
- $t_c(K) \to t_c$; $\boldsymbol{\gamma(K) \to 0}$

# Truncated product rule

## Fix $K$

- Choose 2 edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ at random
- If $\max\{C(v_1), C(v_2)C(v_3), C(v_4)\} \leq K$, then use the edge which minimizes of $\min\{C(v_1)C(v_2), C(v_3)C(v_4)\}$.
- Else use $e_2$.

## Work in progress

- Consider the rescaled and re-centered component sizes

$$\mathbf{C}_K(\lambda) = \left(\frac{1}{n^{2/3}}\mathcal{C}_K^{(i)}\left(t_c(K) + \gamma(K)\frac{\lambda}{n^{1/3}}\right) : i \geq 1\right) \qquad \lambda \in \mathbb{R}$$

  Then we have $(\mathbf{C}_K(\lambda) : \lambda \in \mathbb{R}) \xrightarrow{d} (X(\lambda) : \lambda \in \mathbb{R})$ as $n \to \infty$.

- $t_c(K) \to t_c$; $\boldsymbol{\gamma(K) \to 0}$

# Other questions

## “Natural questions”

- What happens if we start with a configuration other than the empty graph?
- Related to the entrance boundary of the multiplicative coalescent.

## Unnatural next questions

- Scaling limits?

# Other questions

## “Natural questions”

- What happens if we start with a configuration other than the empty graph?
- Related to the entrance boundary of the multiplicative coalescent.

## Unnatural next questions

- Scaling limits?
- Conjecture: Rescale each edge by $n^{-1/3}$
- Largest components converge to random fractals (Gromov-Hausdorff sense), the same limits as for Erdos-Renyii

# Minimal spanning tree

- Connected Graph, put distinct positive edge lengths (road network)
- Want to get a spanning graph,

# Minimal spanning tree

- Connected Graph, put distinct positive edge lengths (road network)
- Want to get a spanning graph, minimal total weight
- Choose spanning tree with minimal total weight: MST

# Minimal spanning tree

- Connected Graph, put distinct positive edge lengths (road network)
- Want to get a spanning graph, minimal total weight
- Choose spanning tree with minimal total weight: MST

- Enormous literature in applied sciences
- Deep connections to statistical physics models of disorder

## Statistical physics models of disorder

### Weak disorder (First passage percolation)

- Weight of a path = sum of weight on edges
- Choose optimal path

# Statistical physics models of disorder

### Weak disorder (First passage percolation)

- Weight of a path = sum of weight on edges
- Choose optimal path

### Strong disorder (Minimal spanning tree)

- Weight of path = max edge on the path
- Choose optimal path

# Adario-Berry, Broutin, Goldschmidt + Miermont

## Minimal spanning tree

- Consider complete graph, each edge given iid continuous edge length
- $\mathcal{M}_n$ minimal spanning tree
- Asymptotics?

# Adario-Berry, Broutin, Goldschmidt + Miermont

## Minimal spanning tree

- Consider complete graph, each edge given iid continuous edge length
- $\mathcal{M}_n$ minimal spanning tree
- Asymptotics?

- For the Erdos-Renyi, the largest components at criticality rescaled by $n^{-1/3}$ converge to limiting random fractals [BBG]

# Adario-Berry, Broutin, Goldschmidt + Miermont

## Minimal spanning tree

- Consider complete graph, each edge given iid continuous edge length
- $\mathcal{M}_n$ minimal spanning tree
- Asymptotics?

- For the Erdos-Renyi, the largest components at criticality rescaled by $n^{-1/3}$ converge to limiting random fractals [BBG]
- This implies that $n^{-1/3}\mathcal{M}_n$ converges to a limiting random fractal [BBGM]
- **Open Problem:** Show that for these models, have same limiting structure