

THE LEFSCHETZ HYPERPLANE THEOREM FOR STACKS

DANIEL HALPERN-LEISTNER

ABSTRACT. We use Morse theory to prove that the Lefschetz Hyperplane Theorem holds for compact smooth Deligne-Mumford stacks over the site of complex manifolds. For $\mathfrak{Z} \subset \mathfrak{X}$ a hyperplane section, \mathfrak{X} can be obtained from \mathfrak{Z} by a sequence of deformation retracts and attachments of high-dimensional finite disc quotients. We use this to derive more familiar statements about the relative homotopy, homology, and cohomology groups of the pair $(\mathfrak{X}, \mathfrak{Z})$. We also prove some preliminary results suggesting that the Lefschetz Hyperplane Theorem holds for Artin stacks as well. One technical innovation is to reintroduce an inequality of Lojasiewicz which allows us to prove the theorem without any genericity or nondegeneracy hypotheses on \mathfrak{Z} .

1. INTRODUCTION

Consider an n -dimensional compact complex manifold X with a positive line bundle L and a global section s with zero locus $Z \subset X$. The Lefschetz hyperplane theorem (henceforth LHT) states that up to homotopy equivalence X is obtained from Z by attaching cells of dimension n or larger, or equivalently that $\pi_i(X, Z) = 0$ for $i < n$. By the homological LHT we refer to the weaker statement that $H_i(X, Z) = H^i(X, Z) = 0$ for $i < n$.

The LHT can fail for singular varieties, although it still holds for projective varieties as long as $X - Z$ is a local complete intersection. See the book [GM88] for an exhaustive treatment of the classical LHT and its generalizations to singular varieties.

To motivate the introduction of stacks, we let $X = \bar{\mathfrak{X}}$ be the coarse moduli space of a Deligne-Mumford (DM) stack \mathfrak{X} . X has local quotient singularities, so Poincaré-Lefschetz duality holds rationally for the pair (X, Z) . Then the fact that $X - Z$ is affine and so $H_i(X - Z) = 0$ for $i > n$ implies that $H_i(X, Z; \mathbb{Q}) = H^i(X, Z; \mathbb{Q}) = 0$ for $i < n$. It is natural to ask if this rational result is the shadow of an integral result for \mathfrak{X} itself. We will answer this question in the affirmative, but to prove it we will need to develop some basic tools of differential topology for stacks.

In Section 2 we show how the ideas of Bott's Morse theoretic proof of the LHT [Bot59] extend to the setting of DM stacks. We prove our main version of the LHT, which states that a stack can be constructed from a hyperplane section by a series of deformation retracts and attachments of high-dimensional disc quotients. Then we show how the usual statements

about relative homotopy groups and integral homology groups follow. The
 25 homotopy groups of a stack are defined to be the homotopy groups of its
 classifying space – this is discussed in more detail below.

In Section 3 we go back and carefully develop Morse theory of DM stacks.
 The main lemmas of Morse theory have been proven for the *underlying space*
 of a DM stack in [Hep09a], but we observe that the proofs work for the stack
 30 itself. In this paper, a cell attachment refers to an honest 2-categorical
 colimit. This is the strongest notion of gluing discussed in [Noo05] – it
 doesn't exist unless one is gluing along embedded substacks, and even then
 some care is required. Thus the main technical hurdle in Section 3 is showing
 that a Morse function on \mathfrak{X} really does give a cell decomposition in the
 35 strongest sense.

Finally, in Section 4 we discuss a purely algebraic proof of the homologi-
 cal LHT for certain smooth stacks which are not DM. The argument applies
 to real differentiable stacks admitting smooth proper maps from a compact
 complex manifold, for instance global quotients of Kähler manifolds by com-
 40 pact Lie groups. We also present some calculations which suggest that the
 homological LHT holds for a much larger class of Artin stacks, although the
 simple algebraic argument does not apply.

Another important point in this note is a classical result on the stability
 of gradient descent flow. In Bott's original paper, he assumes that the zero
 45 locus of s is “non-degenerate” in order to apply Morse theory. Here we show
 that Bott's proof of the LHT works without any assumptions on the section.
 The key observation is that the gradient flow of a function is stable as long
 as it is close enough to a real analytic function near its critical loci. These
 ideas were developed in the work of Lojasiewicz [Loj84], but to the author's
 50 knowledge they are not widely applied in the Morse theory literature. We
 discuss some of Lojasiewicz's results in Appendix A and refer to these ideas
 often throughout the paper.

We refer to the references [BX03, BX11, Hei05, Noo05] for the foundations
 of differentiable and topological stacks. The term Deligne-Mumford(DM)
 55 stack will denote a stack over the site of complex manifolds admitting a
 surjective representable étale map from a complex manifold $X_0 \rightarrow \mathfrak{X}$ such
 that $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is finite. Such stacks are presentable by a proper étale
 groupoid $\mathfrak{X} \simeq [X_1 \rightrightarrows X_0]$. Using a presentation, one can forget the complex
 or differentiable structure of X_0, X_1 and thus define forgetful functors from
 60 stacks over complex manifolds to differentiable stacks to topological stacks.
 We will use these forgetful functors as implicitly as they are used for complex
 manifolds. We will denote the coarse moduli space of \mathfrak{X} by $\tilde{\mathfrak{X}}$. In Section
 4, we will discuss differentiable Artin stacks, which we take to mean a stack
 over the site of smooth manifolds admitting a smooth submersion from a
 65 manifold.

The technical foundation for Morse theory of DM stacks has been laid by
 Hepworth in [Hep09a]. In that paper he develops Morse functions, Riemann-
 ian metrics, integration of vector fields, the strong topology on $C^\infty(\mathfrak{X})$, and

the main theorem of Morse theory for the underlying space $\tilde{\mathfrak{X}}$. We will build
 70 on these foundations, showing that Morse theory gives cell decompositions
 for the underlying stack and not just its coarse moduli space.

I'd like to thank my thesis adviser, Constantin Teleman, for suggesting
 this project to me and for many useful conversations throughout.

2. HYPERPLANE THEOREM FOR DM STACKS

75 Let \mathfrak{X} be an n -dimensional Deligne-Mumford stack over the site of com-
 plex manifolds and \mathfrak{L} a holomorphic line bundle on \mathfrak{X} with hermitian struc-
 ture \mathfrak{h} of type k (defined below). $s \in \Gamma(\mathfrak{X}, \mathfrak{L})$ will be a section of \mathfrak{L} , and
 $\mathfrak{Z} \subset \mathfrak{X}$ will denote the zero locus of s . We assume furthermore that $\tilde{\mathfrak{X}}$, the
 coarse moduli space of \mathfrak{X} treated as a topological stack, is compact.

Choose an étale atlas $X_0 \rightarrow \mathfrak{X}$ such that $\mathfrak{L}|_{X_0}$ admits a nowhere vanishing
 holomorphic section σ . Then the $(1, 1)$ -form

$$\Theta_0 = \bar{\partial}\partial \log \mathfrak{h}(\sigma, \sigma) \in \Omega^{1,1}(X_0)$$

80 descends to a hermitian global section of $\Omega_{\tilde{\mathfrak{X}}}^{1,1}$, the curvature form Θ . In local
 analytic coordinates we can write Θ as $H_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$, and following [Bot59]
 we call $(\mathfrak{L}, \mathfrak{h})$ *type k* if the hermitian matrix $H_{\alpha\beta}$ is positive on a space of
 (complex) dimension k at every point. We call a hermitian line bundle of
 type- n *positive*. On compact complex manifolds, type k line bundles are
 85 $(n - k)$ -ample in the sense of [Tot10], although the converse is not known.

