Θ-STRATIFICATIONS, Θ-REDUCTIVE STACKS, AND APPLICATIONS

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Abstract. These are expanded notes on a lecture of the same title at the 2015 AMS summer institute in algebraic geometry. We give an introduction and overview of the "beyond geometric invariant theory" program for analyzing moduli problems in algebraic geometry. We discuss methods for analyzing stability in general moduli problems, focusing on the moduli of coherent sheaves on a smooth projective scheme as an example. We describe several applications: a general structure theorem for the derived category of coherent sheaves on an algebraic stack; some results on the topology of moduli stacks; and a "virtual non-abelian localization formula" in K-theory. We also propose a generalization of toric geometry to arbitrary compactifications of homogeneous spaces for algebraic groups, and formulate a conjecture on the Hodge theory of algebraic-symplectic stacks.

We present an approach to studying moduli problems in algebraic geometry which is meant as a synthesis of several different lines of research in the subject. Among the theories which fit into our framework: 1) geometric invariant theory, which we regard as the "classification" of orbits for the action of a reductive group on a projective-over-affine scheme; 2) the moduli theory of objects in an abelian category, such as the moduli of coherent sheaves on a projective variety and examples coming from Bridgeland stability conditions; 3) the moduli of polarized schemes and the theory of K-stability.

Ideally a moduli problem, described by an algebraic stack $\mathcal{X}$, is representable by a quasi-projective scheme. Somewhat less ideally, but more realistically, one might be able to construct a map to a quasi-projective scheme $q : \mathcal{X} \to X$ realizing X as the good moduli space [A] of $\mathcal{X}$. Our focus will be on stacks which are far from admitting a good moduli space, or for which the good moduli space map q, if it exists, has very large fibers. The idea is to construct a special kind of stratification of $\mathcal{X}$, called a Θ-*stratification*, in which the strata themselves have canonical modular interpretations. In practice each of these strata is closer to admitting a good moduli space.

Given an algebraic stack $\mathcal{X}$, our program for analyzing $\mathcal{X}$ and "classifying" points of $\mathcal{X}$ is the following:

(1) find a Θ-reductive enlargement $\mathcal{X} \subset \mathcal{X}'$ of your moduli problem (See Definition 2.3),
(2) identify cohomology classes $\ell \in H^2(\mathcal{X}'; \mathbb{Q})$ and $b \in H^4(\mathcal{X}'; \mathbb{Q})$ for which the theory of Θ-stability defines a Θ-stratification of $\mathcal{X}'$ (See §1.2),
(3) prove nice properties about the stratification, such as the boundedness of each stratum.

We spend the first half of this paper (§1 & §2) explaining what these terms mean, beginning with a detailed review of the example of coherent sheaves on a projective scheme. Along the way we discuss constructions and results which may be of independent interest, such as a proposed generalization of toric geometry which replaces fans in a vector space with certain collections of rational polyhedra in the spherical building of a reductive group G (§2.2).

In the second half of this paper we discuss applications of Θ-stratifications. In (§3 & §4) we discuss how to use derived categories to categorify Kirwan's surjectivity theorem for cohomology (See Theorem 3.1), and several variations on that theme. Specifically, we discuss how methods of derived algebraic geometry and the theory of Θ-stratifications can be used to establish structure theorems (Theorem 3.17,Theorem 3.22) for derived categories of stacks with a Θ-stratification, and we use this to prove a version of Kirwan surjectivity for Borel-Moore homology (Corollary 4.1). As an application we show (Theorem 4.3) that the Poincare polynomial for the Borel-Moore homology of the stack of Gieseker semistable sheaves on a K3 surface is independent of the semistability condition (provided it is generic), which leads to Conjecture 4.4 on the Hodge theory of (0-shifted) symplectic derived Artin stacks. Finally in §4 we discuss how the same theory of (derived) Θ-stratifications can be used to establish a "virtual non-abelian" localization formula in K-theory which generalizes other virtual localization formulas for torus actions in K-theory and cohomology.

0.0.1. *Warning.* Many of the results described in this paper, especially in the second half, serve as a summary and announcement of results which will ultimately appear in [HL5] and in the final versions of [HLP2] and [HL1]. We refer the reader to those papers for more precise statements of our main results and a more thorough treatment, including proofs. The reader can find less general preliminary versions of many of the results which will appear in [HL5] in the preprint [HL3].

0.0.2. *Background.* We work throughout over the field of complex numbers for simplicity, although many of the results stated hold over a more general base scheme. For us the phrase "moduli problem" is synonymous with "algebraic stack" in the sense of Artin, which is a sheaf of groupoids on the big étale site of commutative $\mathbb{C}$-algebras such that the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces and there is a smooth surjective morphism from a scheme $X \to \mathcal{X}$. We will assume that $\mathcal{X}$ is "quasi-geometric," meaning the diagonal is quasi-affine. Many commonly studied moduli problems are quasi-geometric.

0.0.3. *Author's note.* I thank the organizers of the 2015 AMS Summer Institute in Algebraic Geometry for a lively conference and for inviting me to contribute to these proceedings. I thank Daniel Pomerleano, Pablo Solis, and an anonymous referee for many useful comments on the first versions of this manuscript. The work described here was partially supported by the NSF MSPRF award DMS-1303960.

1. The Harder-Narasimhan problem

1.1. **Motivational example: the Harder-Narasimhan filtration for coherent sheaves.** Fix a projective scheme X over $\mathbb{C}$. Our goal shall be to *classify* all coherent sheaves on X and how they vary in families. The story summarized here serves as the template for the theory of Θ-stratifications, which seeks to extend this picture to more general moduli problems.

