Let $X \subset \mathbb{P}^4$ be the smooth Fermat quintic threefold, that is, the vanishing set of the homogeneous polynomial $X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5$. Let $(\xi, \eta) \in \mathbb{C}^2$ be a point satisfying $\xi^5 + \eta^5 + 1 = 0$, with $\xi \neq 0$, and let $\varepsilon \in \mathbb{C}$ be a root of $\varepsilon^5 + 1 = 0$. Clearly, the morphism

$$\begin{array}{rcl} f : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^4 \\ [S, T] & \longmapsto & [\xi S, \eta S, S, \varepsilon T, T] \end{array}$$

is an isomoprhism onto a line contained in X. Write $f^*\mathcal{T}_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3)$, with $a_1 \geq a_2 \geq a_3$. We want to compute a_1, a_2 and a_3. It is simpler to analyze $f^*\Omega^1_X \simeq \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3)$.

The following two exact sequences of sheaves on $\mathbb{P}^1$

$$f^*\Omega^1_X \longrightarrow \Omega^1_{\mathbb{P}^1} \longrightarrow 0$$

and

$$0 \longrightarrow f^*\mathcal{I}_X \longrightarrow f^*\Omega^1_{\mathbb{P}^4} \longrightarrow f^*\Omega^1_X \longrightarrow 0$$

can be rewritten as

$$\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow 0$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3) \to 0$$

From the first sequence we deduce that $-a_1 \leq -2$, from the second one that $-a_1, -a_2, -a_3 \geq -2$ and $a_1 + a_2 + a_3 = 0$. Thus $a_1 = 2$ and then again from the second sequence we deduce that $a_2, a_3 \leq 1$ and $a_2 + a_3 = -2$. Thus the only possible values of $(-a_2, -a_3)$ are $(-1, 3)$, $(0, 2)$ and $(1, 1)$. The tensor product $f^*\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is thus isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2 - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3 - 1)$. If we show that $h^0\big(\mathbb{P}^1, f^*\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\big) = 3$, then the only possibility for $(-a_2, -a_3)$ is $(-1, 3)$. Let us therefore compute the global sections of $f^*\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$, which we think of as the sections of $f^*\Omega^1_X$ vanishing at a point.

Let $U_i \subset \mathbb{P}^4$ be the standard open set $X_i \neq 0$; the open sets $f^{-1}(U_4)$ and $f^{-1}(U_2)$ cover $\mathbb{P}^1$. Let $x_i = \frac{X_i}{X_4}$ and $y_j = \frac{X_j}{X_2}$. We have

$$\begin{aligned} f^*\Omega^1_X|_{U_4} &\simeq \left(\bigoplus_{i=0}^{3} \mathbb{C}[s]dx_i\right) \Big/ \mathbb{C}[s]\Big(\xi^4 s^4 dx_0 + \eta^4 s^4 dx_1 + s^4 dx_2 + \varepsilon^4 dx_3\Big) \simeq \\ &\simeq \mathbb{C}[s]dx_0 \oplus \mathbb{C}[s]dx_1 \oplus \mathbb{C}[s]dx_2 \\ f^*\Omega^1_X|_{U_2} &\simeq \left(\bigoplus_{i \in \{0,1,3,4\}} \mathbb{C}[t]dy_i\right) \Big/ \mathbb{C}[t]\Big(\xi^4 dy_0 + \eta^4 dy_1 + \varepsilon^4 t^4 dy_3 + t^4 dy_4\Big) \simeq \\ &\simeq \mathbb{C}[t]dy_1 \oplus \mathbb{C}[t]dy_3 \oplus \mathbb{C}[t]dy_4 \end{aligned}$$

since dx_3 and dy_0 can be expressed in terms of the remaining differentials (remember that we assume that $\xi \neq 0$). The transition between the two charts is given by

$$\begin{aligned} dx_0 &\mapsto d\left(\frac{X_0}{X_2}\frac{X_2}{X_4}\right) = d\left(y_0 y_4^{-1}\right) = y_4^{-1}dy_0 - y_4^{-2}y_0 dy_4 = \frac{dy_0}{t} - \frac{\xi}{t^2}dy_4 \\ dx_1 &\mapsto \frac{dy_1}{t} - \frac{\eta}{t^2}dy_4 \\ dx_2 &\mapsto d\left(\frac{X_2}{X_4}\right) = d\left(y_4^{-1}\right) = \frac{dy_4}{t^2} \end{aligned}$$

We are interested in computing the dimension of the space of triples of polynomials $\left(p_0(s), p_1(s), p_2(s)\right)$ such that $p_0(s)dx_0 + p_1(s)dx_1 + p_2(s)dx_2$ extends to a polynomial on the open set $f^{-1}(U_2)$. Remember also that we want to only consider polynomials that vanish at one point, say $s = 0$ (without this condition we would be computing the global sections of $f^*\Omega^1_X$, not the global sections of $f^*\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$). The three conditions on the polynomials coming from the above transition functions are

$$\begin{aligned} -\xi^{-4}\eta^4 p_0\left(\frac{1}{t}\right) + p_1\left(\frac{1}{t}\right) &= 0 \\ \deg p_0(s) &\leq 3 \\ -\xi p_0\left(\frac{1}{t}\right) - \eta p_1\left(\frac{1}{t}\right) + p_2\left(\frac{1}{t}\right) &= 0 \end{aligned}$$

Thus knowing $p_0(s)$ determines the remaining polynomials; note that if $p_0(0)$ vanishes, then also $p_1(0)$ and $p_2(0)$ vanish. Since $p_0(s)$ is any polynomial of degree at most three with a zero at $s = 0$, we conclude that the space of global sections of $f^*\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ has dimension 3 and finally that

$$f^*\mathcal{T}_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$$