This notion is the right one from the perspective of descent. $\Omega^{p,q}$ is a
 sheaf on the Artin site of complex manifolds, so the construction of Θ above
 works when $X_0 \rightarrow \mathfrak{X}$ is a surjective submersion and not necessarily étale. In
 addition, the property of being positive definite on a k -dimensional subspace
 90 is Artin-local, i.e. if $\varphi : X \rightarrow Y$ is a surjective submersion, then a hermitian
 form on Y is type k if and only if its pullback to X is type k . Thus we can
 define a type k holomorphic line bundle on a smooth *Artin* stack.

Lemma 1. *Let \mathfrak{X} be a smooth DM stack of dimension n which can be pre-
 sented as a global quotient and whose coarse moduli space is projective. Then
 95 a line bundle \mathfrak{L} is positive if and only if \mathfrak{L}^k descends to an ample line bundle
 on $\tilde{\mathfrak{X}}$ for some k .*

Proof. The paper [Bai57] proves that a sufficiently high power of a positive
 line bundle defines a projective embedding $\tilde{\mathfrak{X}} \hookrightarrow \mathbb{C}\mathbb{P}^N$.

Conversely, assume $\mathfrak{X} = [X/G]$ for a smooth quasiprojective X and re-
 100 ductive G .¹ \mathfrak{L} is positive if and only if \mathfrak{L}^k is positive for any $k > 0$, so we
 may assume that \mathfrak{L} is pulled back from a very ample invertible sheaf on
 $\tilde{\mathfrak{X}}$. There is a smooth G -equivariant compactification $X \subset \hat{X}$ such that \mathfrak{L}
 extends to a G -ample bundle on \hat{X} and $X = \hat{X}^{ss}(\mathfrak{L})$ [Tel00]. Let $K \subset G$ be

¹This property is fairly common. For algebraic stacks it is equivalent to \mathfrak{X} having
 enough locally free sheaves [?].

a maximal compact subgroup and $\Phi : \hat{X} \rightarrow \mathfrak{k}^*$ the moment map corresponding to the linearization \mathcal{L} . $\Phi^{-1}(0)$ is a manifold with locally free K action, and $\mathfrak{X} \simeq [\Phi^{-1}(0)/K]$, and a K -invariant positive Hermitian structure on \mathcal{L} descends to a positive Hermitian structure on \mathfrak{X} (see [GS82], Section 3). \square

Furthermore, if $q : \mathfrak{X} \rightarrow \bar{\mathfrak{X}}$ is the projection to the coarse moduli space, then $q_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\bar{\mathfrak{X}}}$, so an invertible sheaf on $\bar{\mathfrak{X}}$ has the same global sections when pulled back to \mathfrak{X} .

Corollary 2.1. *Let $\mathfrak{Z} \subset \mathfrak{X}$ be the zero locus of a section of a positive line bundle, then topologically $\mathfrak{Z} = q^{-1}(H)$ for some hyperplane section $H \subset \bar{\mathfrak{X}}$.*

One can think of the generalization of the LHT to type k line bundles as a “large fiber” generalization as discussed in [GM88].

Definition 2.2. Let $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$ is a map of stacks. Choose complex manifolds X_0 and X'_0 fitting into the commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & X'_0 \\ \text{surjective} \downarrow & & \downarrow \text{étale} \\ \mathfrak{X} & \xrightarrow{\phi} & \mathfrak{X}' \end{array}$$

We define the *minimal rank* of ϕ to be $\min_{x \in X_0} \text{rank}_{\mathbb{C}}(D_x\phi_0 : T_x X_0 \rightarrow T_{\phi_0(x)} X'_0)$. It does not depend on the choice of X_0 and X'_0 .

A map having minimal rank r implies that all fibers have codimension at least r . Of course the converse does not hold: a finite map need not be étale, for instance.

Example 2.3. If $X \rightarrow X'$ is a G -equivariant map of complex manifolds with G action, then the minimal rank of $X/G \rightarrow X'/G$ is the same as $X \rightarrow X'$.

If \mathcal{L} is a positive invertible sheaf on \mathfrak{X}' , and $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$ has minimal rank r , then $\phi^*\mathcal{L}$ is type r on \mathfrak{X} . Hence if $H \subset \bar{\mathfrak{X}}'$ is a hyperplane section, then the LHT implies that the relative homotopy groups $\pi_i(\mathfrak{X}, \phi^{-1}(H))$, interpreted as the relative homotopy groups of the classifying spaces, vanish for $i < r$.

Now we will prove the LHT by analyzing the “Morse” function $f = \mathfrak{h}(s, s) \in C^\infty(\mathfrak{X})$. We refer to [Hep09a] for a discussion of smooth functions on smooth DM stacks. In particular, he defines the strong topology on $C^\infty(\mathfrak{X})$ and proves that Morse functions are dense. Define the closed topological substack $\mathfrak{X}^{\leq \epsilon} = f^{-1}[0, \epsilon]$ and the open differentiable substack $\mathfrak{X}^{< \epsilon} = f^{-1}[0, \epsilon)$. Note that $\mathfrak{X}^{\leq 0} = f^{-1}(\{0\}) = \mathfrak{Z}$ is precisely the vanishing locus of s .

Fixing a Riemannian metric on \mathfrak{X} , we first show that f satisfies a Łojasiewicz inequality near its global minimum. This property of f guarantees that the gradient descent flow of f retracts $\mathfrak{X}^{\leq \epsilon}$ onto \mathfrak{Z} for sufficiently small ϵ , but we will not make this precise until Section 3.

Claim 2.4. *For a sufficiently small $\epsilon > 0$, there are constants $C > 0$ and $\rho \in (0, 1)$ such that*

$$|f|^\rho \leq C|\nabla f| \tag{L}$$

on the substack \mathfrak{X}^ϵ .

Proof. Note that it suffices to verify this inequality on $\bar{\mathfrak{X}}^\epsilon$. By compactness of $\bar{\mathfrak{Z}}$ it suffices to verify (L) around each $p \in \bar{\mathfrak{Z}}$. Choose an étale coordinate patch $U \rightarrow \mathfrak{X}$ containing $p \in \bar{\mathfrak{Z}}$ such that $\mathfrak{L}|_U$ is trivial. Then $f_U = \mathfrak{h}(s, s) = h \cdot |s|^2$ is the product of a nonvanishing smooth function h with the real analytic function $|s|^2$, and thus by Theorem A.5 below the inequality (L) holds in a neighborhood of p . \square

Next we will perturb f to be Morse without affecting its values on $\mathfrak{X}^{\epsilon/2}$, so that our new function will continue to satisfy (L). Let $g : \mathfrak{X} \rightarrow [0, 1]$ be a smooth function such that $g|_{\mathfrak{X}^{\epsilon/2}} = 0$ and $g|_{\mathfrak{X} - \mathfrak{X}^\epsilon} = 1$.

Claim 2.5. *There is a small open neighborhood $U \subset C^\infty(\mathfrak{X})$ containing 0 such that for any $\eta \in U$, the new function $f' = f + g\eta$ has the properties*

- f' has no critical points on $\mathfrak{X}^\epsilon - \mathfrak{X}^{\epsilon/2}$
- f' is positive on $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$
- the hermitian form $\bar{\partial}\partial \log f'$ is positive on a space of dimension k at every point in $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$.

Proof. Multiplication by g induces a continuous map $C^\infty \mathfrak{X} \rightarrow C^\infty \mathfrak{X}$, so it suffices to prove that the set of η' for which $f + \eta'$ satisfies the above properties is open in $C^\infty \mathfrak{X}$ and contains 0.

