Limited choice and randomness in evolution of networks
Lecture 2

Cornell Probability summer school
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Preferential attachment

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- Global characteristics such as spectral distribution of adjacency matrix?
- Variants such as limited choice or non-local preferential attachment. Analysis?

Outline of the talk

- Preferential attachment model
- Continuous time embedding
- Global results
- Convergence of local neighborhoods
- Construction of `sin`-trees (single infinite path).

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- Convergence of local neighborhoods
- Construction of `sin`-trees (single infinite path).
- Asymptotic degree distribution
- Random adjacency matrices and Spectra (Arnab Sen, Steve Evans, SB)
- Preferential attachment with choice (Omer Angel, Robin Pemantle, SB)

Basic model

Method of growing trees

Method of recursively growing random trees

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- $D(v, n) =$ out-degree of node v at time n.

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   (a) Multiplicative fitness: $f(k) = f_v(k+1)$.
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4. **Sublinear Pref Attachment:** $f(k) = (k+1)^\alpha$, $0 < \alpha < 1$

Main math idea

Simple idea [Karlin-Athreya]

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- Want to selected vertex i with probability proportional to d_i
- Simple way: Let X_i independent rate d_i exponential r.v.s
- Let J be index

$$X_J = \min_{1 \le i \le m} X_i$$

- $\mathbb{P}(J = i) \propto d_i$

Point process corresponding to attractiveness function f

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- Corresponding continuous time branching process $\mathcal{F}(t)$:
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Key connection

$T_n = \inf\{t : \mathcal{F}(t) = n\}$ **then** $\mathcal{F}(T_n) \stackrel{d}{=} \mathcal{T}_n^f$.

Branching process theory

Asymptotics

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- Processes grow exponentially: $|\mathcal{F}(t)| \sim e^{\lambda t}$
- Here λ is a very important characteristic : called the *Malthusian rate of growth*
- Given by the formula:

$$\mathbb{E}(\mathcal{P}(T_\lambda)) = 1$$

$T_\lambda \sim \exp(\lambda)$.

Exact result

More precisely, under technical conditions ($\mathbb{E}(\mathcal{P}(T_\lambda) \log^+ \mathcal{P}(T_\lambda)) < \infty$)

$$\frac{|\mathcal{F}(t)|}{e^{\lambda t}} \longrightarrow_{a.s.} W \qquad W > 0$$

$$T_n \sim \tfrac{1}{\lambda} \log n \pm O_P(1)$$

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root degree asymptotics

$\deg_n(\rho) = \mathcal{P}(\frac{1}{2}\log n + O_P(1)) \sim O_P(e^{\frac{1}{2}\log n}) = O_P(\sqrt{n})$

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More refined analysis gives

$$\frac{\text{deg}_n(\rho)}{\sqrt{n}} \xrightarrow{a.s.} Z \qquad Z \text{ has explicit recursive construction}$$

.

Maximal degree

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$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{\max \mathsf{deg}_n}{\sqrt{n}} > K_\epsilon\right) < \epsilon$$

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- With a bit more work, possible to deduce distributional convergence for the maximal degree.
- Example of interesting results: Sublinear pref attachment $f(k) = (k+1)^\alpha$

$$\frac{\mathsf{deg}_n(\rho)}{(\log n)^{\frac{1}{1-\alpha}}} \xrightarrow{P} \left(\frac{1}{\theta(\alpha)}\right)^{\frac{1}{1-\alpha}}$$

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Thus

$$\frac{B_{h_n}}{h_n} \leq \frac{T_n}{h_n} \leq \frac{B_{h_n+1}}{h_n}$$

Now use the fact that

$$\frac{T_n}{\frac{1}{\lambda}\log n} \xrightarrow{P} 1 \Rightarrow \frac{h_n}{\log n} \xrightarrow{P} C_{\texttt{model}}$$