Fix an ample invertible sheaf $\mathcal{O}_X(1)$ of Neron-Severi class $H \in NS(X)_{\mathbb{R}}$. Given a coherent sheaf E on X, the Grothendieck-Riemann-Roch theorem implies that the Hilbert function this is a polynomial of degree $d = \dim(\mathrm{Supp}(E))$ whose coefficients can be expressed explicitly in terms of the Chern classes of E and the class H, $P_E(n) = \chi(X, E \otimes \mathcal{O}_X(n)) = \sum_k a_k(E) \frac{n^k}{k!}$. For a flat family of coherent sheaves over a scheme S, which by definition is a coherent sheaf on $X \times S$ which is flat over S, the Hilbert polynomial of the restriction E_s to each fiber is locally constant on S. We consider the moduli functor $\mathcal{C}oh(X)_P$, which is a contravariant functor from schemes to groupoids defined by

$$\mathcal{C}oh(X)_P(T) = \{F \in \mathrm{Coh}(X \times T)\, | F \text{ is flat over } S, \text{ and } P_{E_s}(t) = P(t), \forall s \in S\}$$

Similarly we let $\mathcal{C}oh(X)_{\dim \leq d}$ be the stack of families of coherent sheaves whose support has dimension $\leq d$. We will use the symbol $\mathrm{Coh}(X)$ to denote the abelian category of coherent sheaves on X, rather than the stack.

$\mathcal{C}oh(X)_P$ is an algebraic stack locally of finite type over $\mathbb{C}$, and in fact it can locally be described as a quotient of an open subset of a quot-scheme by the action of a general linear group. $\mathcal{C}oh(X)_P$ is not representable. In fact it is not even *bounded*, meaning there is no finite type $\mathbb{C}$-scheme Y with a surjection $Y \to \mathcal{C}oh(X)_P$, i.e. there is no finite type Y parameterizing a flat family of coherent sheaves such that every isomorphism class in $\mathrm{Coh}(X)$ appears as some fiber.

Definition 1.1. We consider the following polynomial invariants of a coherent sheaf $E \in \mathrm{Coh}(X)_{\dim \leq d}$

$$\mathrm{rk}(E) := P_E = \sum_{k=0}^{d} a_k(E) \frac{n^k}{k!}, \qquad \deg(E) := \sum_{k=0}^{d} (d-k) a_k(E) \frac{n^k}{k!}$$

We also define the *polynomial slope* $\nu(E) := \deg(E)/\mathrm{rk}(E)$, which is a well defined rational function of n of $E \neq 0$ because $\mathrm{rk}(E) \neq 0$. A coherent sheaf $E \in \mathrm{Coh}(X)_{\dim \leq d}$ is H-semistable if $\nu(F) \leq \nu(E)$ for all proper subsheaves $F \subset E$, by which we mean $\nu(F)(n) \leq \nu(E)(n)$ for all $n \gg 0$. E is *unstable* if it is not semistable.

Remark 1.2. If $d' = \dim(\mathrm{Supp}(E))$, then elementary manipulations show that as $n \to \infty$ we have

$$\nu(E) = d - d' + d' \frac{a_{d'-1}(E)}{a_{d'}(E)} \frac{1}{n} + O(\frac{1}{n^2}) \tag{1}$$

and in particular $\lim_{n \to \infty} \nu(E) = d - d'$. This implies that a semistable sheaf must be pure (i.e. have no subsheaves supported on a subscheme of lower dimension). Furthermore if we let $\nu_{\leq d'}(E)$ denote the slope of E regarded as an object of $\mathrm{Coh}(X)_{\dim \leq d'}$, then for any $d > d'$ we have $\nu_{\leq d'}(E) = \nu_{\leq d}(E) - d + d'$. So the notion of semistability for $E \in \mathrm{Coh}(X)_{\dim \leq d'}$ agrees with that for E regarded as an object of $\mathrm{Coh}(X)_{\dim \leq d}$.

This definition of semistability in terms of polynomial slopes coincides with Gieseker's notion of semistability, as developed and studied for arbitrary coherent sheaves by Simpson [S4] (See also [HL6]). More precisely, the notion above is a slight reformulation of the polynomial Bridgeland stability condition discussed in [B3, §2], which is itself a reformulation of Rudakov's reformulation [R] of Simpson/Gieseker stability. One defines polynomial Bridgeland stability with respect to a polynomial valued "central charge" for $E \in \mathrm{Coh}(X)$

$$Z(E) = \sum_k \rho_k a_k(E) \frac{n^k}{k!},$$

where ρ_k are some fixed complex numbers in $\mathbb{H} \cup \mathbb{R}_{<0}$ for which $\frac{1}{\pi}\arg(\rho_0) > \frac{1}{\pi}\arg(\rho_1) > \cdots$. Then $E \in \mathrm{Coh}(X)$ is said to be semistable with respect to Z if for every subsheaf $F \subset E$, one has $\frac{1}{\pi}\arg(Z(F)(n)) \leq \frac{1}{\pi}\arg(Z(E)(n))$ for all $n \gg 0$. If we choose $\rho_k = k - \dim(X) + i \in \mathbb{C}$, so that $\nu(E) = -\Re(Z(E))/\Im(Z(E))$ as functions of n, then $\frac{1}{\pi}\arg(Z(F)(n)) \leq \frac{1}{\pi}\arg(Z(E)(n))$ if and only if $\nu(F)(n) \leq \nu(E)(n)$, hence this polynomial Bridgeland stability condition coincides with Definition 1.1.

For a family of coherent sheaves parametrized by S, the set of points $s \in S$ for which E_s is semistable is open, hence we can define an open substack $\mathcal{C}oh(X)_P^{H-\mathrm{ss}} \subset \mathcal{C}oh(X)_P$ parameterizing families of semistable sheaves.