Now choose a map $V \rightarrow \mathfrak{X}$ and an open subset $X_0 \subset V$ such that $X_0 \rightarrow \mathfrak{X}$ is surjective étale and $\text{cl}(X_0) \rightarrow \mathfrak{X}$ is proper. Then $C^\infty(\mathfrak{X})$ is identified as a vector space with invariant functions in $C^\infty(X_0)$, and one can show that for such an atlas the strong topology on $C^\infty(\mathfrak{X})$ is the subspace topology in $C^\infty(X_0)$. All three properties are open conditions on the derivatives of $(f + \eta)|_{X_0}$ up to order 2, and they are satisfied by f , so the claim follows. \square

Morse functions form a dense open subset of $C^\infty(\mathfrak{X})$ [Hep09a], so there is some $\eta \in U$, where $U \subset C^\infty \mathfrak{X}$ is defined in Claim 2.5, such that $f + \eta$ is Morse. Consider the new function $f' = f + g\eta$. f' will satisfy all of the properties of Claim 2.5, and in addition it will be Morse on the substack $\mathfrak{X} - \bar{\mathfrak{Z}}$, because f' contains no critical points in $\mathfrak{X}^\epsilon - \bar{\mathfrak{Z}}$ and f' agrees with $f + \eta$ on $\mathfrak{X} - \mathfrak{X}^\epsilon$. From this point on we replace our function f with the new function f' .

Next we consider the index of the critical points of f away from $\bar{\mathfrak{Z}}$. For a critical point c , the tangent space $T_c \mathfrak{X}$ and the Hessian $H_c f$ are defined in [Hep09a] using the groupoid $U \times_{\mathfrak{X}} U \rightrightarrows U$, where $U \rightarrow \mathfrak{X}$ is an étale coordinate patch containing c . The tangent space splits (non-canonically) as a representation of Aut_c , $T_c \mathfrak{X} \simeq T_c \mathfrak{X}_- \oplus T_c \mathfrak{X}_+$ on which $H_c f$ is negative

175 definite and positive definite respectively. The index, ind_c is the isomorphism class of the representation $T_c\mathfrak{X}_-$ of Aut_c .

Of course in a coordinate patch around c , the classical observation that the Hessian is negative definite on a space of dimension k still applies.

Claim 2.6. [Bot59] *At every critical point $c \in \bar{X} - \bar{Z}$, the index representation has $\dim(\text{ind}_c) \geq k$.*

Thus we have a function $f : \mathfrak{X} \rightarrow \mathbb{R}$ that is Morse on the open substack $\mathfrak{X} - \mathfrak{Z}$ and such that $\mathfrak{Z} \subset \mathfrak{X}$ is a *stable* global minimum of f . Now from the results of Morse theory in Section 3, we have the main theorem:

Theorem 2.7. *As a topological stack, \mathfrak{X} can be obtained from \mathfrak{Z} by a finite sequence of deformation retracts and gluings of $[D^r/G]$ along $[S^{r-1}/G]$, where $r \geq k$. Here we can assume for each cell attachment that $G = \text{Aut}_c$ acting linearly on the unit disc in $T_c\mathfrak{X}_-$ for some $c \in \tilde{\mathfrak{X}} - \tilde{\mathfrak{Z}}$.*

Remark 2.8. We only need \mathfrak{L} to be type k on the complement $\mathfrak{X} - \mathfrak{Z}$. Geometrically, if \mathfrak{Z} is the preimage of a hyperplane section $H \subset \tilde{\mathfrak{X}}'$ under a map $\phi : \mathfrak{X} \rightarrow \tilde{\mathfrak{X}}'$, we only need k to be the minimal rank of the restriction $\mathfrak{X} - \mathfrak{Z} \rightarrow \tilde{\mathfrak{X}}' - H$. In particular ϕ can have large fibers over H .

We will define more carefully what a deformation retract and a cell attachment are in Section 3 below, but suffice it so say that many corollaries about the various topological invariants follow.

195 Singular homology and cohomology are defined from the singular chain bi-complex of a simplicial manifold presenting \mathfrak{X} . The reader is referred to [Beh04] and [BX03] for an in-depth discussion and the equivalence between this and other notions of cohomology (i.e. de Rham or sheaf cohomology). One thing that's not described in [Beh04] is that H^i and H_i are homotopy 200 functors, but this follows from the usual argument, the key fact being the Poincaré lemma: by a local-to-global spectral sequence the projection $p : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ induces an isomorphism in homology and cohomology.

The homotopy groups are defined as $\pi_i(\mathfrak{X}) = \pi_i(\mathbb{B}\mathfrak{X})$ for a classifying space $\mathbb{B}\mathfrak{X}$. We follow Noohi's treatment in [Noo08], in which the classifying 205 space is characterized up to weak equivalence by a universal property. He uses the "Haefliger-Milnor" construction to show the existence of classifying spaces in general, but in this note all stacks will admit presentations in which X_1 and X_0 are both metrizable – they are hoparacompact in the language of [Noo08] – in which case $\mathbb{B}\mathfrak{X}$ is the geometric realization of the simplicial manifold $X_p = X_0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X_0$ (see [Seg68]) defined using an atlas $X_0 \rightarrow \mathfrak{X}$. 210

We recall the properties of $\mathbb{B}\mathfrak{X}$ from [Noo08]. First there is an epimorphism $\varphi : \mathbb{B}\mathfrak{X} \rightarrow \mathfrak{X}$ which is a universal weak equivalence, meaning that the base change along any map $T \rightarrow \mathfrak{X}$ is a weak equivalence. This implies that

φ induces isomorphisms on homology and cohomology, in other words the
 215 homology of $\mathbb{B}\mathfrak{X}$ agrees with the double complex homology defined above.²

Also the construction $\mathbb{B}\mathfrak{X}$ is a *functor* from the category of (hoparacompact) stacks to the homotopy category of paracompact spaces. Finally, for $\mathfrak{Z} \subset \mathfrak{X}$ an embedding, we can construct $\mathbb{B}\mathfrak{Z}$ such that $\mathbb{B}\mathfrak{Z} \subset \mathbb{B}\mathfrak{X}$ is a subspace. Thus we define the relative homotopy groups $\pi_i(\mathfrak{X}, \mathfrak{Z}) := \pi_i(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z})$. We
 220 omit base points from our notation, but the following theorem applies for any base point in $\mathbb{B}\mathfrak{Z}$.

Corollary 2.9. *We have $\pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. Hence from the long exact sequence of a pair $\pi_i(\mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X})$ is an isomorphism for $i < k - 1$ and surjective for $i = k - 1$.*

Proof. For a triple of topological stacks $\mathfrak{Z} \subset \mathfrak{X} \subset \mathfrak{X}'$ we have a long exact sequence

$$\cdots \rightarrow \pi_i(\mathfrak{X}, \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{X}) \rightarrow \cdots$$

225 so if $\pi_i(\mathfrak{X}', \mathfrak{X}) = \pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$, then $\pi_i(\mathfrak{X}', \mathfrak{Z}) = 0$ as well. Thus the corollary follows by induction on the sequence of cell attachments and deformation retracts described in Theorem 2.7, once we verify that $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for $i < k$ when $\mathfrak{X} \subset \mathfrak{X}'$ is a disc attachment or deformation retract.

For a deformation retract, the inclusion $\mathbb{B}\mathfrak{X} \rightarrow \mathbb{B}\mathfrak{X}'$ is a homotopy equivalence, so $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for all i .
 230

Next let $\mathfrak{X}' = \mathfrak{X} \cup_{[S/G]} [D/G]$ be a 2-categorical colimit, where D is a disc of dimension $\geq k$, S its boundary sphere, and G a finite group acting linearly on the pair (D, S) . Let $A \subset D$ be the complement of an open disc of half the radius of D , and consider the substack $\mathfrak{X} \cup_{[S/G]} [A/G] \subset \mathfrak{X}'$.