Local asymptotics

Conceptual point

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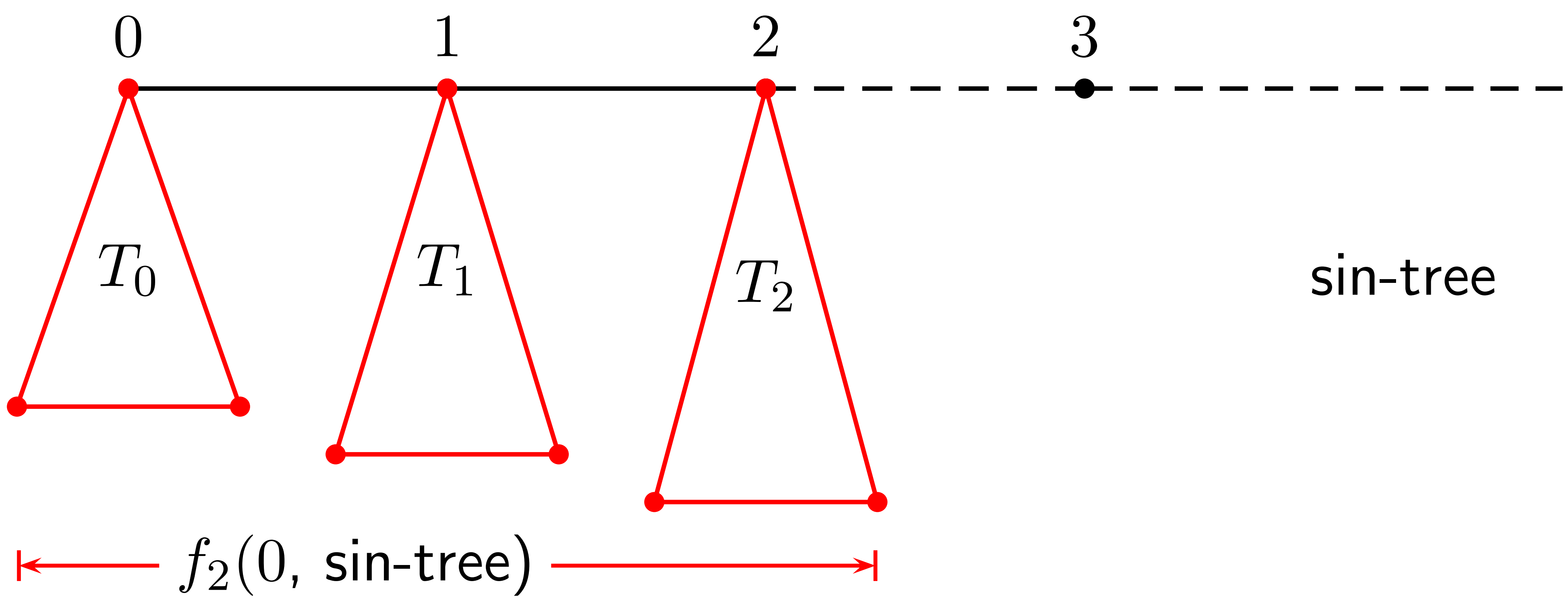
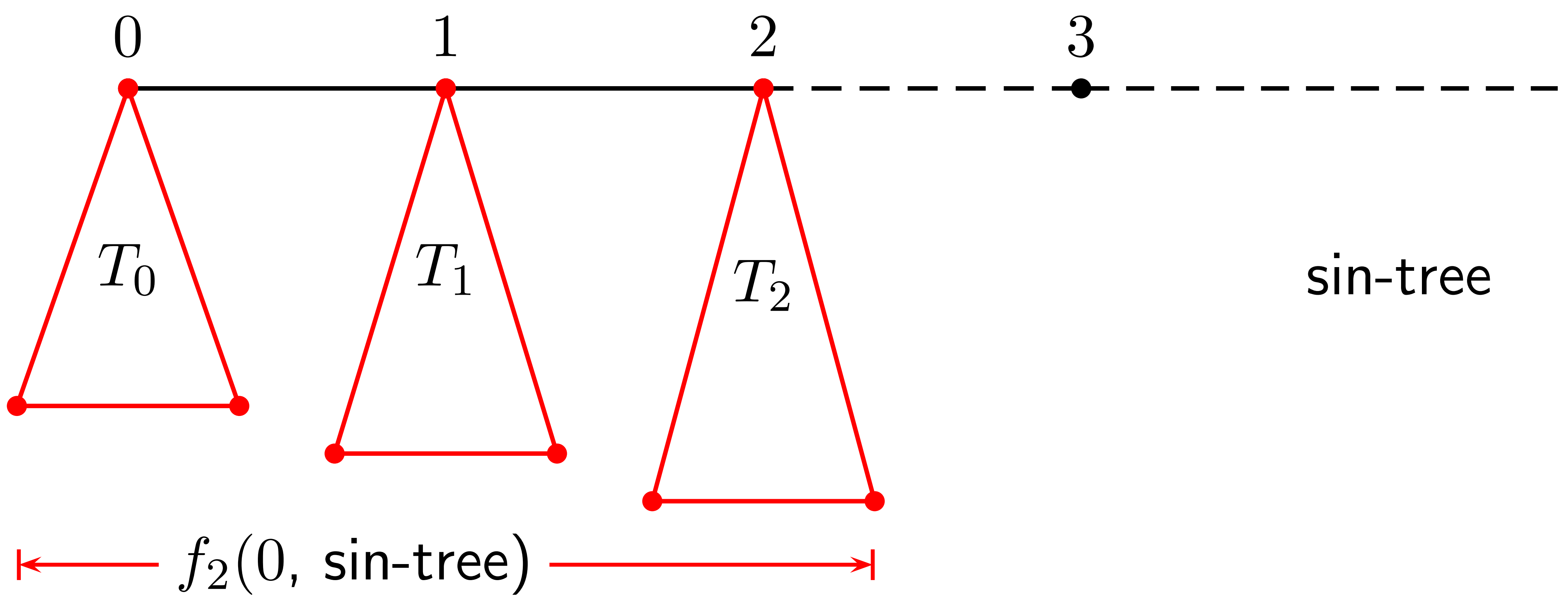
Suggests tree "below" random node looks like $\mathcal{F}(T_\lambda)$ i.e. branching process run for random exponential amount of time.