Theorem 1.3. [HL6, Theorem 4.3.4] *For every integer valued polynomial $P \in \mathbb{Q}[T]$ of degree $\leq \dim X$ and every ample class $H \in NS(X)_{\mathbb{R}}$, the stack of H-semistable coherent sheaves on X admits a projective good moduli space $\mathcal{C}oh(X)_P^{H-\mathrm{ss}} \to M(X)_P^{H-\mathrm{ss}}$.*

We regard the scheme $M(X)_P^{H-\mathrm{ss}}$ as a solution of the classification problem for H-semistable sheaves: $M(X)_P^{H-\mathrm{ss}}$ does not quite represent the moduli problem, but the fibers of the map $\mathcal{C}oh(X)_P^{H-\mathrm{ss}} \to M(X)_P^{H-\mathrm{ss}}$ can be described fairly explicitly as "S-equivalence" classes of semistable sheaves. This is not a complete classification of points of $\mathcal{C}oh(X)_P$, however, because we have discarded the closed substack of unstable sheaves, which is not even finite type. For the purposes of this paper, we are interested in the rest of the classification of coherent sheaves on X, and the structure of the unstable locus.

Theorem 1.4. *If $E \in \mathrm{Coh}(X)_{\dim \leq d}$ is H-unstable, then there is a* unique *filtration $E_N \subset E_{N-1} \subset \cdots \subset E_0 = E$ called the* Harder-Narasimhan (HN) *filtration such that*

(1) $\mathrm{gr}_i(E_\bullet) := E_i/E_{i+1}$ *is semistable for all i, and*
(2) $\nu(\mathrm{gr}_0(E_\bullet)) < \nu(\mathrm{gr}_1(E_\bullet)) < \cdots < \nu(\mathrm{gr}_N(E_\bullet))$ *for $n \gg 0$.*

We refer to the tuple of Hilbert polynomials $\alpha = (P_{\mathrm{gr}_0^{HN}(E)}, \ldots, P_{\mathrm{gr}_N^{HN}(E)})$ as the Harder-Narasimhan (HN) type *of E.*

The moduli functor $\mathcal{S}_\alpha$ which parametrizes families of unstable sheaves of HN type $\alpha = (P_0, \ldots, P_N)$ along with their HN filtration *is an algebraic stack. The map which forgets the filtration $\mathcal{S}_\alpha \to \mathcal{C}oh(X)_P$ is a locally closed embedding of algebraic stacks, and we have a map of stacks*

$$\mathcal{S}_\alpha \to \mathcal{C}oh(X)_{P_0}^{H-\mathrm{ss}} \times \cdots \times \mathcal{C}oh(X)_{P_N}^{H-\mathrm{ss}}, \quad \textit{sending} \quad [E] \mapsto (\mathrm{gr}_0^{HN}(E), \ldots, \mathrm{gr}_N^{HN}(E)).$$

For the existence and uniqueness of the Harder-Narasimhan filtration, see [HL6, Theorem 1.3.4], and for the modular interpretation of the Harder-Narasimhan strata, see [N1]. An algebraic stratification of this kind was first studied for vector bundles on a curve in [S3], and for arbitrary G-bundles on a curve in [B4]. Note that the sheaves $\mathrm{gr}_i^{HN}(E)$ must be pure of dimension decreasing in i.

What this theorem means is that for any family of coherent sheaves E over S, there is a finite stratification by locally closed subschemes $S = \bigcup_\alpha S_\alpha$ determined by the HN type of the fiber E_s. For each stratum S_α the family $E|_{S_\alpha}$ is determined by a map $S_\alpha \to \mathcal{C}oh(X)_{P_0}^{H-\mathrm{ss}} \times \cdots \times \mathcal{C}oh(X)_{P_N}^{H-\mathrm{ss}}$ classifying the family $\mathrm{gr}_\bullet^{HN}(E_s)$, as well as some linear extension data encoding how $E|_{S_\alpha}$ can be reconstructed from $\mathrm{gr}_\bullet^{HN}(E|_{S_\alpha})$. We refer to the stratification $\mathcal{C}oh(X)_P = \mathcal{C}oh(X)_P^{H-\mathrm{ss}} \cup \bigcup \mathcal{S}_\alpha$ as the Harder-Narasimhan-Shatz stratification. We regard Theorem 1.4 as a type of classification of coherent sheaves on X, as every coherent sheaf has been related in a controlled way to a point on some quasi-projective scheme.

Remark 1.5. Another important property of the Harder-Narasimhan filtration is the fact that there is a total ordering on HN types (See [N1, Section 2]) such that the closure of a stratum $\mathcal{S}_\alpha$ is contained in $\bigcup_{\beta \geq \alpha} \mathcal{S}_\beta$. We build this into our definition of Θ-stratifications in Definition 2.21 below.

1.1.1. *Canonical weights for the Harder-Narasimhan filtration.* A $\mathbb{Q}$-weighted filtration of $E \in \mathrm{Coh}(X)$ is a filtration $0 \subsetneq E_N \subsetneq \cdots \subsetneq E_0 = E$ along with a choice of rational weights $w_0 < w_1 < \cdots < w_p$. The second property of the HN filtration of an unstable bundle E in Theorem 1.4 suggests that we regard the HN filtration as a $\mathbb{Q}$-weighted filtration by choosing some $n \gg 0$ and assigning weight $w_i = \nu(\mathrm{gr}_i^{HN}(E))(n)$. For reasons which will be clear below, we will only be interested in filtrations up to simultaneously rescaling the weights $(E_\bullet, w_\bullet) \mapsto (E_\bullet, kw_\bullet)$ for some $k > 0$, so it suffices (by clearing denominators) to consider only $\mathbb{Z}$-weighted filtrations.

