Equivariant geometry and Calabi-Yau manifolds

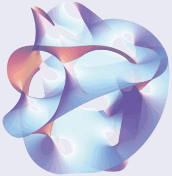
Daniel Halpern-Leistner

January 24, 2017

Overview

Calabi-Yau manifolds

A rich and interesting class of complex manifolds, studied intensely in differential geometry, algebraic geometry, and high energy physics



(Wikimedia Commons)

Mirror symmetry predicts that certain invariants of Calabi-Yau manifolds are unchanged under birational modification.

New ideas from equivariant geometry have led to the first significant progress on this question in 15 years.

Projective manifolds

We will consider the geometry of a projective complex manifold $X \subset \mathbb{P}^n$.[1]

Example: Hypersurfaces

Vanishing locus of a homogeneous polynomial. For instance, we can consider the "Fermat quintic"

$$X = \left\{ [z_0 : \cdots : z_4] \in \mathbb{P}^4 \middle| z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \right\}$$

X is a smooth compact complex manifold of complex dimension 3, real dimension 6.

[1] As a reminder: $\mathbb{P}^n$ can be thought of as the set of lines in $\mathbb{C}^{n+1}$, or more concretely as *non-zero* $n+1$-tuples $[z_0 : \cdots : z_n]$ up to rescaling $[tz_0 : \cdots : tz_n]$.

Birational modification

Analogous to surgery of smooth manifolds

Definition

A *birational equivalence* of projective manifolds $X \dashrightarrow X'$ is an isomorphism $U \to U'$ of algebraic open subsets $U \subset X$ and $U' \subset X'$.[a]

[a] An algebraic open set is the complement of a closed subvariety.

Classifying projective manifolds up to birational equivalence is a huge question in algebraic geometry.

Example: A basic but complicated question

Is X birationally equivalent to $\mathbb{P}^n$, i.e. does X admit an algebraic coordinate chart?

Geometric invariants and birational modification

First tool for classifying varieties: the *canonical line bundle* $K_X := \Omega_X^d$, the bundle of holomoprhic d-forms, where $d = \dim_{\mathbb{C}} X$.

Definition

The integers $P_n := \dim H^0(X, K_X^{\otimes n})$ are birational invariants, called *plurigenera*.

Can construct other invariants using the cotangent bundle Ω_X^1, but most geometric invariants change under birational modification, such as:

- Cohomology groups $H^*(X; \mathbb{C})$
- Hodge numbers $h^{p,q}(X) := \dim H^q(X; \Omega_X^p)$, for $q > 0$

Calabi-Yau manifolds

Definition

X is *Calabi-Yau* if $K_X = \Omega^d_X$ is trivial, where $d = \dim_{\mathbb{C}} X$; i.e. there exists a holomorphic volume form.

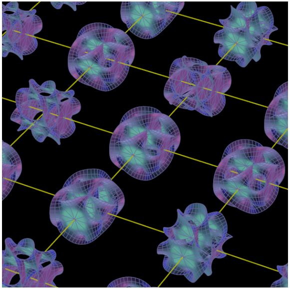
- Examples: elliptic curves ($\dim_{\mathbb{C}} = 1$), $K3$ surfaces ($\dim_{\mathbb{C}} = 2$), the Fermat quintic hypersurface ($\dim_{\mathbb{C}} = 3$)
- Interest in differential geometry: existence of Kähler metrics with vanishing Ricci curvature
- Interest in birational geometry: all of the Plurigenera $P_n = 1$.

From this point forward: only consider birational modifications $X \dashrightarrow X'$ of **Calabi-Yau manifolds.**

Predictions from physics

String theory describes spacetime as $\mathbb{R}^4 \times$ (compact Calabi-Yau 3-fold).

Mirror symmetry: Calabi-Yau's come in pairs, with a correspondence between invariants of $X \leftrightarrow X^{mir}$



(courtesy wolfram.com)

Philosophy: Given $X \dashrightarrow X'$, the corresponding mirror manifolds $X^{mir}, (X')^{mir}$ will be *deformation equivalent*

- Leads to the prediction that $H^*(X;\mathbb{Q}) \simeq H^*(X';\mathbb{Q})$
- $H^*(X;\mathbb{Q})$ should carry a representation of the fundamental group of the "complexified Kähler moduli space" of X.

Birational invariance of cohomology

Theorem (Batyrev '95, Kontsevich '95, Denef-Loeser '98)

If $X \dashrightarrow X'$ *are birationally equivalent Calabi-Yau manifolds, then* $H^*(X;\mathbb{Q}) \simeq H^*(X';\mathbb{Q})$*, and in fact* $h^{p,q}(X) = h^{p,q}(X')$*.*

Can think of cohomology classes in $H^*(X;\mathbb{Q})$ as the *characteristic classes* of holomorphic vector bundles on X.