235 First observe that $A \rightarrow [A/G]$ and $D \rightarrow [D/G]$ are regular coverings, so $\pi_i([A/G]) \rightarrow \pi_i([D/G])$ is an isomorphism for $i < k - 1$ and surjective for $i = k - 1$; hence $([D/G], [A/G])$ is $k - 1$ -connected. The interiors of $\mathfrak{X} \cup_{[S/G]} [A/G]$ and $[D/G]$ cover \mathfrak{X}' , so excision for homotopy groups implies that $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G])$ is $k - 1$ -connected as well.

240 Finally, the equivariant map $D \rightarrow D$ compressing A onto the boundary sphere S is equivariantly homotopic to the identity through a homotopy fixing S . Thus using the universal property of $\mathfrak{X} \cup_{[S/G]} [D/G]$ we have a homotopy equivalence of pairs $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G]) \simeq (\mathfrak{X}', \mathfrak{X})$, so we deduce $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for $i < k$. \square

245 Recall that $H_*(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z}) = H_*(\mathfrak{X}, \mathfrak{Z})$ and likewise for H^* , so

Corollary 2.10. *For the relative homology and cohomology, with integer coefficients, we have $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. As a consequence $H^i(\mathfrak{X}) \rightarrow H^i(\mathfrak{Z})$ is an isomorphism for $i < k - 1$ and injective for $i = k - 1$.*

²If X_p is a simplicial presentation for \mathfrak{X} and X'_p the pulled-back presentation of $\mathbb{B}\mathfrak{X}$, then $X'_p \rightarrow X_p$ is a weak equivalence for all p , so the spectral sequence $E_1^{p,q} = H^q(X_p) \Rightarrow H^{p+q}(\mathfrak{X})$ implies that $H^*(\mathfrak{X}) \rightarrow H^*(\mathbb{B}\mathfrak{X})$ is an isomorphism.

250 **2.1. Topology of the underlying space.** For completeness we describe the consequences for the topology of the underlying spaces $\bar{\mathfrak{X}} \subset \bar{\mathfrak{X}}$. In the introduction we mention that the homological LHT holds with rational coefficients for $\bar{\mathfrak{X}} \subset \bar{\mathfrak{X}}$, but in fact we only need to invert the orders of the automorphism groups of points of $\bar{\mathfrak{X}}$.

Proposition 2.11. *The underlying space $\bar{\mathfrak{X}}$ can be obtained from $\bar{\mathfrak{X}}$, up to homotopy equivalence, by attaching finitely many disc quotients D^r/G along the boundary S^{r-1}/G . Here, as before, $r \geq k$ and $G = \text{Aut}_c$ acting linearly on the unit disc $D^r \subset T_c\bar{\mathfrak{X}}_- \subset T_c\bar{\mathfrak{X}}$.*

Using the Morse function f constructed above, we know that $\bar{\mathfrak{X}}^{\epsilon}$ deformation retracts onto $\bar{\mathfrak{X}}$ for sufficiently small ϵ . From this point the proposition is a direct application of the Morse lemmas 7.5-7.7 in [Hep09a], using the estimate on the index of the critical points of f .

Corollary 2.12. *$H_i(\bar{\mathfrak{X}}, \bar{\mathfrak{X}}; \mathbb{Z}[1/N]) = H^i(\bar{\mathfrak{X}}, \bar{\mathfrak{X}}; \mathbb{Z}[1/N]) = 0$ for $i < k$ as long as N is a common multiple of the orders of the isotropy groups of $\bar{\mathfrak{X}}$ (of which there are finitely many because $\bar{\mathfrak{X}}$ is compact).*

265 *Proof.* This follows from the fact that $H_i(D/G, S/G; \mathbb{Z}[1/N]) = 0$ for $i < k$, which is really fact about group homology. One could also deduce the claim directly from Corollary 2.10 and the isomorphism $H_*(\bar{\mathfrak{X}}; \mathbb{Z}[1/N]) \rightarrow H_*(\bar{\mathfrak{X}}; \mathbb{Z}[1/N])$. \square

3. MORSE THEORY OF DELIGNE-MUMFORD STACKS

270 Now we will prove the main theorem of the Morse theory for DM stacks used in the proof of Theorem 2.7.

Theorem 3.1. *Let $f : \bar{\mathfrak{X}} \rightarrow \mathbb{R}$ be a smooth nonnegative function on a smooth Deligne-Mumford stack $\bar{\mathfrak{X}}$ achieving its global minimum on the closed topological substack $\bar{\mathfrak{X}} = f^{-1}\{0\}$. Assume that*

- 275
- $\bar{f} : \bar{\mathfrak{X}} \rightarrow \mathbb{R}$ is proper,
 - f satisfies a Łojasiewicz inequality (L) at every $p \in \bar{\mathfrak{X}}$, and
 - f is Morse on the open substack $\bar{\mathfrak{X}} - \bar{\mathfrak{X}}$, with critical points p_1, p_2, \dots, p_r .

Then $\bar{\mathfrak{X}}$ can be obtained from $\bar{\mathfrak{X}}$ by a finite sequence of deformation retracts and 2-categorical attachments of $[D_i/\text{Aut}_{p_i}]$ along $[S_i/\text{Aut}_{p_i}]$ where $D_i \subset T_{p_i}\bar{\mathfrak{X}}_-$ is a small disc with boundary S_i .

We fix a Riemannian metric g on $\bar{\mathfrak{X}}$, and we will study the flow along the vector field $-\nabla f$.³

Fact 3.2. [Hep09b] Given a vector field ξ on $\bar{\mathfrak{X}}$ with compact support, there is a representable morphism $\Phi : \bar{\mathfrak{X}} \times \mathbb{R} \rightarrow \bar{\mathfrak{X}}$ such that for any étale map

³On a DM stack a vector field corresponds to a consistent choice of vector field ξ_U on U for all étale $U \rightarrow \bar{\mathfrak{X}}$. The tangent bundle of an Artin stack is more complicated [Hep09b]

$M \rightarrow \mathfrak{X}$, the base change

$$\begin{array}{ccc} M' & \longrightarrow & \mathfrak{X} \times \mathbb{R} \\ \Phi' \downarrow & & \downarrow \Phi \\ M & \longrightarrow & \mathfrak{X} \end{array}$$

satisfies $D\Phi'(\frac{\partial}{\partial t})_{M'} = \xi_M$. Furthermore there is an isomorphism $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$, and the flow morphism Φ is uniquely determined by e_Φ .

285 The map Φ is a group action of \mathbb{R} up to 2-isomorphism, and so it induces an \mathbb{R} action on the underlying space \mathfrak{X} and an \mathbb{R} action up to homotopy on the classifying space $\mathbb{B}\mathfrak{X}$.

We will use existence of flows to deconstruct the topology of \mathfrak{X} in three steps. First we show that $\mathfrak{X}^{\epsilon^l}$ deformation retracts onto \mathfrak{Z} for sufficiently
290 small ϵ . Then we show that if the interval $[a, b]$ contains no critical values of f , then \mathfrak{X}^{b^l} deformation retracts onto \mathfrak{X}^{a^l} , and finally we show how the topology changes as a crosses a critical value.