For the discussion that follows, we shall turn our attention to the coarser notion of slope stability. We say that E is *slope unstable* if there is a subsheaf $F \subset E$ such that $\nu(F) > \nu(E)$ to first order in $1/n$, by which we mean there is $0 < \epsilon \ll 1$ such that $\nu(F)(n) > \nu(E)(n) + \epsilon/n$ as $n \to \infty$. E is said to be slope semistable if it is not slope unstable, and Gieseker semistability implies slope semistability.

Remark 1.6. This notion is slightly stronger than the notion of slope semistability referred to as μ-stability in [S4, HL6]. In particular the formula (1) implies that a slope semistable sheaf in our sense must be pure, whereas a μ-semistable sheaf in [HL6, Section 1.6] is allowed to contain subsheaves of codimension ≥ 2. For a pure sheaf, though, our notion of slope semistability agrees with μ-stability as defined in [HL6, Definition 1.6.3] (compare the formula in [HL6, Example 1.6.5] with (1)).

Any slope unstable coherent sheaf E admits a unique Harder-Narasimhan filtration as in Theorem 1.4, the only difference being that in the context of slope semistability we make the stronger requirement that $\nu(\mathrm{gr}_i(E)) < \nu(\mathrm{gr}_{i+1}(E))$ must hold to first order in $1/n$. In fact the HN filtration with respect to slope stability is obtained by starting with the HN filtration of Theorem 1.4 and deleting the term E_i if $\nu(\mathrm{gr}_i(E_\bullet)) = \nu(\mathrm{gr}_{i-1}(E_\bullet))$ to first order in $1/n$. An analog of Theorem 1.4 holds, leading to a Harder-Narasimhan-Shatz stratification of $\mathrm{Coh}(X)_P$ with respect to μ-stability (See [HL6, Section 1.6] or [HL1] for a full treatment).

Let $E \in \mathrm{Coh}(X)_{\dim \le d}$, and define $D = \deg(E) \in \mathbb{Q}[n]$ and $R = \mathrm{rk}(E) \in \mathbb{Q}[n]$. For any $\mathbb{Z}$-weighted filtration $(E_\bullet, w_\bullet)$, define the numerical invariant

$$\mu(E_\bullet, w_\bullet) = \lim_{n\to\infty} \frac{\sum_i \left(\deg(\mathrm{gr}_i(E_\bullet))R - \mathrm{rk}(\mathrm{gr}_i(E_\bullet))D\right) w_i}{\sqrt{\sum w_i^2 \,\mathrm{rk}(\mathrm{gr}_i(E_\bullet))}} \in \mathbb{R} \cup \{\pm\infty\}. \tag{2}$$

One can show that a coherent sheaf is slope semistable if and only if for all $\mathbb{Q}$-weighted filtrations of E, $\mu(E_\bullet, w_\bullet) \le 0$. Furthermore, we have:

Theorem 1.7. [HL1, Z, GSZ] *Let $E \in \mathrm{Coh}(X)_{\dim \le d}$ be pure of dimension d which is unstable with respect to slope stability. Then among all $\mathbb{Z}$-weighted filtrations $(E_\bullet, w_\bullet)$ of E, there is a unique (up to rescaling weights) one which maximizes the numerical invariant $\mu(E_\bullet, w_\bullet)$. The filtration $E_\bullet$ is the Harder-Narasimhan filtration of E with respect to slope semistability, and $w_i = a_{d-1}(\mathrm{gr}_i(E_\bullet))/a_d(\mathrm{gr}_i(E_\bullet))$ up to overall scale, which is proportional to the leading coefficient in $\nu(\mathrm{gr}_i(E_\bullet))$ as $n \to \infty$.*

What is remarkable about Theorem 1.7 is that it admits a formulation which makes no reference to the structure of the abelian category $\mathrm{Coh}(X)$ but only to the geometry of the stack $\mathcal{X} = \mathcal{C}oh(X)$. Thus one obtains a framework, the theory of Θ-stability and Θ-stratifications, for generalizing this classification to other examples of moduli problems.

Remark 1.8. We expect that a suitable generalization of the theory of Θ-stability below should recover the notion of Gieseker semistability and should produce the full Harder-Narasimhan filtration for an unstable bundle. For instance, it is natural to regard the value of the numerical invariant μ as a function of n and attempt to maximize $\mu(E_\bullet, w_\bullet)$ as $n \to \infty$. It quickly becomes apparent that in order to obtain the full Harder-Narasimhan filtration in this way, one should regard the weights of the filtration w_i as functions of n as well. Such a theory has yet to be developed.

1.2. **Numerical invariants and the Harder-Narasimhan problem.** We now formulate a notion of Θ-stability and HN filtrations which generalizes our discussion of slope semistability of coherent sheaves. We focus on a special set of "test stacks," the most important of which is $\Theta := \mathrm{Spec}(\mathbb{C}[t])/\mathbb{G}_m$, where the coordinate t has weight -1 with respect to $\mathbb{G}_m$. We develop our theory of stability by considering maps out of Θ.

Example 1.9. Our motivation for considering maps out of Θ is that the groupoid of maps $f : \Theta \to \mathcal{C}oh(X)_P$ is equivalent to the groupoid whose objects consist of a coherent sheaf $f(1) = [E] \in \mathcal{C}oh(X)_P$ along with a $\mathbb{Z}$-weighted filtration of E [HL1]. Maps $*/\mathbb{G}_m \to \mathcal{C}oh(X)_P$ classify $\mathbb{Z}$-graded coherent sheaves, and the restriction of a map $f : \Theta \to \mathcal{C}oh(X)_P$ to $\{0\}/\mathbb{G}_m$ classifies $\mathrm{gr}_\bullet(E)$.

