I learned this computation from A. J. de Jong.

Theorem. *Let n and d be integers, and suppose that $n \geq 1$. Then*

$$\begin{aligned} \mathrm{H}^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d)\big) &= \mathbb{C}[x_0, \ldots, x_n]_d \\ \mathrm{H}^i\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d)\big) &= 0 \quad , \quad 1 \leq i \leq n-1 \\ \mathrm{H}^n\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d)\big) &= \Big(\tfrac{1}{x_0 \cdot \ldots \cdot x_n} \mathbb{C}[x_0^{-1}, \ldots, x_n^{-1}]\Big)_d \end{aligned}$$

Remark. In the above statement and in the proof below, $\mathbb{C}$ stands for any commutative ring with unit. It is not required to be noetherian.

Proof. Let $\mathcal{F} := \oplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)$. Clearly, $\mathcal{F}$ is a quasi-coherent sheaf on $\mathbb{P}^n$. Let $\mathcal{U} = \{U_i\}_{i=0}^n$ be the open affine cover of $\mathbb{P}^n$ such that $U_i = \mathbb{P}^n \setminus V(x_i)$.

Fact. Open affine covers are acyclic for the global section cohomology and thus the Čech complex associated to $\mathcal{U}$ actually computes $\mathrm{H}^i\big(\mathbb{P}^n, \mathcal{F}\big)$.

We have

$$\mathcal{C}^m\big(\mathcal{U}, \mathcal{F}\big) =$$

$$= \bigoplus_d \bigoplus_{i_0 < \ldots < i_m} \Big(\mathbb{C}[x_0, \ldots, x_n, \frac{1}{x_{i_0} \ldots x_{i_m}}]\Big)_d =$$

$$= \bigoplus_{i_0 < \ldots < i_m} \mathbb{C}[x_0, \ldots, x_n, \frac{1}{x_{i_0} \ldots x_{i_m}}]$$

and there is a natural $\mathbb{Z}^{n+1}$–grading, where the monomial $x_0^{e_0} \cdot \ldots \cdot x_n^{e_n}$ has degree $(e_0, \ldots, e_n)$. Moreover, it is clear that the differential respects the grading; this allows us to forget the grading for now and recover all the information later. We have

$$\mathcal{C}^m\big(\mathcal{U}, \mathcal{F}\big) = \bigoplus_{\underline{e} \in \mathbb{Z}^{n+1}} K^m_{\underline{e}}$$

where of course $K^m_{\underline{e}}$ is the direct sum of all terms involving monomials of degree $\underline{e}$. Let $p(\underline{e}) = \#\big\{i \in \{0, \ldots, n\} \big| \ e_i < 0\big\} - 1$. If $m < p(\underline{e})$, then $K^m_{\underline{e}} = 0$, since monomials in $\mathcal{C}^m\big(\mathcal{U}, \mathcal{F}\big)$ are allowed exactly $m+1$ negative exponents. We can write:

$$K^\bullet_{\underline{e}} : \quad \cdots \longrightarrow \underset{p(\underline{e})-1}{0} \longrightarrow \underset{p(\underline{e})}{\mathbb{C}} \longrightarrow \underset{p(\underline{e})+1}{\mathbb{C}^{n-p(\underline{e})}} \longrightarrow \underset{p(\underline{e})+2}{\mathbb{C}^{\binom{n-p(\underline{e})}{2}}} \longrightarrow \cdots \longrightarrow \underset{n}{\mathbb{C}} \longrightarrow \underset{n+1}{0} \longrightarrow \cdots$$

where a generator for the first non-zero group from the left is $\underline{x}^{\underline{e}}$, as a section on the largest intersection $U_{\underline{e}}$ of open sets in $\mathcal{U}$ where it is regular; generators for the second non-zero group from the left are $\underline{x}^{\underline{e}}$, as sections on the intersection $U_{\underline{e}} \cap U_i$, with U_i not already appearing in $U_{\underline{e}}$, ... Thus $K^\bullet_{\underline{e}}$ is the standard Koszul complex which only has cohomology if $p(\underline{e}) = -1$ or $p(\underline{e}) = n$:

$$p(\underline{e}) = 0 \qquad K^\bullet_{\underline{e}} : \qquad \underset{-1}{\mathbb{C}} \dashrightarrow \underset{0}{\mathbb{C}^{n-p(\underline{e})}} \longrightarrow \underset{1}{\mathbb{C}^{\binom{n-p(\underline{e})}{2}}} \longrightarrow \cdots$$

$$p(\underline{e}) = n \qquad K^\bullet_{\underline{e}} : \qquad \cdots \longrightarrow \underset{n-1}{0} \longrightarrow \underset{n}{\mathbb{C}} \longrightarrow \underset{n+1}{0} \longrightarrow \cdots$$

This completes the proof. ∎