First of all what does a deformation retract mean for stacks? Say $j : \mathfrak{Z} \rightarrow \mathfrak{X}$ is a closed immersion, then a *deformation retract* consists of a homotopy
295 $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ and a projection $\pi : \mathfrak{X} \rightarrow \mathfrak{Z}$, along with 2-morphisms

$$\begin{array}{ccc} \mathfrak{Z} \times I & \xrightarrow{j \times \text{id}} & \mathfrak{X} \times I \\ \text{pr}_1 \downarrow & \nearrow & \downarrow h \\ \mathfrak{Z} & \xrightarrow{j} & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} \mathfrak{X} & \xrightarrow{h_0} & \mathfrak{X} \\ \downarrow & \text{id} & \downarrow \\ \mathfrak{X} & \xrightarrow{h_1} & \mathfrak{X} \\ \downarrow & j \circ \pi & \downarrow \end{array}$$

This definition is a categorification of the definition of deformation retract of spaces (or what's called a strong deformation retract by some authors), and implies that j is a homotopy equivalence. Both the geometric realization
300 functor and the underlying space functor respect products, so if \mathfrak{X} deformation retract onto \mathfrak{Z} in the above sense, then $\tilde{\mathfrak{X}}$ deformation retracts onto $\tilde{\mathfrak{Z}}$ and $\mathbb{B}\mathfrak{X}$ is homotopy equivalent to $\mathbb{B}\mathfrak{Z}$.

Remark 3.3. A strict deformation retract of topological groupoids induces a deformation retract of stacks in the above sense.

Remark 3.4. A morphism factoring through a substack does so uniquely
305 up to unique isomorphism, so the morphism π could be omitted from the definition of a deformation retract, instead just requiring that $h_1 : \mathfrak{X} \rightarrow \mathfrak{X}$ factor through $\mathfrak{Z} \hookrightarrow \mathfrak{X}$.

Thus if $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ is a deformation retract of \mathfrak{X} onto \mathfrak{Z} , and $\mathfrak{X}' \subset \mathfrak{X}$ is a substack (not necessarily closed) containing \mathfrak{Z} such that $h|_{\mathfrak{X}' \times I} : \mathfrak{X}' \times I \rightarrow \mathfrak{X}$
310 factors through \mathfrak{X}' , then \mathfrak{X}' deformation retracts onto \mathfrak{Z} as well via this restricted h .

Remark 3.5. For an embedding of topological spaces $j : Z \rightarrow X$, we can form the mapping cylinder $M(j) := X \times \{0\} \cup Z \times [0, 1] \subset X \times [0, 1]$. If

$j : \mathfrak{Z} \rightarrow \mathfrak{X}$ is an embedding of topological stacks, we choose a presentation
 315 for \mathfrak{X} and form the mapping cylinder $\mathfrak{M}(j) = [M(j_1) \rightrightarrows M(j_0)] \subset \mathfrak{X} \times [0, 1]$.
 Morita equivalent presentations give Morita equivalent mapping cylinders
 because the mapping cylinder construction for spaces respects fiber products
 and epimorphisms. Furthermore, $\mathfrak{M}(j) \cong \mathfrak{X} \times \{0\} \cup_{\mathfrak{Z} \times \{0\}} \mathfrak{Z} \times [0, 1]$ is a pushout
 in the 2-category of topological stacks. Thus we can simplify our definition
 320 of a deformation retract further as *a map $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ such that h_1 factors
 through \mathfrak{Z} , along with an isomorphism $h|_{\mathfrak{M}(j)} \cong \text{pr}_1|_{\mathfrak{M}(j)}$.*

We are now ready to prove the

Lemma 2. *For $\epsilon > 0$ sufficiently small, \mathfrak{X}^ϵ contains no critical points of
 325 f , and the gradient descent flow of f provides a deformation retract of \mathfrak{X}^ϵ
 onto \mathfrak{Z} .*

Proof. The key idea is to find, for ϵ small enough, an atlas for \mathfrak{X}^ϵ that
 is *equivariant* for the action of the semigroup $[0, \infty)$. Start with an étale
 atlas $a : X_0 \rightarrow \mathfrak{X}$. $f|_{X_0}$ still satisfies the inequality (L) at every point in
 $Z_0 = (f_{X_0})^{-1}\{0\}$.

330 By Corollary A.4 there is an open subset $U_0 \subset X_0$ containing Z_0 such
 that $(-\nabla f)_{X_0}$ is integrable to a flow $\Phi_0 : U_0 \times [0, \infty) \rightarrow U_0$ which extends
 uniquely to a deformation retract $h_0 : U_0 \times [0, \infty) \rightarrow U_0$ onto Z_0 .

The étale map $U_0 \rightarrow \mathfrak{X}$ is equivariant with respect to the action of the
 semigroup $[0, \infty)$, so $[0, \infty)$ acts on $U_1 := U_0 \times_{\mathfrak{X}} U_0$ as well, and we would
 like to lift the action to ∞ , i.e. find a lift

$$\begin{array}{ccc}
 U_1 \times [0, \infty) & \xrightarrow{\Phi_1} & U_1 \\
 \downarrow & \dashrightarrow^{\exists! h_1} & \downarrow (s,t) \\
 U_1 \times [0, \infty) & \xrightarrow{(s,t)} & U_0 \times U_0 \times [0, \infty) \xrightarrow{h_0 \times h_0} U_0 \times U_0
 \end{array}$$

One shows the existence and uniqueness of h_1 using the fact that, because
 \mathfrak{X} is DM, $U_1 \rightarrow U_0 \times U_0$ is a proper immersion.

335 It suffices to show existence and uniqueness of h_1 in a neighborhood of
 each point $(\infty, p) \in [0, \infty) \times U_1$. If $(x, y) \in U_0 \times U_0$ is the image of (∞, p) , then
 there is a compact neighborhood N of (x, y) such that $(s, t)^{-1}(N)$ is a disjoint
 union of connected compact subsets, each mapped homeomorphically onto
 its image by (s, t) . Finally for a neighborhood of (∞, p) whose image in
 340 $U_0 \times U_0$ is contained in N there exists a unique lift.

Thus we get a map of groupoids $h : [0, \infty) \times (U_1 \rightrightarrows U_0) \rightarrow (U_1 \rightrightarrows U_0)$
 providing a deformation retract of $U_1 \rightrightarrows U_0$ onto the subgroupoid $Z_1 \rightrightarrows$
 Z_0 . By Remark 3.3 h , gives a deformation retract of the open substack
 $\mathfrak{U} = [U_1 \rightrightarrows U_0]$ onto \mathfrak{Z} . Because \mathfrak{Z} is compact, \mathfrak{X}^ϵ is contained in \mathfrak{U} for
 345 sufficiently small ϵ , so by Remark 3.4 the result follows. \square

Next we study what happens when there are no critical points between a
 and b .

Proposition 3.6. *If f has no critical points in $\mathfrak{X}^{[a,b]}$, then there is a deformation retract of $\mathfrak{X}^{[a,b]}$ onto \mathfrak{X}^a .*

Proof. Let ξ be a vector field with compact support such that $\xi \cdot f \leq 0$ and $\xi \cdot f = -1$ on $\mathfrak{X}^{[a,b]}$. Let $\Phi : \mathfrak{X} \times \mathbb{R} \rightarrow \mathfrak{X}$ be the flow of ξ and form the composition

$$h : \mathfrak{X}^{[a,b]} \times [0, 1] \xrightarrow{(\text{id}_{\mathfrak{X}^{[a,b]}}, \max(0, t(f-a)))} \mathfrak{X}^{[a,b]} \times [0, \infty) \xrightarrow{\Phi} \mathfrak{X}$$

350 \bar{f} decreases along the flow lines of $\bar{\Phi}$ and decreases with constant rate 1 in $\mathfrak{X}^{[a,b]}$, so it follows that $h : \mathfrak{X}^{[a,b]} \times [0, 1] \rightarrow \mathfrak{X}$ factors through $\mathfrak{X}^{[a,b]}$ and that h_1 factors through \mathfrak{X}^a .⁴ The isomorphism $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$ induces an isomorphism $h|_{\mathfrak{M}(j)} \simeq \text{pr}_1|_{\mathfrak{M}(j)}$ because $\max(0, t(f-a)) = 0$ on the substack $\mathfrak{M}(j) \subset \mathfrak{X}^{[a,b]} \times I$. \square

355 **Remark 3.7.** An argument identical to that in theorem 3.1 of [Mil63] or theorem 7.5 of [Hep09a] can also be used to show that $\mathfrak{X}^a \simeq \mathfrak{X}^b$ as stacks.