Sin-tree

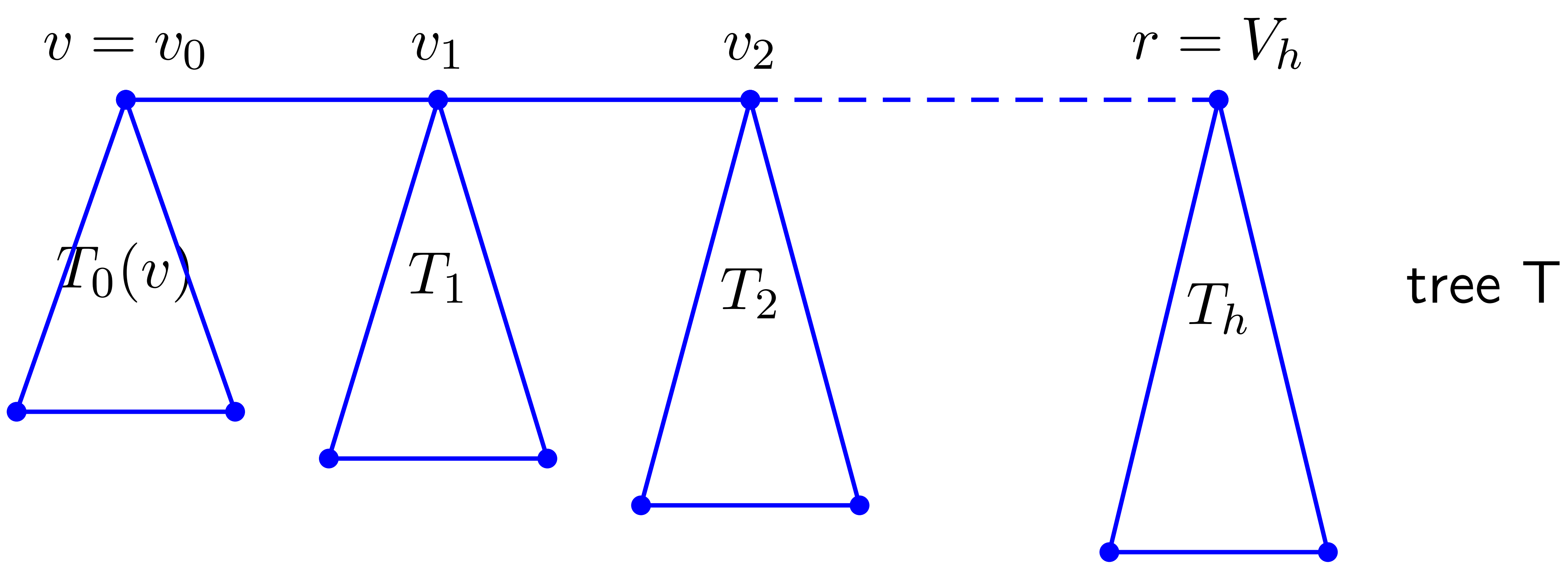




Can think of random sin-trees

Tuesday, July 24, 2012

T is a tree with root r. Given a vertex v, there exists a unique path $v_0 = v, v_1, ..., v_h = r$ from v to the root.



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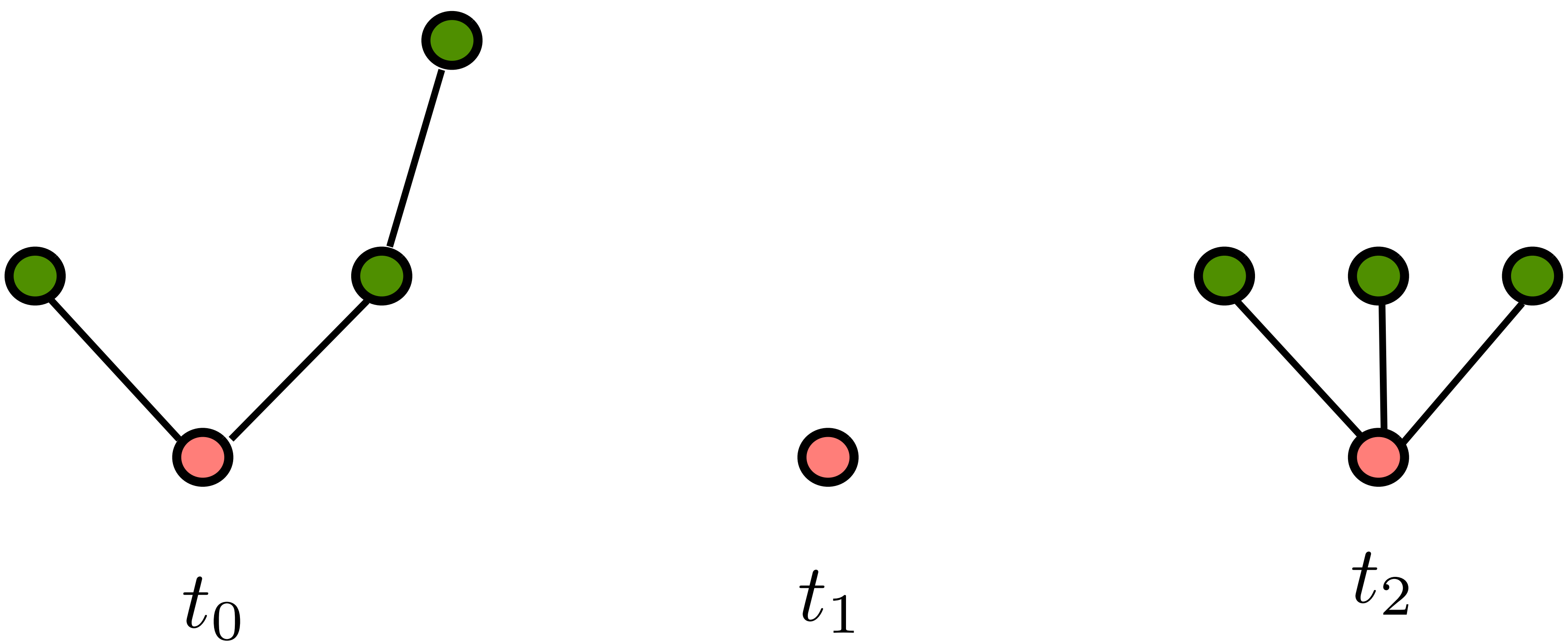
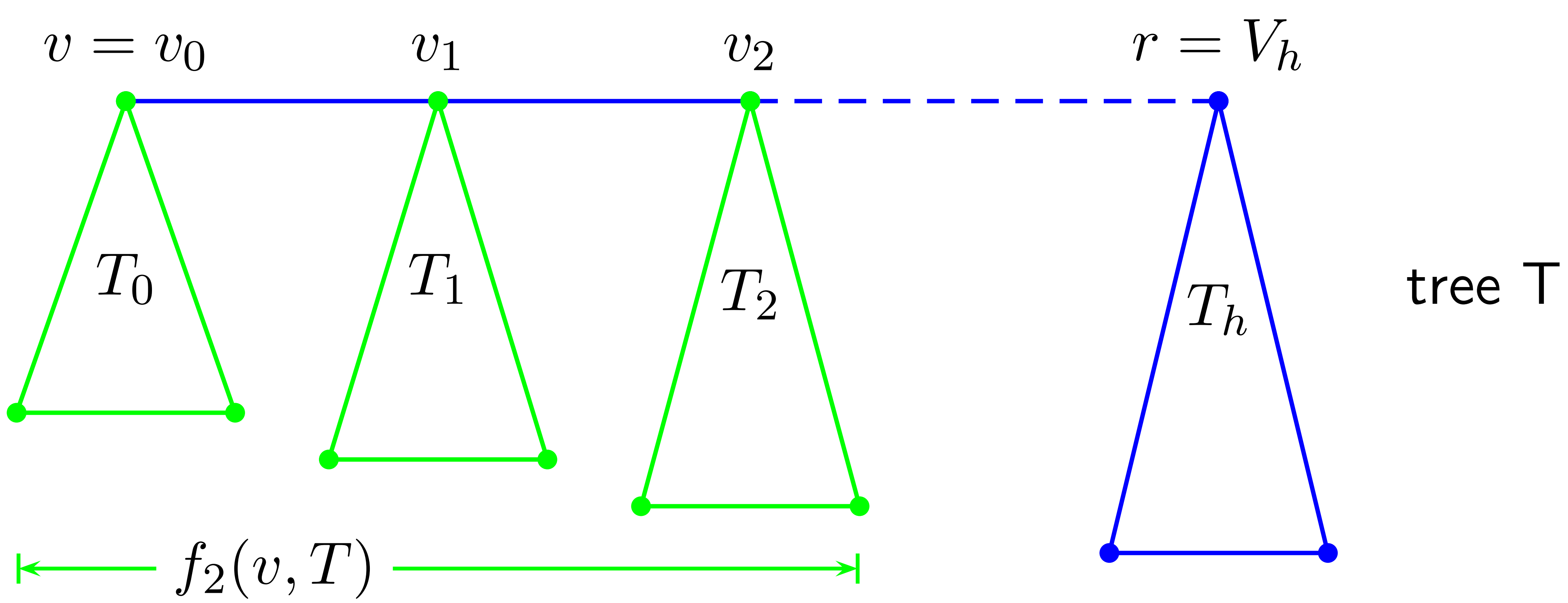


