Abstracts

Θ-reductive moduli problems, stratifications, and applications

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For most moduli problems in algebraic geometry, the existence of a quasi-projective fine moduli space fails in myriad ways. The language of stacks is necessary to deal with the issue that objects can have (finite or infinite) automorphism groups, but even then the moduli problem can be "too big."

Example 1. Let $\mathscr{M}_{R,D}$ be the moduli of vector bundles over a smooth curve C or rank R and degree D. This algebraic stack is locally finite type but not quasi-compact: indeed the quantity $\dim H^0(C,E)$ is semicontinuous and obtains arbitrarily large values, so the moduli functor can not receive a surjective map from a quasi-projective scheme.

The solution to this problem is the Harder-Narasimhan (HN) filtration: Every unstable bundle admits a unique filtration whose associated graded pieces are semistable with slopes arranged in decreasing order. This leads to the Shatz stratification

$$\mathscr{M}_{R,D} = \mathscr{M}^{ss}_{R,D} \cup \bigcup \mathcal{S}_\alpha \tag{1}$$

where $\mathscr{M}^{ss}_{R,D}$ admits a projective good moduli space, and $\mathcal{S}_\alpha$ denotes the moduli of vector bundles of a fixed HN type (indexed by the rank and degrees of the associated graded pieces $\alpha = (r_1, \ldots, r_p; d_1, \ldots; d_p)$). Assigning a bundle to its associated graded defines a map $\mathcal{S}_\alpha \to \mathcal{Z}_\alpha := \mathscr{M}^{ss}_{r_1,d_1} \times \ldots \times \mathscr{M}^{ss}_{r_p,d_p}$ whose fibers are affine spaces.

We present a program for "solving" other moduli problems in this manner, by introducing a type of stratification which we call a Θ-stratification.

0.1. **Θ-reductive stacks.** Our main character is the algebraic stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$. A vector bundle on Θ is the same as a vector space with a weighted descending filtration. This leads to the observation that the mapping stack $\underline{\mathrm{Map}}(\Theta, B\mathrm{GL}_n)$ is algebraic – it is an infinite disjoint union of quotients of partial flag varieties by GL_n. In fact, this is a special case of a more general result

Theorem 1. *Let $\mathcal{X}$ be a (derived) locally finite type algebraic stack with quasi-affine diagonal over a field. Then $\underline{\mathrm{Map}}(\Theta, \mathcal{X})$ is a locally finite type algebraic stack with quasi-affine diagonal.*

Example 2. In the example of $\mathscr{M}_{R,D}$, a T-point of $\underline{\mathrm{Map}}(\Theta, \mathscr{M}_{R,D})$ a vector bundle E on $C \times T$ along with a flat family of weighted descending filtrations, i.e. sequence $\cdots E_{w+1} \subset E_w \subset \cdots \subset E$ of vector bundles which stabilizes to E on the right and 0 on the left, and such that $\mathrm{gr}_w E_\bullet = E_w / E_{w+1}$ is a vector bundle for all w.

Evaluation of a map $\Theta \to \mathcal{X}$ at the point $1 \in \mathbb{A}^1$ defines a map of algebraic stacks $\mathrm{ev}_1 : \underline{\mathrm{Map}}(\Theta, \mathcal{X}) \to \mathcal{X}$ which corresponds to forgetting the data of the filtration in the previous example.

Definition 1. [7] Let $\mathcal{X}$ be a locally finite type stack with quasi-affine diagonal over a field. Then $\mathcal{X}$ is Θ-*reductive* if for any finite type k-scheme T over $\mathcal{X}$, the connected components of $T \times_{\mathcal{X}} \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ are proper over T.

Example 3. If X is affine and G is a reductive group acting on X, then $\mathcal{X} = X/G$ is Θ-reductive. However, this fails for more general quasi-projective X.

Example 4. Let X be a projective scheme, and fix a t-structure on $D^b\mathrm{Coh}(X)$ satisfying certain properties (Noetherian, generic flatness, boundedness of generalized Quot-spaces; see [7]). Then the moduli stack of flat families of objects in $D^b\mathrm{Coh}(X)^\heartsuit$ is Θ-reductive. In particular, the usual moduli stack of flat families of coherent sheaves is Θ-reductive.

The stack $\mathscr{M}_{R,D}$ is not Θ-reductive, but it is an open substack of the Θ-reductive stack $\underline{\mathrm{Coh}}(C)$, the moduli of flat families of coherent sheaves on C. We hope to study many more moduli problems by finding natural enlargements which are Θ-reductive and then constructing Θ-stratifications as follows.