Proposition 3.8. *Let $c \in \mathbb{R}$ be a critical value of f , and assume that f is Morse near $f^{-1}\{c\}$. Denote the critical points in $f^{-1}\{c\}$ by p_1, \dots, p_r . Then for ϵ sufficiently small there is a closed topological substack*

$$\mathfrak{X}^{c-\epsilon] \subset \mathfrak{X}' \subset \mathfrak{X}^{c+\epsilon]}$$

such that $\mathfrak{X}^{c+\epsilon]}$ deformation retracts onto \mathfrak{X}' , and

$$\mathfrak{X}' \simeq \mathfrak{X}^{c-\epsilon] \cup \coprod_i [S_i / \text{Aut}_{p_i}] \coprod \coprod_i [D_i / \text{Aut}_{p_i}],$$

where D_i and S_i are the disc and sphere of radius ϵ in the index representation of Aut_{p_i} .

360 *Proof.* Again, the idea of the proof is identical to the discussion in [Mil63] or [Hep09a], one just has to check that the argument gives a cell attachment of stacks and not just underlying spaces.

For each p_i , identify an open substack containing p_i with $[U_{p_i} / \text{Aut}_{p_i}]$, where U_{p_i} is a ball around the origin in $T_{p_i}\mathfrak{X}$, and coordinates have been chosen in $T_{p_i}\mathfrak{X} = T_{p_i}\mathfrak{X}_+ \oplus T_{p_i}\mathfrak{X}_-$ such that the function $f|_{U_i} = c + |u_+|^2 - |u_-|^2$. Choose an ϵ smaller than the radii of all of these Morse coordinate patches, and define the closed substack of $\mathfrak{X}^{c+\epsilon]}$

$$\mathfrak{X}' = \{f \leq c - \epsilon\} \cup \bigcup_i \{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$$

Where we have used the slightly informal notation $\{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$ for the closed substack $[D_i / \text{Aut}_{p_i}]$.

365 Theorem 16.9 of [Noo05] gives a criterion for checking that \mathfrak{X}' is in fact a 2-categorical union. First of all, \mathfrak{X}' is manifestly a gluing in the language of [ibid.], in that $\mathfrak{X}' \setminus \coprod_i [S_i / \text{Aut}_{p_i}] \simeq \mathfrak{X}^{c-\epsilon] \sqcup \coprod_i [(D_i \setminus S_i) / \text{Aut}_{p_i}]$. The

⁴A map $\psi : \mathfrak{Y} \rightarrow \mathfrak{X}$ factors through $\mathfrak{X}^{[a,b]}$ if and only if $f \circ \psi : \mathfrak{Y} \rightarrow \mathbb{R}$ factors through $(-\infty, b]$, so it suffices to check on the level of underlying spaces.

second criterion – the existence of an atlas $a : X'_0 \rightarrow \mathfrak{X}'$ such that the pair of invariant subspaces $a^{-1}([S_i/\text{Aut}_{p_i}]) \subset a^{-1}([D_i/\text{Aut}_{p_i}])$ satisfies the local left lifting property (LLLP) with respect to the class of covering maps – is tautologically satisfied because every pair satisfies the LLLP with respect to local homeomorphisms.⁵ So we have really verified that a union of two substacks of a DM stack is always a 2-categorical union.

The rest of the argument for proceeds exactly as in [Mil63] and [Hep09a]. One introduces an auxiliary function F such that: F agrees with f outside of the Morse coordinate patches (on which $F \leq f$), $\{F \leq c + \epsilon\} = \mathfrak{X}^{c+\epsilon}$, and $F(p_i) < c - \epsilon$. Then one uses Proposition 3.6 to retract $\mathfrak{X}^{c+\epsilon}$ onto $\{F \leq c - \epsilon\}$, a substack which differs from $\mathfrak{X}^{c-\epsilon}$ only within the Morse coordinate patches. Finally one constructs a manifestly Aut_{p_i} equivariant deformation retract of $\{F \leq c - \epsilon\}$ onto \mathfrak{X}' in each $[U_i/\text{Aut}_{p_i}]$ which restricts to the identity on $\mathfrak{X}^{c-\epsilon} \cap [U_i/\text{Aut}_{p_i}]$. These can then be glued to give the global deformation retract of $\{F \leq c - \epsilon\}$ onto \mathfrak{X}' . \square

4. HOMOLOGICAL LEFSCHETZ THEOREM FOR ARTIN STACKS

The LHT for Artin⁶ stacks is beyond the current reach of Morse theoretic techniques. The theory of integration of vector fields on Artin stacks has been worked out in [Hep09b], but Morse theory has not been developed yet.

In this section we describe a local-to-global proof of the homological LHT that applies to differentiable Artin stacks admitting a presentation by compact manifolds.

In the complex category, proper Artin stacks are close to being gerbes, so in order to have a version of the LHT with broader applications, we will leave the category of complex stacks. For instance, the following theorem applies to a global quotient of a compact complex manifold by a compact Lie group.

Theorem 4.1. *Let \mathfrak{L} be a hermitian line bundle on a differentiable Artin stack \mathfrak{X} , and let s be a section. Assume that there exists a proper submersion $X_0 \rightarrow \mathfrak{X}$ from a compact complex manifold X_0 such that the pullback $L_0 = \mathfrak{L}|_{X_0}$ is holomorphic of type k and the section $s|_{X_0}$ is holomorphic.*

As before let $\mathfrak{Z} \subset \mathfrak{X}$ be the topological substack on which s vanishes. Then $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$.

Proof. From the proper submersion $X_0 \rightarrow \mathfrak{X}$ we get a closed, full, saturated sub-groupoid $Z_\bullet \subset X_\bullet$ presenting $\mathfrak{Z} \subset \mathfrak{X}$. Let $\{X_p\}$ and $\{Z_p\}$ denote the respective simplicial nerves.

We can form the smooth function $f = |s|^2$ on \mathfrak{X} , and as before \mathfrak{Z} is precisely the zero locus of f . In particular Z_p is the zero locus of $f|_{X_p} =$

⁵Although it is not necessary here, one could even find an atlas $a : X'_0 \rightarrow \mathfrak{X}'$ for which $a^{-1}([D_i/\text{Aut}_{p_i}]) = D_i$ and likewise for S_i

⁶An Artin stack is just a differentiable stack admitting a surjective submersion from a manifold

405 $(f|_{X_0})|_{X_p}$. Additionally Z_p is the preimage of Z_0 under any of the $p + 1$ simplicial maps $X_p \rightarrow X_0$.

Now the function $f|_{X_0}$ is the norm squared of a section of a hermitian line bundle of type k . Thus as in Section 2 above we can perturb $f|_{X_0}$ to a function which we call ϕ_0 which

- 410
- satisfies a Lojasiewicz inequality near its global minimum Z_0 ,
 - is Morse(-Bott) away from Z_0 with critical points of index $\geq k$.

Define the functions $\phi_p = \phi_0|_{X_p}$ using the 0^{th} simplicial face map $X_p \rightarrow X_0$ to restrict to X_p . Note that ϕ_0 is no longer invariant (does not necessarily descend to a smooth function on \mathfrak{X}), so we must be explicit about which
 415 simplicial maps we use. Both bulleted properties are preserved when one restricts along a surjective submersion, so the ϕ_p are functions with Lojasiewicz global minima $Z_p = \phi_p^{-1}\{0\}$ that are Morse-Bott away from Z_p with critical manifolds of index $\geq k$. It follows from non-degenerate Morse theory that $H_i(X_p, Z_p) = H^i(X_p, Z_p) = 0$ for all $i < k$.