t_0
t_1
t_2

Decompose tree into a sequence of finite rooted subtrees or fringes $(T_0(v), T_1(v), T_2(v), \ldots)$. For each $k \geq 1$,

$$\frac{1}{n}\sum_{v\in T} 1(f_k(v,T) = (t_0, t_1, ..., t_k)) \xrightarrow{P} \mathbb{P}_\mu(f_k(0,\mathcal{T}) = (t_0, t_1, ..., t_k)).$$



Convergence in probability fringe sense

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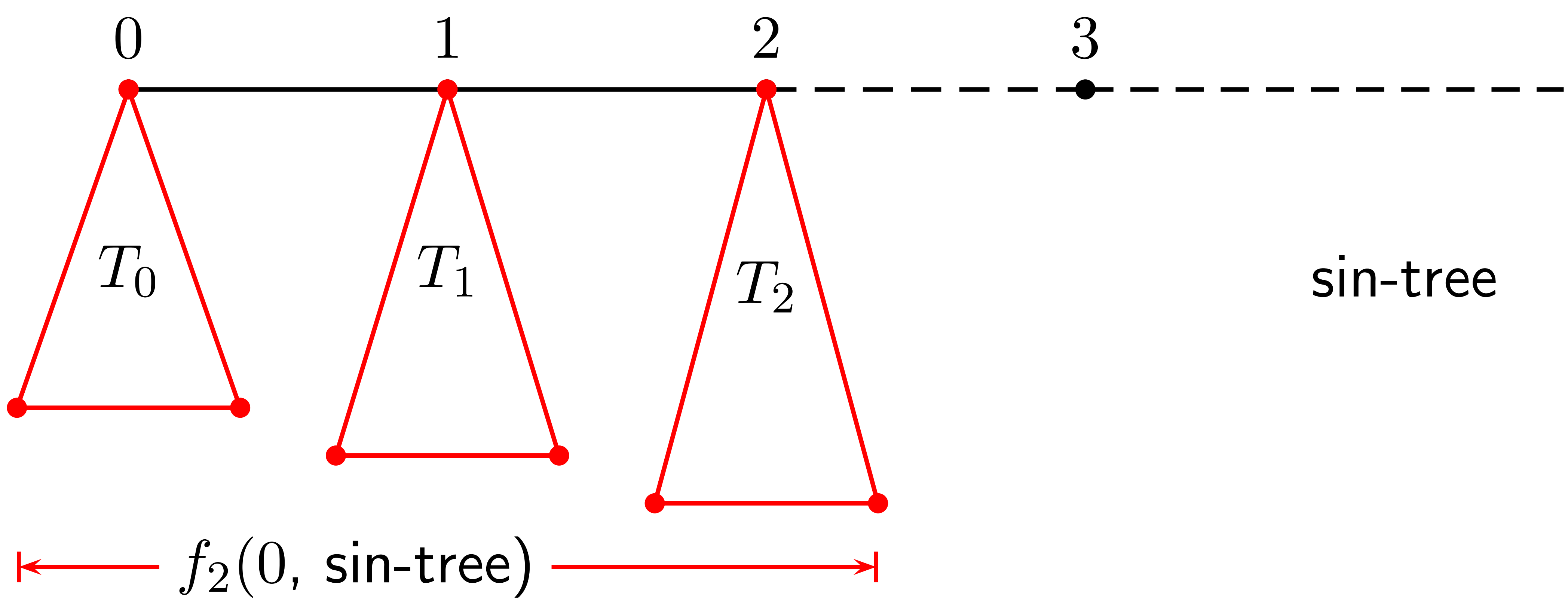

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- Conditional on the sequence $(S_n)_{n\geq 0}$
  1. $\mathcal{F}_{X_0}$: continuous time branching process driven by $\mathcal{P}$ observed up to time X_0.
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  0 designated as the root. $\mathcal{F}_{X_0}$ to be rooted at 0 and for $n \geq 1$ consider $\mathcal{F}_{S_n,S_{n-1}}$ to be rooted at n.
- $\mathbf{f}_k(\mathcal{T}^{\mathtt{sin}}_{\alpha,\mu,\mathcal{P}}) = (\mathcal{F}_{X_0}, \mathcal{F}_{S_1,S_0}, \mathcal{F}_{S_2,S_3}, \ldots, \mathcal{F}_{S_k,S_{k-1}})$

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Setting

- For the convergence of spectral distribution can take general families of trees satisfying sin-tree convergence.

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Setting

- For the convergence of spectral distribution can take general families of trees satisfying sin-tree convergence.
- For maximal eigen value convergence talking about preferential attachment with $f(v,n) = \text{Deg}(v,n) + a$.

Main result

Theorem (SB, Evans, Sen 08)

(a) Consider a sequence of trees converging in fringe since to a random infinite `sin`-tree. Then there exists a model dependent probability distribution function F such that

$$d(F_n, F) \xrightarrow{P} 0$$

as $n \to \infty$.