Thus for a general algebraic stack $\mathcal{X}$ we regard a map $f : \Theta \to \mathcal{X}$ as a "filtration" of the point $f(1) \in \mathcal{X}$. The groupoid of maps $B\mathbb{G}_m \to \mathcal{X}$ consists of points $p \in \mathcal{X}(\mathbb{C})$ along with a homomorphism of algebraic groups $\mathbb{G}_m \to \mathrm{Aut}(p)$, and isomorphisms are isomorphisms $p_1 \simeq p_2$ so that the induced map $\mathbb{G}_m \to \mathrm{Aut}(p_1) \to \mathrm{Aut}(p_2)$ is the given homomorphism for p_2. Given a filtration $f : \Theta \to \mathcal{X}$ the associated graded is the restriction $f_0 : \{0\}/\mathbb{G}_m \to \mathcal{X}$. In [HL1] we explicitly describe maps $\Theta \to \mathcal{X}$ for several additional example stacks $\mathcal{X}$:

Example 1.10. If $\mathcal{X} = X/G$ is a global quotient stack, then the groupoid of maps $f : \Theta \to \mathcal{X}$ is equivalent to the groupoid whose objects are pairs (λ, x) where $\lambda : \mathbb{G}_m \to G$ is a one-parameter subgroup and $x \in X(\mathbb{C})$ is a point such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists. Isomorphisms are generated by conjugation $(\lambda, x) \mapsto (g\lambda g^{-1}, gx)$ for $g \in G$ and $(\lambda, x) \mapsto (\lambda, px)$ for $p \in P_\lambda = \{g \in G | \lim_t \lambda(t) g \lambda(t^{-1}) \text{ exists}\}$.

Example 1.11. If $\mathcal{X}$ is the moduli of flat families of projective schemes along with a relatively ample invertible sheaf, a filtration $f : \Theta \to \mathcal{X}$ classifies a $\mathbb{G}_m$-equivariant family $X \to \mathbb{A}^1$. These are the "test-configurations" studied in the theory of K-stability.

The notion of Θ-stability will depend on a choice of rational cohomology classes in $H^2(\mathcal{X}; \mathbb{Q})$ and $H^4(\mathcal{X}; \mathbb{Q})$, where by cohomology we mean the cohomology of the analytification of $\mathcal{X}$ [N2]. Concretely if $\mathcal{X} = X/G$, then $H^*(\mathcal{X}; \mathbb{Q}) \simeq H^*_G(X; \mathbb{Q})$. In particular one can compute $H^*(\Theta; \mathbb{Q}) \simeq \mathbb{Q}[q]$ with $q \in H^2(\Theta; \mathbb{Q})$ the Chern class of $\mathcal{O}_\Theta\langle 1\rangle$, by which we denote the invertible sheaf on Θ corresponding to the free graded $\mathbb{C}[t]$-module generated by an element of degree -1.

Definition 1.12. A *numerical invariant* on a stack $\mathcal{X}$ is a function which assigns a real number $\mu(f)$ to any non-constant map $f : \Theta \to \mathcal{X}$ in the following way: given the data of

- a cohomology class $\ell \in H^2(\mathcal{X}; \mathbb{Q})$, and
- a class $b \in H^4(\mathcal{X}; \mathbb{Q})$ which is *positive definite* in the sense that for any $p \in \mathcal{X}(\mathbb{C})$ and non-trivial homomorphism $\mathbb{G}_m \to \mathrm{Aut}(p)$, corresponding to a map $\lambda : B\mathbb{G}_m \to \mathcal{X}$, the class $\lambda^* b \in H^4(B\mathbb{G}_m; \mathbb{Q}) \simeq \mathbb{Q}$ is positive,

the numerical invariant assigns $\mu(f) = f^*\ell/\sqrt{f^*b} \in \mathbb{R}$.

Remark 1.13. In [HL1] we study a more general notion of a numerical invariant – for instance we can allow μ to take values in a more general totally ordered set Γ such as $\mathbb{R} \cup \{\pm\infty\}$, or we can allow b to be positive semi-definite – but Definition 1.12 suffices for the purpose of exposition.

Lemma 1.14 ([HL1, Lemma 4.12])**.** *There are classes $\ell \in H^2(\mathcal{C}oh(X)_P)$ and $b \in H^4(\mathcal{C}oh(X)_P)$ such that the numerical invariant of Definition 1.12 is given by the formula (2).*

Note that the stack Θ has a ramified covering $z \mapsto z^n$ for every integer $n > 0$. This scales $H^2(\Theta)$ by n and scales $H^4(\Theta)$ by n^2, so a numerical invariant $\mu(f)$ is invariant under pre-composing $f : \Theta \to \mathcal{X}$ with a ramified covering of Θ. For instance, pre-composing a map $\Theta \to \mathcal{C}oh(X)_P$ with a ramified cover of Θ amounts to rescaling the weights of the corresponding weighted descending filtration. We can thus formulate

Definition 1.15. Let μ be a numerical invariant on a stack $\mathcal{X}$. A point $p \in \mathcal{X}$ is called *μ-unstable* if there is a map $f : \Theta \to \mathcal{X}$ with $f(1) \simeq p$ such that $\mu(f) > 0$. A *Harder-Narasimhan (HN) filtration* of an unstable point p is a map $f : \Theta \to \mathcal{X}$ along with an isomorphism $f(1) \simeq p$ which maximizes $\mu(f)$ among all such maps with $f(1) \simeq p$.

We refer to the question of existence and uniqueness, up to pre-composing f with a ramified cover of Θ, of a HN filtration for any unstable point in $\mathcal{X}$ as the "Harder-Narasimhan problem" associated to $\mathcal{X}$ and μ.