0.2. **Θ-stratifications.** It turns out that among all weighted descending filtrations of an unstable vector bundle the numerical invariant

$$\mu(f : \Theta \to \mathscr{M}_{R,D}) = \frac{\sum_w w\,(R\deg(\mathrm{gr}_w E_\bullet) - D\mathrm{rk}(\mathrm{gr}_w E_\bullet))}{\sqrt{\sum_w w^2 \mathrm{rk}(\mathrm{gr}_w E_\bullet)}}$$

is maximized by a unique (up to simultaneous rescaling) choice of weights on the Harder-Narasimhan filtration, which lets us canonically identify points on $\mathcal{S}_\alpha \subset \mathscr{M}_{R,D}$ with points on the mapping stack. On an arbitrary stack $\mathcal{X}$, one can construct a function generalizing the function μ from a pair of cohomology classes in $H^2(\mathcal{X};\mathbb{Q})$ and $H^4(\mathcal{X};\mathbb{Q})$.

In general, we define a Θ-stratification to be an open substack $\mathcal{S} \subset \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ such that $\mathrm{ev}_1 : \mathcal{S} \to \mathcal{X}$ is a locally closed immersion (satisfying some additional nice properties). Note that in general $\underline{\mathrm{Map}}(\Theta, \mathcal{X})$ will have many more connected components than $\mathcal{X}$, and $\mathcal{S}$ plays the role of the disjoint union of Shatz strata.

Theorem 2. [3] *Let $\mathcal{X}$ be a Θ-reductive stack. Then any locally convex, bounded numerical invariant defines a Θ-stratification of $\mathcal{X}$.*

The notion of a Θ-stratification is a simultaneous generalization of the Shatz stratification as well as the canonical stratification of the unstable locus in GIT. Our theorem leads to new examples of Θ-stratifications, not known to be related to GIT, such as a stratification of the stack of flat families of objects in the heart of a Bridgeland stability condition on the derived category of a $K3$ surface.

0.3. **Some applications.** Kirwan's surjectivity theorem [8] says that for a smooth (local) quotient stack with Θ-stratification the restriction $H^*(\mathcal{X};\mathbb{Q}) \to H^*(\mathcal{X}^{ss};\mathbb{Q})$ is surjective. This leads to beautifully explicit formulas [9] expressing the difference in the Poincare polynomial of $\mathcal{X}$ and $\mathcal{X}^{ss}$ as a sum of contributions from each stratum. Recently these results have been categorified to a structure theorem [2, 1] for the derived category $D^b(\mathcal{X})$, where a direct sum decomposition of $H^*(\mathcal{X})$ is categorified by an infinite semiorthogonal decomposition of $D^b(\mathcal{X})$.

Using the modular interpretation as a mapping stack allows one to generalize this result beyond the smooth global quotient situation.

Theorem 3. [5, 6] *Let $\mathcal{X}$ be a locally finite type derived*[1] *algebraic stack with a quasi-affine diagonal. If $\mathcal{X}$ has a derived Θ-stratification, then there is an infinite semiorthogonal decomposition*

$$D^{-}\mathrm{Coh}(\mathcal{X}) = \langle \ldots, D^{-}\mathrm{Coh}(\mathcal{X}^{ss}), \mathcal{A}_{\alpha}^{w_\alpha}, \mathcal{A}_{\alpha}^{w_\alpha+1}, \ldots, \mathcal{A}_{\alpha'}^{w_{\alpha'}}, \ldots \rangle$$

where $\mathcal{A}_\alpha^w \simeq D^{-}\mathrm{Coh}(\mathcal{Z}_\alpha)^w$. When $\mathcal{X}$ is quasi-smooth, then a version of this theorem holds with $D^b\mathrm{Coh}$ instead of $D^{-}\mathrm{Coh}$.

Here $\mathcal{Z}_\alpha$ are the "centers" of the strata (generalizing the example of the Shatz stratification above), the category $D^{-}\mathrm{Coh}(\mathcal{Z}_\alpha)^w$ is the full subcategory with weight w with respect to a canonical generic $\mathbb{G}_m$-stabilizer in $\mathcal{Z}_\alpha$. Algebraic symplectic stacks always satisfy the $D^b\mathrm{Coh}$ version of the theorem, so we have

Corollary 3.1. *Let $\mathcal{X}$ be an algebraic symplectic stack with a Θ-stratification. Then $K(D^b\mathrm{Coh}(\mathcal{X})) \to K(D^b\mathrm{Coh}(\mathcal{X}^{ss}))$ is a split surjection.*

Specializing to global quotients over the ground field $\mathbb{C}$, it is possible to recover the Atiyah-Segal equivariant topological K-theory from $D^b\mathrm{Coh}(X/G)$ [4]. This leads to some surprising implications for the topology of singular stacks:

Corollary 3.2. *Let $\mu : X \to \mathfrak{g}^*$ be an algebraic moment map for a Hamiltonian action of a reductive group G on a projective-over-affine algebraic symplectic variety X, all over $\mathbb{C}$. Let $X_0 = \mu^{-1}(0)$, and let $G_c \subset G$ be a maximal compact subgroup. Then $K_{G_c}^{top}(X_0) \to K_{G_c}^{top}(X_0^{ss})$ is a split surjection.*

References

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Reporter: Luca Tasin

[1] Even if $\mathcal{X}$ is a classical (non-derived) stack, the derived structure on the strata $\mathcal{S}_\alpha \subset \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ will differ from the naive structure as a classical locally closed substack.