Natural question: "categorification"

Is the equivalence of cohomology groups $H^*(X;\mathbb{Q}) \simeq H^*(X';\mathbb{Q})$ the shadow of an equivalence of categories?

$$\{\text{Vector bundles on } X\} \simeq \{\text{Vector bundles on } X'\}$$

Homological invariants of projective manifolds

The answer is no – we need to modify the question slightly...

The *derived category* of X, $D^b(X)$, is an enlargement of the category $Vect(X)$ of holomorphic vector bundles on X. Consists of *complexes* of vector bundles

$$\cdots \to E^{i-1} \xrightarrow{d} E^i \xrightarrow{d} E^{i+1} \to \cdots, \qquad d^2 = 0$$

- It's possible to recover $H^*(X;\mathbb{Q})$ from $D^b(X)$, and one can think of $D^b(X)$ itself as a richer kind of cohomology theory.
- $D^b(X)$ encodes information about many other geometric invariants (K-theory, Chow groups, etc..)

D-equivalence conjecture

Applying same philosophy from mirror symmetry, but this time using *homological mirror symmetry*, leads to...

D-equivalence conjecture, Bondal-Orlov ('95)

If X and X' are birationally equivalent Calabi-Yau manifolds, then

$$D^b(X) \simeq D^b(X').$$

- One of the motivating conjectures in the study of derived categories
- Piece of a broader set of conjectures and results relating birational geometry and derived categories

Progress on the D-equivalence conjecture

Originally studied in dimension 2 (Mukai, '81,'87), and for the simplest kind of birational modifications in higher dimensions.

Theorem (Bridgeland '00)

A birational modification of 3-dimensional compact Calabi-Yau manifolds $X \dashrightarrow X'$ induces an equivalence $D^b(X) \simeq D^b(X')$.

This has been basically the state of the art for compact Calabi-Yau's.

Remark

Some progress for holomorphically convex but non-compact algebraic symplectic manifolds (Bezrukavnikov and Kaledin '03-'05). Using ideas from geometric representation theory, and specifically "quantization in positive characteristic."

The new state of the art

Major source of examples of birational modifications of Calabi-Yau manifolds: **moduli spaces**

Moduli spaces of sheaves on a $K3$ surface, S

For any generic algebraic Kähler class $H \in H^2(S;\mathbb{C})$, $\exists$ a smooth compact Calabi-Yau moduli space M_H parameterizing "*Gieseker H-semistable*" coherent sheaves on S.

- Varying H leads to birational modifications $M_H \dashrightarrow M_{H'}$.

New approach using **equivariant geometry** leads to the first new cases of the D-equivalence conjecture in higher dimensions:

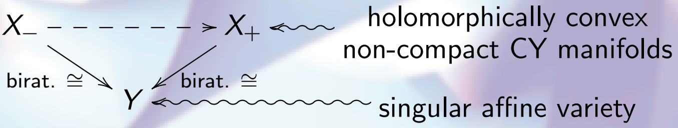
Theorem (HL)

If X is a projective Calabi-Yau manifold which is birationally equivalent to M_H for some generic H, then $D^b(X) \simeq D^b(M_H)$.

Overview

- Calabi-Yau manifolds are of interest in many subjects, especially in birational geometry.
- The D-equivalence conjecture predicts that the derived category is a birational invariant for Calabi-Yau manifolds.
- There has been recent progress on this conjecture using equivariant geometry.

Remainder of talk: discuss examples of the "local version" of the D-equivalence conjecture illustrating the role of equivariant geometry



Example: resolution of the ordinary double point

Let's focus on the simplest example: the 3 dimensional "ordinary double point" singularity.

$$Y = \left\{ \begin{bmatrix} u, w \\ v, z \end{bmatrix} \middle| \det = 0 \right\} \subset \mathbb{C}^4$$

There are two smooth (non-compact) Calabi-Yau's mapping birationally $X_\pm \to Y$, constructed as **quotients (i.e. orbit spaces)**:
Consider the map $V := \mathbb{C}^4 \to Y$ given by,

$$(x_0, x_1, y_0, y_1) \mapsto \begin{bmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{bmatrix},$$

which is invariant for the $\mathbb{C}^*$-action

$$t \cdot (x_0, x_1, y_0, y_1) = (t x_0, t x_1, t^{-1} y_0, t^{-1} y_1).$$

Equivariant surgery

For any $c \in \mathbb{R}$ we have a degenerate Morse function on $\mathbb{C}^4$:

$$\Phi_c(x_0, x_1, y_0, y_1) = (\overbrace{|x_0|^2 + |x_1|^2 - |y_0|^2 - |y_1|^2}^{\text{moment map } \mu} - c)^2$$

Degenerate critical locus at global minimum $\Phi_c = 0$, and one additional critical point at $(0,0,0,0)$. For $c \neq 0$, $X_c := \Phi_c^{-1}(0)/U(1)$ is a smooth manifold.