Remark 1.16. Jochen Heinloth has also independently considered the general notion of semi-stability in terms of maps out of Θ that we present here, and he has introduced a beautiful method for showing that the semi-stable locus is separated in certain situations [H1]. We will focus primarily on the *unstable* locus here, although we hope in the future to connect the two stories.

Example 1.17. When $\mathcal{X} = X/G$, Definition 1.15 provides an intrinsic formulation of the Hilbert-Mumford criterion for instability in geometric invariant theory by letting $l = c_1(L)$ for some G-ample invertible sheaf L and letting $b \in H^4(X/G; \mathbb{R})$ come from a positive definite invariant bilinear form $b \in \mathrm{Sym}^2((\mathfrak{t}^\vee_{\mathbb{R}})^W) \simeq H^4(*/G; \mathbb{R})$.

Example 1.18. In the context of Example 1.11, one can find classes (See [HL1, Section 4.2]) in H^2 and H^4 such that μ is the normalized Futaki invariant of [D2]. Thus Θ-stability is a formulation of Donaldson's "infinite dimensional GIT" in the context of algebraic geometry.

2. Θ-reductive stacks

Here we introduce a certain kind of moduli stack, which we refer to as a Θ-reductive stack. These moduli stacks are natural candidates to admit Θ-stratifications, as we shall discuss. By way of introduction, consider the main examples

Θ-reductive	**Not Θ-reductive**
X/G, where G is reductive and X is *affine*	X/G, where G is reductive and X is *projective*
$\mathcal{C}oh(X)$, where X is a projective scheme, as well as many other examples of stacks which classify objects of some abelian category $\mathcal{A}$ [2]	the stack of vector bundles, or even the stack of torsion free sheaves on a proper scheme X
Any of the stacks in Example 2.8 below	

Note that in both cases a stack on the right hand side naturally admits an open immersion into a stack of the kind in the left column. If X is a G-projective scheme with G-linearized very ample bundle L, then X/G is an open substack of $\operatorname{Spec}(\bigoplus_{n\geq 0} \Gamma(X, L^n))/\mathbb{G}_m \times G$, the quotient of the affine cone over X. In fact, a close reading of the original development of geometric invariant theory [MFK] reveals that many statements in projective GIT are proved by immediately reducing to a statement on the affine cone. We shall refer to such an open embedding informally as an *enlargement* of a moduli problem.

Recall that for two stacks $\mathcal{X}, \mathcal{Y}$, one can always define the mapping stack

$$\mathcal{M}ap(\mathcal{Y}, \mathcal{X}) : T \mapsto \{\text{maps } T \times \mathcal{Y} \to \mathcal{X}\}.$$

When $\mathcal{Y} = Y$ is a projective scheme and $\mathcal{X} = X$ is a quasi-projective scheme, one can use Hilbert schemes to construct a quasi-projective scheme representing the mapping stack explicitly. More generally, many situations have been established in which the stack $\mathcal{M}ap(\mathcal{Y}, \mathcal{X})$ is algebraic: when $\mathcal{Y}$ is a proper scheme, algebraic space, or Deligne-Mumford stack [O], and when $\mathcal{X}$ is a quasi-compact stack with affine diagonal [L]. In [HLP2] we develop with Anatoly Preygel a theory of "cohomological properness" for algebraic stacks, and we prove an algebraicity result for mapping stacks out of cohomologically proper stacks. This notion and some of its applications are closely related to the notion of "coherent completeness" in [AHR]. The stack Θ is cohomologically proper, and we have:

Theorem 2.1. [HLP2] *Let $\mathcal{X}$ be a locally finite type algebraic stack with a quasi-affine diagonal. Then $\mathcal{M}ap(\Theta, \mathcal{X})$ is a locally finite type algebraic stack, and the evaluation map $\operatorname{ev}_1 : \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{X}$ which restricts a map to the open subset $* \simeq (\mathbb{A}^1 - \{0\})/\mathbb{G}_m$ is relatively representable by locally finite type algebraic spaces.*

Remark 2.2. If $\mathcal{X}$ is a derived stack, then [HLP2] establishes that the derived mapping stack $\mathcal{M}ap(\Theta, \mathcal{X})$ is algebraic as well. Even when $\mathcal{X}$ is classical, we will have to regard $\mathcal{X}$ and $\mathcal{M}ap(\Theta, \mathcal{X})$ as derived stacks for some applications of Θ-stratifications (see §3.3.1 below).

Definition 2.3. Let $\mathcal{X}$ be a locally finite type algebraic stack with a quasi-affine diagonal. Then $\mathcal{X}$ is Θ-*reductive* if the map $\operatorname{ev}_1 : \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{X}$ satisfies the valuative criterion for properness. We say that $\mathcal{X}$ is *strongly Θ-reductive* if for any finite type ring R and any R-point of $\mathcal{X}$, the connected components of the fibers of ev_1 are proper over $\operatorname{Spec}(R)$.

Example 2.4. In [HL1, Section 1], we show that $\mathcal{M}ap(\Theta, X/G) = \bigsqcup Y_\lambda/P_\lambda$, where the disjoint union is over conjugacy classes of one parameter subgroups $\lambda : \mathbb{G}_m \to G$ and Y_λ is the disjoint union of Bialynicki-Birula strata associated to λ (compare Example 1.10). When X is affine and G is reductive, the map ev_1 factors as the closed immersion $Y_\lambda/P_\lambda \hookrightarrow X/P_\lambda$ followed by the proper fibration $X/P_\lambda \to X/G$ with fiber G/P_λ. Therefore ev_1 is proper on every connected component, so X/G is strongly Θ-reductive in this case.