420 From the bicomplex computing $H_*(\mathfrak{X}, \mathfrak{Z})$ we have a homological spectral sequence

$$E_{p,q}^1 = H_q(X_p, Z_p) \Rightarrow H_{p+q}(\mathfrak{X}, \mathfrak{Z}) \quad (1)$$

so the vanishing of $H_i(X_p, Z_p)$ implies $H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. A similar spectral sequence implies the result for cohomology. \square

Example 4.2. This theorem applies to global quotients even when the
 425 group does not act holomorphically. For instance let $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}\mathbb{P}^l$ by complex conjugation $[z_0 : \cdots : z_l] \mapsto [\bar{z}_0 : \cdots : \bar{z}_l]$. By identifying $\mathbb{C}^{l+1} = \mathbb{C} \otimes \mathbb{R}^{l+1}$, real subspaces of \mathbb{R}^{l+1} induce complex subspaces of \mathbb{C}^{l+1} , and thus we have an embedding $\mathbb{R}\mathbb{P}^l \subset \mathbb{C}\mathbb{P}^l$ as the fixed locus of G .

Complex conjugation acts equivariantly on $\mathcal{O}_{\mathbb{C}\mathbb{P}^l}(1)$, and an invariant holomorphic section corresponds to a form with real coefficients $s = r_0 z_0 +$
 430 $\cdots r_l z_l$. The zero locus of s is a hypersurface $H \subset \mathbb{C}\mathbb{P}^l$ induced by a real hypersurface in \mathbb{R}^{l+1} , and by a real coordinate change we may assume that $H = \{[0 : * : \cdots : *]\}$.

Let $p = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathbb{P}^l$. Then $\mathbb{C}\mathbb{P}^l - \{p\}$ deformation retracts equivariantly onto H , so we have

$$\begin{aligned} H_G^i(\mathbb{C}\mathbb{P}^l, H) &= H_G^i(\mathbb{C}\mathbb{P}^l, \mathbb{C}\mathbb{P}^l - \{p\}) = H_G^i(\mathbb{C}^l, \mathbb{C}^l - \{0\}) \\ &\simeq H^{i-l}(\mathbb{B}[*/\mathbb{Z}/2]; \Omega) \end{aligned}$$

In this last expression, we have used the Thom isomorphism for the bundle
 435 $[\mathbb{C}^l/G] \rightarrow [*/G]$, and Ω is a local system on $\mathbb{B}[*G]$ which depends on the orientability of this bundle. Thus we have $H_G^i(\mathbb{C}\mathbb{P}^l, H) = 0$ for $i < l$.

Example 4.3. Non-holomorphic group actions also arise naturally as subgroups of the group of unit quaternions acting by left multiplication on $\mathbb{P}(\mathbb{H}^l)$, where choose either i, j , or k as the complex structure on \mathbb{H}^l .

440 **Example 4.4.** Let X be a projective variety of dimension n and $\mathfrak{X} \rightarrow X$ a gerbe for a compact group G , and let $Z \subset X$ be a hyperplane section. The

theorem implies that the cohomology of \mathfrak{X} agrees with the cohomology of the restricted gerbe $\mathfrak{Z} \rightarrow Z$ in degree $< n - 1$.

In this case we can also see this by comparing the Leray-Serre spectral sequence $E_2^{p,q} = H^p(X; H^q(BG)) \Rightarrow H^{p+q}(\mathfrak{X})$ and the corresponding sequence for $\mathfrak{Z} \rightarrow Z$. The result follows because, by the classical LHT, the restriction map on the E^2 page $H^p(X; H^q(BG)) \rightarrow H^p(Z; H^q(BG))$ is an isomorphism for $p < n - 1$.

Unlike Theorem 4.1, the Leray-Serre argument in the example above does not require G to be compact, which suggests that Theorem 4.1 might hold for a much larger class of Artin stacks. We also observe that the homological LHT also applies to global quotients by reductive groups, giving further evidence for a more general statement than Theorem 4.1.

Corollary 4.5. *Let G be a compact Lie group and let the complexification $G_{\mathbb{C}}$ act on a compact complex manifold X , and let $(\mathfrak{L}, \mathfrak{h})$ be a $G_{\mathbb{C}}$ -equivariant hermitian line bundle of type k with invariant global section s vanishing on $Z \subset X$, then $H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = H^i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = 0$ for $i < k$.*

Proof. $G_{\mathbb{C}}$ deformation retracts onto G , and so the pair $(X \times (G_{\mathbb{C}})^{\times p}, Z \times (G_{\mathbb{C}})^{\times p})$ deformation retracts onto $(X \times (G)^{\times p}, Z \times (G)^{\times p})$ for any number of factors p . Thus by the spectral sequence (1), $H_i([X/G], [Z/G]) \rightarrow H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}])$ is an isomorphism, and $H_i([X/G], [Z/G]) = 0$ for $i < k$ by Theorem 4.1. \square

APPENDIX A. STABILITY OF GRADIENT DESCENT FLOW

We have used Morse theory to prove the LHT without requiring the section of the line bundle to be non-degenerate as in [Bot59]. In Lemma 2, we reduce the problem of showing that $(\mathfrak{X}, \mathfrak{Z})$ is a neighborhood deformation retract to the purely local statement that the gradient descent flow of a function near its minimum is *stable*. The key technique is due to Lojasiewicz [Loj63, Loj65, Loj84], and we review it here. He proved that real analytic functions satisfy the inequality (L), and applied it to show that analytic subvarieties of \mathbb{R}^n are neighborhood deformation-retracts. Many authors have extended his work, for instance in [KMP00].

Definition A.1. Let X be a Riemannian manifold and f a smooth function on X . We say that f satisfies a *Lojasiewicz inequality* at a point p if there are constants $\rho \in (0, 1)$ and $C > 0$ such that

$$|f - f(p)|^{\rho} \leq C|\nabla f| \tag{L}$$

in some neighborhood of p

As a consequence of (L), f takes the value $f(p)$ at any critical point near p . In particular if Z is the global minimal set of f and (L) holds at every point of Z , then Z is isolated from the rest of the critical locus of f .

Proposition A.2. *If f achieves its global minimum along $Z \subset X$ and (L) holds at $p \in Z$, then there is an open set $U \subset X$ containing p such that*

- 480 (1) the flow $\phi_t(x)$ of $-\nabla f$ is defined for all $x \in U$ and all $t \geq 0$, and $\phi_t(x) \in U$.
- (2) The map $U \times [0, \infty) \rightarrow U$ given by $(x, t) \mapsto \phi_t(x)$ extends uniquely to a continuous map $U \times [0, \infty] \rightarrow U$. In particular the flow of $-\nabla f$ deformation retracts U onto $U \cap Z$.

Remark A.3. The proposition is equally valid for a local minimum. Also, 485 the argument below shows that any gradient like vector field for f will have the same stability property.