(b) Let $\gamma_a = a + 2$. Then for the linear preferential attachment model

$$\left(\frac{\lambda_1}{n^{1/2\gamma_a}}, \frac{\lambda_2}{n^{1/2\gamma_a}}, \ldots, \frac{\lambda_k}{n^{1/2\gamma_a}} \right) \xrightarrow{d} \nu_k$$

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Spectral distribution turns out to be a local property of random node, maximal eigen values, local property about the root

Spectral distribution: Method of proof

Stieltjes transform

$$s(z) = \int_{\mathbb{R}} \frac{1}{x - z}$$

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For eigen value distribution

$$\begin{aligned} s(z) &= \frac{1}{n} \mathrm{Tr}(A - zI)^{-1} \\ &= \frac{1}{n} \sum_{v=1}^{n} R_{vv}(z) \end{aligned}$$

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$$= \frac{1}{n}\sum_{v=1}^{n} R_{vv}(z)$$

$$R_{vv}(z) = \frac{1}{-z + \sum_1^{N(v)} R_{v_i v_i}(z) + R^{\mathtt{big}}_{A(v)}(z)}$$

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- Fix $Im(z) > 1$. Iterate the above expansion d times. Get a continued fraction upto d terms and some error term.

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- Not hard to see that for $Im(z) > 1$, this implies that $s_n(z)$ "depends" on the first K terms
- Fringe convergence of the random trees tells you what happens upto distance K for any fixed K
- So not hard to show that there exists a fixed Stieltjes transform $s(z)$ such that

$$s_n(z) \xrightarrow{P} s(z)$$

Properties and questions

- Sufficient conditions for a point $a \in \mathbb{R}$ to be an atom of limiting F
- Implies that for most standard models, limiting F has dense set of atoms

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- Implies that for most standard models, limiting F has dense set of atoms
- **Open Question:** Does limiting F have absolutely continuous part?

At this point, fair question

- If one can embed things in a continuous time all good things happen
- Can one expect such behavior generally?

Power of choice in random trees [D'Souza, Mitzenmacher]

Model: Motivation and construction

- Usual pref. attachment: Basic assumption: every new vertex has knowledge of entire network
- Each stage new vertex chooses 2 vertices uniformly at random
- Connect to vertex with maximal degree **amongst** the ones chosen (breaking ties with probability $1/2$)
- Model which incorporates randomness as well as limited choice
- Let $\mathcal{T}_n$ denote the tree on n vertices

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Theorem (Angel, Pemantle, SB)

There exists a rooted limiting random tree $\mathcal{T}_\infty$, described by Jagers-Nerman stable age distribution theory such that such that $\mathcal{T}_n$ converges locally $\mathcal{T}_\infty$.

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- Rate one poisson process, marking each with probability $p_0/2$, time of first point: $X_0 \sim \exp(p_0/2)$
- So probability not a leaf: $1 - p_0 = \mathbb{P}(T > X_0)$

Description of the limit tree

Recursive construction of the degree

- Let p_0 limiting fraction of leaves
- Define $q_0 = p_0/2$
- Then p_0 obtained by doing the following: Let $T \sim \exp(1/2)$ and $X_0 \sim \exp(q_0)$. Then
$$1 - p_0 = \mathbb{P}(T > X_0)$$
-
$$p_0 = \frac{\sqrt{5}-1}{2}$$
- General, having obtained p_k, get p_{k+1} by solving
$$1 - (p_0 + \cdots + p_{k+1}) = \mathbb{P}(X_0 + \cdots X_{k+1} > T)$$
where
$$X_{k+1} \sim \exp(p_0 + \cdots + p_k + \frac{p_{k+1}}{2})$$

Description of $\mathcal{T}_\infty$

- After having obtained p_i, let $L_i = \sum_{j=0}^{i} X_j$
- Consider the point process $\mathcal{P}_{\max} = (L_0, L_1, \ldots)$
- Define

$$\mu_{\max}(0,t) = \mathbb{E}(\#i : L_i < t)$$

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Theorem

- Then $\mathcal{T}_\infty$ is the Jagers-Nerman stable age distribution tree with offspring distribution $\mathcal{P}_{\max}$, age distribution $\exp(1/2)$ and time to nearest ancestor $\nu_{\max}$
- Implies convergence of global functionals as well such as the spectral distribution of adjacency matrix