$X_c \to Y$ **is birational equivalence.**

Video: As c varies, $\Phi_c^{-1}(0)$ undergoes a surgery which is $U(1)$-equivariant.

The complex structure on X_c

Theorem (Special case of the Kirwan-Ness theorem)

Define open subsets of $V \simeq \mathbb{C}^4$:

$$V_+^{\mathrm{ss}} := \{(x_0, x_1) \neq 0\} \quad \textit{and} \quad V_-^{\mathrm{ss}} := \{(y_0, y_1) \neq 0\}.$$

Then $\mathbb{C}^*$ acts freely on $V_\pm^{\mathrm{ss}}$, and

$$\Phi_c^{-1}(0)/U(1) \simeq \begin{cases} V_+^{ss}/\mathbb{C}^*, & \text{if } c > 0 \\ Y, & \text{if } c = 0 \\ V_-^{ss}/\mathbb{C}^*, & \text{if } c < 0 \end{cases}$$

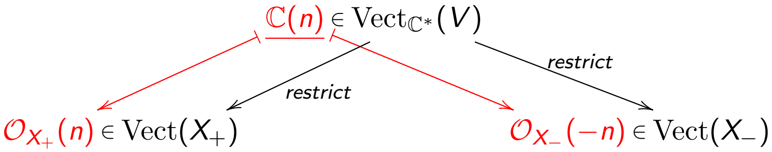
- The metric structure on $\Phi_c^{-1}(0)/U(1)$ depends on c, but the complex structure does not.
- We denote $X_\pm = X_c$ where $\pm = \mathrm{sign}(c)$.

Equivalences between the derived categories, I

Key tool: Equivariant vector bundles, $\mathrm{Vect}_G(V)$

For any $U \in \mathrm{Rep}(G)$, let $\underline{U}$ denote the trivial vector bundle $V \times U \simeq V \times \mathbb{C}^n$ over V. G acts on the fiber as well as the base, giving $\underline{U}$ the structure of an *equivariant vector bundle*.

Naive way to compare categories $\mathrm{Vect}(X_\pm)$: **restrict equivariant vector bundles** on V to $V^{\mathrm{ss}}_\pm/\mathbb{C}^* \simeq X_\pm$.



Idea: lift then restrict; but does not work for all $\mathcal{O}_{X_+}(n)$ at once.

Equivalences between derived categories, II

Definition

For any $\delta \in \mathbb{R}$, let $\mathcal{M}(\delta) \subset D^b_{\mathbb{C}^*}(V)$ be the category of complexes of equivariant vector bundles built from $\underline{\mathbb{C}(n)}$ for $n \in \delta + [-1, 1]$.

By a result of Beilinson, any two consecutive $\mathcal{O}_{X_\pm}(n)$ are enough to build *any* complex, and in fact we have

Theorem (Hori-Herbst-Page, Segal '09)

For δ generic, the restriction functor is an equivalence

$$\mathcal{M}(\delta) \xrightarrow{\simeq} D^b(V^{ss}_\pm / \mathbb{C}^*) \simeq D^b(X_\pm).$$

For any generic δ, this leads to an equivalence

$$F_\delta : D^b(X_-) \simeq \mathcal{M}(\delta) \simeq D^b(X_+).$$

The general picture - geometric invariant theory (GIT)

One can construct a quotient for *any* subvariety $X \subset \mathbb{P}^n \times \mathbb{C}^m$ with an action of a compact Lie group K with complexification G.

Technical remark: GIT parameters, general case

The GIT parameter is an equivariant Kähler class $c \in H^2_G(X;\mathbb{R})$. This defines an energy function $\Phi_c : X \to \mathbb{R}_{\geq 0}$, and the quotient space $\Phi_c^{-1}(0)/K = X^{\mathrm{ss}}/G$ has an algebraic structure as well.

General principle: construct derived equivalences by verifying

Theorem (Theorem template)

There is a category $\mathcal{M}(\delta) \subset D^b_G(X)$ depending on $\delta \in H^2_G(X;\mathbb{R})$ such that for generic δ and c restriction is an equivalence

$$\mathcal{M}(\delta) \xrightarrow{\simeq} D^b(\Phi_c^{-1}(0)/K).$$

General developments in equivariant derived categories

New tool (Ballard-Favero-Katzarkov '12,HL '12)

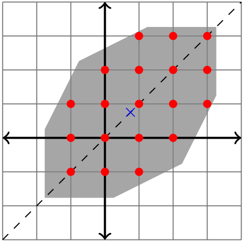
A general structure theorem for the category of equivariant complexes $E^\bullet \in D^b_G(X)$, relating $D^b_G(X)$ to $D^b(\Phi_c^{-1}(0)/K)$.