Example 2.5. Let $\mathrm{Spec}(R) \to \mathcal{X} = \mathcal{C}oh(X)_P$ classify a $\mathrm{Spec}(R)$-flat family of coherent sheaves F on $X \times \mathrm{Spec}(R)$. For any R-scheme T, the T-points of the algebraic space $Y = \mathcal{M}ap(\Theta, \mathcal{X}) \times_{\mathcal{X}} \mathrm{Spec}(R)$ classify $\mathbb{Z}$-weighted filtrations of the coherent sheaf $F|_{X \times T}$ whose associated graded is flat over T. Thus the connected components of Y can be identified with generalized flag schemes of F over $\mathrm{Spec}(R)$, which are proper over $\mathrm{Spec}(R)$.

On the other hand, when $\mathcal{X}$ is the open substack of $\mathcal{C}oh(X)_P$ parameterizing locally free sheaves, then the corresponding flag scheme for $E \in \mathrm{Coh}(X)$ parameterizes flat families of filtrations whose associated graded is also locally free. This defines an open and non-closed subscheme of the flag scheme of E regarded as a coherent sheaf, which is therefore not proper. Hence the moduli of locally free sheaves is not Θ-reductive.

Given a point $p : \mathrm{Spec}(R) \to \mathcal{X}$ we regard the connected components of the locally finite type algebraic space $\mathcal{M}ap(\Theta, \mathcal{X})_p := \mathcal{M}ap(\Theta, \mathcal{X}) \times_{\mathrm{ev}_1, \mathcal{X}, p} \mathrm{Spec}(R)$ as "flag spaces" (they are not necessarily varieties) for the moduli problem $\mathcal{X}$, following the previous example. Thus a stack is strongly Θ-reductive if and only if it is Θ-reductive and its flag spaces are quasi-compact.

Example 2.6. When $\mathcal{X}$ parametrizes objects in an abelian category $\mathcal{A} \subset \mathrm{DCoh}(X)$ for a projective scheme X, the locally finite type algebraic spaces $\mathcal{M}ap(\Theta, \mathcal{X})_p$ satisfy the valuative criterion for properness [HL1, Section 4], generalizing the example of $\mathcal{C}oh(X)_P$. This should hold for stacks classifying objects in more general abelian categories as well.

Given a Θ-reductive moduli problem, we can produce new Θ-reductive moduli problems via the following:

Lemma 2.7 ([HL1, Section 3])**.** *Let $\mathcal{Y} \to \mathcal{X}$ be a representable affine morphism. If $\mathcal{X}$ is a Θ-reductive or strongly Θ-reductive stack, then so is $\mathcal{Y}$.*

Example 2.8. There are many natural moduli problems which are affine over the stack $\mathcal{C}oh(X)$ for a projective scheme X. For instance one can consider the moduli stack $\mathcal{Y}$ of flat families of coherent algebras on X, along with the map $\mathcal{Y} \to \mathcal{C}oh(X)$ which forgets the algebra structure. The fiber over a given $[F] \in \mathcal{C}oh(X)$ consists of an element of the vector space $\mathrm{Hom}_X(F \otimes F, F)$, corresponding to the multiplication rule, satisfying a finite set of polynomial equations, corresponding to the associativity and identity axioms. Other examples of stacks which are affine over $\mathcal{C}oh(X)$ include the stack of coherent modules over a fixed quasi-coherent sheaf of algebras on X. This includes as a special case the stack of (not necessarily semistable) Higgs bundles, which for smooth X can be regarded as the stack of coherent sheaves of modules over the algebra $\mathrm{Sym}_X(TX)$.

2.0.1. *The main advantage of Θ-reductive stacks.* The primary importance of Definition 2.3 is that the existence and uniqueness question for Harder-Narasimhan filtrations is well behaved for points in a Θ-reductive stack. Below we will introduce a formal notion of numerical invariant μ on the stack $\mathcal{X}$, as well as what it means for a numerical invariant to be *bounded*. Before introducing the necessary machinery, however, let us state the main result that we are heading towards:

Proposition 2.9. *Let $\mathcal{X}$ be a stack which is Θ-reductive, and let $l \in H^2(\mathcal{X}; \mathbb{Q})$ and $b \in H^4(\mathcal{X}; \mathbb{Q})$ with b positive definite. Assume that the numerical invariant $\mu(f) = f^*l/f^*b$ is bounded (Definition 2.19). Then any unstable point $p \in \mathcal{X}$ has a unique Harder-Narasimhan filtration: i.e. there is a map*

$f : \Theta \to \mathfrak{X}$ with an isomorphism $f(1) \simeq p$ which maximizes $\mu(f)$, and this pair is unique up to ramified coverings $\Theta \to \Theta$.

When $\mathfrak{X} = X/G$ is a global quotient stack, then boundedness holds automatically for any numerical invariant. So, if X is affine and G is reductive, this recovers Kempf's theorem on the existence of canonical destabilizing one parameter subgroups in GIT [K1].

2.1. **The degeneration space: a generalization of the spherical building of a group.** In order to explain the boundedness hypothesis and the proof of Proposition 2.9, we introduce the notion of the degeneration space, a topological space associated to any point p in an algebraic stack $\mathfrak{X}$. We shall see that any non-constant map $f : \Theta \to \mathfrak{X}$ with an isomorphism $f(1) \simeq p$, up to ramified coverings of Θ, corresponds to a point on the degeneration space, and it is therefore a useful tool for analyzing the Harder-Narasimhan problem.