Proof. For simplicity we will assume that $f \geq 0$ and $f(p) = 0$. Let $\phi_t(x_0)$ be an integral curve starting at $x_0 \in X$. As long as $\phi_t(x_0)$ stays in a region in which (L) holds, an arc length integral gives a uniform bound

$$\text{dist}(x_0, \phi_t(x_0)) \leq \frac{C}{1-\rho} f(x_0)^{1-\rho} \quad \text{for } t \geq 0 \quad (2)$$

490 Start with a relatively compact open ball $B_{r'}$ of radius r' around p on which the inequality (L) holds. Now we can find a smaller ball $B_r \subset B_{r'}$ such that the difference in radii $r' - r$ is larger than $Cf(x_0)^{1-\rho}/(1-\rho)$ for any point $x_0 \in B_r$. It follows from (2) that any flow line starting in B_r stays within $B_{r'}$ under the (positive time) flow of $-\nabla f$, and thus by 495 the escape lemma [Lee03] the flow is defined on B_r for all $t \geq 0$. We let $U = \bigcup_{t \geq 0} \phi_t(B_r)$.

Inequality (L) implies that $f(\phi_t(x_0)) \rightarrow 0$ as $t \rightarrow \infty$. Combining this with (2) shows that $\phi_t(x_0)$ is Cauchy as $t \rightarrow \infty$ and remains in a compact region, so $\phi_\infty(x) = \lim_{t \rightarrow \infty} \phi_t(x)$ is a well defined function on U . In fact 500 (2) shows that $(x, t) \mapsto \phi_t(x)$ is uniformly continuous as $t \rightarrow \infty$, and so the gradient descent flow extends uniquely to a continuous map $U \times [0, \infty] \rightarrow U$. Finally, $\phi_s(\phi_\infty(x)) = \lim_{t \rightarrow \infty} \phi_{s+t}(x) = \phi_\infty(x)$, so $\phi_\infty(U) \subset Z$ and the map constructed above is a deformation retract of U onto $U \cap Z$.

□

505 An immediate corollary of Proposition A.2 is the global statement that

Corollary A.4. *Let f be a smooth nonnegative function on X with global minimum $Z = f^{-1}\{0\} \subset X$. If the inequality (L) holds at each $p \in Z$, then there is an open neighborhood U of Z on which the negative gradient flow extends uniquely to a map $U \times [0, \infty] \rightarrow U$ which is a deformation retract 510 of U onto Z .*

In addition if Z is compact then X^{cl} deformation retracts onto Z for sufficiently small ϵ .

We have shown above that the Lojasiewicz inequality implies a stable gradient flow, but we are left with the question of which functions satisfy 515 (L). First of all, functions satisfying (L) have the following properties.

- (1) if h is a smooth function with $h(p) \neq 0$ and f satisfies (L) with $f(p) = 0$, then so does hf

- 520 (2) Let f and g satisfy (L) with $f(p) = g(p) = 0$. If the angle between ∇f and ∇g is bounded away from π near p then $f + g$ satisfies (L),⁷ and if the angle between ∇f^2 and ∇g^2 is bounded away from π then fg satisfies (L).
- 525 (3) if $\pi : X \rightarrow Y$ is an open mapping near $p \in X$ and $f \circ \pi$ satisfies (L) at p then f satisfies (L) at $\pi(p) \in Y$. If π is submersive at p , then the converse is true as well. Letting π be the identity map shows that (L) is independent of the metric.
- 530 (4) if $\pi : X \rightarrow Y$ is proper and surjective onto a neighborhood of $q \in Y$ and $f \circ \pi$ satisfies (L) at every point in the fiber $\pi^{-1}\{q\}$, then f satisfies (L) at q .

One might wonder in light of property (2) above if arbitrary algebraic combinations of functions will still satisfy (L), but this turns out to be false. For instance if $\rho : \mathbb{R} \rightarrow \mathbb{R}$ vanishes to all orders at 0, then $f(x) = (1 + \rho)x - x = \rho x$ does not satisfy (L) at 0 even though $(1 + \rho)x$ and x do. Thus while inequality (L) has convenient dynamical properties, it lacks algebraic properties. It is thus surprising that many functions do satisfy the Lojasiewicz inequality:

540 **Theorem A.5.** *Let f be smooth function on X which in some choice of local coordinates is a non-vanishing smooth function times a real analytic function which vanishes at p , then f satisfies property (L) at p .*

Proof. Using real analytic resolution of singularities there is a proper surjective analytic map $\pi : Y \rightarrow X$ such that $f \circ \pi = h(y)y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ where y_1, \dots, y_n are local analytic coordinates on Y and h is a smooth function which does not vanish near $\pi^{-1}\{p\}$. Applying property (2) above, or by a direct calculation, one shows that the monomial $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ satisfies (L) along its zero locus, and thus by property (1) so does $f \circ \pi$. Finally by property (4) f satisfies (L) at p . \square

REFERENCES

- 550 [Bai57] W. L. Baily, *On the imbedding of V -manifolds in projective space*, Amer. J. Math. **79** (1957), 403–430. MR 0100104 (20 #6538)
- [Beh04] K. Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 249–294 (electronic). MR MR2172499 (2006i:14016)
- 555 [Bot59] Raoul Bott, *On a theorem of Lefschetz*, Michigan Math. J. **6** (1959), 211–216. MR MR0215323 (35 #6164)
- [BX03] Kai Behrend and Ping Xu, *S^1 -bundles and gerbes over differentiable stacks*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 2, 163–168. MR MR1969572 (2004i:58025)
- [BX11] ———, *Differentiable stacks and gerbes*, J. Symplectic Geom. **9** (2011), no. 3, 285–341. MR 2817778 (2012k:53045)
- 560

⁷More precisely the condition is $(\nabla f, \nabla g) \geq (-1 + \epsilon)|\nabla f||\nabla g|$ in a neighborhood of p for some small ϵ

- [GM88] Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR MR932724 (90d:57039)
- 565 [GS82] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538. MR 664118 (83m:58040)
- [Hei05] J. Heinloth, *Notes on differentiable stacks*, Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005, Universitätsdrucke Göttingen, Göttingen, 2005, pp. 1–32. MR 2206877 (2007c:19009)
- 570 [Hep09a] Richard Hepworth, *Morse inequalities for orbifold cohomology*, Algebr. Geom. Topol. **9** (2009), no. 2, 1105–1175. MR MR2511141
- [Hep09b] ———, *Vector fields and flows on differentiable stacks*, Theory Appl. Categ. **22** (2009), 542–587. MR MR2591949
- [KMP00] Krzysztof Kurdyka, Tadeusz Mostowski, and Adam Parusiński, *Proof of the gradient conjecture of R. Thom*, Ann. of Math. (2) **152** (2000), no. 3, 763–792. MR MR1815701 (2002d:37028)
- 575 [Lee03] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003. MR MR1930091 (2003k:58001)
- [Loj63] S. Lojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les Équations aux Dérivées Partielles (Paris, 1962), Éditions du Centre National de la Recherche Scientifique, Paris, 1963, pp. 87–89. MR 0160856 (28 #4066)
- 580 [Loj65] ———, *Ensembles semi-analytiques*, Preprint, I.H.E.S., Bures-sur-Yvette (1965).
- [Loj84] ———, *Sur les trajectoires du gradient d'une fonction analytique*, Geometry seminars, 1982–1983 (Bologna, 1982/1983), Univ. Stud. Bologna, Bologna, 1984, pp. 115–117. MR 771152 (86m:58023)
- 585 [Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR MR0163331 (29 #634)
- [Noo05] B. Noohi, *Foundations of Topological Stacks I*, ArXiv Mathematics e-prints (2005).
- 590 [Noo08] ———, *Homotopy types of topological stacks*, ArXiv e-prints (2008).
- [Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112. MR MR0232393 (38 #718)
- [Tel00] Constantin Teleman, *The quantization conjecture revisited*, Ann. of Math. (2) **152** (2000), no. 1, 1–43. MR 1792291 (2002d:14073)
- 595 [Tot10] B. Totaro, *Line bundles with partially vanishing cohomology*, ArXiv e-prints (2010).

UC BERKELEY

E-mail address: danielhl@math.berkeley.edu