- Structure theorem reflects the Morse stratification (i.e. gradient descent stratification) of X under Φ_c.
- Can functorially "lift" a complex on $\Phi_c^{-1}(0)/K$ to a G-equivariant complex on X which satisfies certain "weight bounds" at the critical points of Φ_c.
- As c varies, the Morse stratification under Φ_c changes, and one can use the structure theorem to compare the derived categories of different GIT quotients.

Example: Magic windows theorem for linear representations

Consider: reductive group G, *self dual* representation V, and $\delta \in M_{\mathbb{R}}^W$, where M =weight lattice of G and W =Weyl group.

$G = \mathrm{GL}_2$, $V = T^* Sym^3(\mathbb{C}^2)$:



Define: $\mathcal{M}(\delta) \subset D^b_G(V)$ to be the subcategory of complexes of equivariant bundles built from $\underline{U}$ where U is a representation of G whose character lies in a certain polytope $\delta + \Sigma_V$.

Theorem (Magic windows, HL-Sam '16)

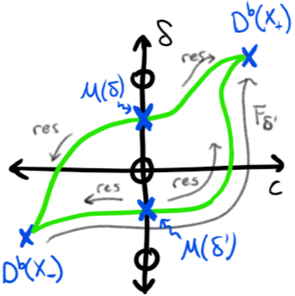
If V is a self dual linear representation of a reductive group G, then for δ and c generic the restriction functor induces an equivalence $\mathcal{M}(\delta) \simeq D^b(\Phi_c^{-1}(0)/K)$. Hence **all generic GIT quotients of V are derived equivalent**.

Organizing data: the Kähler moduli space

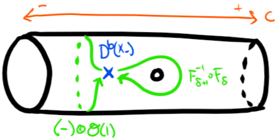
Idea from physics

The categories $D^b(\Phi_c^{-1}(0)/K)$ can be assembled into a "local system of categories" over a complex manifold $\mathcal{K} = \mathcal{K}_{V/G}$.

In the original $\mathbb{C}^4/\mathbb{C}^*$ example, $H^2_{\mathbb{C}^*}(V;\mathbb{C}) \simeq \mathbb{C} = \{c + i\delta\}$:



$\mathcal{K} := (\mathbb{C} \setminus \{\text{non-generic } \delta\})\,/i\mathbb{Z}$



$\pi_1(\mathcal{K})$ acts by autoequivalences of $D^b(X_\pm)$.

The complexified Kähler moduli space II

For a self-dual linear representation V of G, the $\mathcal{K}$ has the form

$$\mathcal{K} = \left(\begin{array}{l} \text{complement of complex} \\ \text{hyperplane arrangement in } M_{\mathbb{C}}^W \end{array} \right) / iM^W.$$

Theorem (HL-Sam '16)

There is a local system of triangulated categories over $\mathcal{K}$ whose stalk at $c + i\delta$ is $D^b(\Phi_c^{-1}(0)/K)$ for generic c.

The groups $\pi_1(\mathcal{K})$ are generalizations of affine braid groups

- Actions of affine braid groups on derived categories are used to construct knot homology theories (Cautis-Kamnitzer-Licata, '11).
- Even on the level of K-theory, the representations constructed are potentially new and interesting.

Just the beginning...

- With Davesh Maulik and Andrei Okounkov, I am using these methods to categorify representations of quantum affine algebras on the K-theory of quiver varieties.
  - Related to Bezrukavnikov and collaborators' study of quantizations of symplectic resolutions, generalizing Springer theory.
- Key engine powering the proof of main theorem:
  - A new approach to analyzing moduli problems in algebraic geometry, "beyond geometric invariant theory" program. key words: derived algebraic geometry, algebraic stacks.
  - Reduce *D*-equivalence conjecture for moduli spaces to the linear examples discussed earlier.

Conclusions

The geometry of **Calabi-Yau manifolds** is a rich subject, with connections to differential geometry and physics. The *D*-**equivalence conjecture**, that the derived category is a birational invariant of Calabi-Yau manifolds, is a motivating conjecture in the theory of derived categories and birational geometry.

- New techniques in **equivariant geometry** have led to the first new instances of the D-equivalence conjecture in several years.
- Many connections with geometric representation theory to explore, and more applications of general techniques in store.

Thanks!