Let $\Theta^n = \mathbb{A}^n/\mathbb{G}_m^n$ and let $\mathbf{1} = (1, \ldots, 1)$ be the dense point and $\mathbf{0} = (0, \ldots, 0)$ be the origin. For any stack $\mathfrak{X}$ we call a map $f : \Theta^n \to \mathfrak{X}$ *non-degenerate* if the map on automorphism groups $\mathbb{G}_m^n \to \operatorname{Aut} f(0)$ has finite kernel. For any $p \in \mathfrak{X}(\mathbb{C})$ consider the set[3]

$$\operatorname{Deg}(\mathfrak{X}, p)_n := \{\text{non-degenerate maps } f : \Theta^n \to \mathfrak{X} \text{ with an isomorphism } f(1) \simeq p\}.$$

We identify a full-rank $n \times k$ matrix $[a_{ij}]$ with nonnegative integer coefficients with a non-degenerate map $\phi : \Theta_k \to \Theta_n$ with an isomorphism $\phi(\mathbf{1}) \simeq \mathbf{1}$ given in coordinates by $z_i \mapsto z_1^{a_{1i}} \cdots z_n^{a_{ni}}$, and this assignment is in fact a bijection between such matrices and maps of this kind. For any such map $\phi : \Theta^k \to \Theta^n$, pre-composition gives a restriction map $\operatorname{Deg}(\mathfrak{X}, p)_n \to \operatorname{Deg}(\mathfrak{X}, p)_k$, and this data fits together into a combinatorial object referred to as a *formal fan* in [HL1]. In addition we can associate an injective linear map $\mathbb{R}^k_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ to such a $\phi = [a_{ij}]$, and this induces a closed embedding

$$\phi_* : \Delta^{k-1} = (\mathbb{R}^k_{\geq 0} \setminus \{0\})/\mathbb{R}^\times_{>0} \hookrightarrow \Delta^{n-1} = (\mathbb{R}^n_{\geq 0} \setminus \{0\})/\mathbb{R}^\times_{>0}$$

Definition 2.10. We define the *degeneration space* $\mathcal{D}eg(\mathfrak{X}, p)$ of the point $p \in \mathfrak{X}(\mathbb{C})$ to be the union of simplices Δ^{n-1}_f for all $f \in \operatorname{Deg}(\mathfrak{X}, p)_n$, glued to each other along the closed embeddings $\phi_* : \Delta^{k-1}_{\phi^* f} \hookrightarrow \Delta^{n-1}_f$ for any $f \in \operatorname{Deg}(\mathfrak{X}, p)_n$ and any map $\phi : \Theta^k \to \Theta^n$ corresponding to a full-rank $n \times k$ matrix with nonnegative integer entries.

This construction is quite similar to the geometric realization of a semi-simplicial set. In addition to being glued along faces, however, simplices can be glued to each other along any closed linear sub-simplex with rational vertices. This potentially leads to non-Hausdorff topologies, but that does not happen in the simplest examples (See [HL1, Section 2.2] for a more detailed discussion of these examples):

Example 2.11. Let T be a torus and N its co-character lattice. Then $\operatorname{Deg}(*/T, *)_n$ is equivalent to the set of linearly independent n-tuples $(v_1 \ldots v_n) \in N^n$. If X is a toric variety under the action of T, Σ is the fan in $N_\mathbb{R}$ describing X, and $p \in X$ is a generic point, then $\operatorname{Deg}(X/T, p)_n \subset \operatorname{Deg}(*/T, *)_n$ is the set of linearly independent n-tuples $(v_1, \ldots, v_n) \in N^n$ which are contained in some cone of Σ. The degeneration space $\mathcal{D}eg(X/T, p)$ is homeomorphic to $(\operatorname{Supp}(\Sigma) \setminus \{0\})/\mathbb{R}^\times_{>0}$, where Supp denotes the support [HL1, Proposition 2.20]. In fact, it is possible to reconstruct the original fan Σ from the formal subfan $\operatorname{Deg}(X/T, p)_\bullet \subset \operatorname{Deg}(*/T, *)_\bullet$: it suffices to recover the maximal cones of Σ, which are the maximal cones among all $\sigma \subset N_\mathbb{R}$ for which any linearly independent n-tuple in σ lies in $\operatorname{Deg}(X/T, p)_n$.

[3] A priori this is a groupoid, but it is equivalent to a set because any automorphism of a map $f : \Theta^n \to \mathfrak{X}$ is uniquely determined by its restriction to the dense open substack $\{\mathbf{1}\}$. This follows from the fact that the projection from the inertia stack of $\mathfrak{X}$ to $\mathfrak{X}$ is a separated map.

Example 2.12. For a general group, $\mathrm{Deg}(*/G,*)_n$ classifies group homomorphisms with finite kernel $\phi : \mathbb{G}_m^n \to G$ up to conjugation by an element of

$$G_\phi = \{g \in G | \lim_{t\to 0} \phi(t^{a_1},\ldots,t^{a_n}) g \phi(t^{a_1},\ldots,t^{a_n})^{-1} \text{ exists } \forall a_i \geq 0\}.$$

The correspondence assigns $\phi : \mathbb{G}_m^n \to G$ to the composition $f_\phi : \Theta^n \to */\mathbb{G}_m^n \to */G$. This in turn corresponds to a rational simplex $\Delta_\phi \to \mathcal{D}eg(*/G,*)$, which is a priori just a continuous map, but it is a closed embedding in this case [HL1, Proposition 2.24]. When G is semisimple, we can use simplices of this form to construct a homeomorphism $\mathcal{D}eg(BG,*) \simeq \mathrm{Sph}(G)$ [HL1, Proposition 2.22] with the spherical building of G, i.e. the simplicial complex associated to the partially ordered set of parabolic subgroups of G ordered under inclusion.

Example 2.13. More generally, if X is a scheme with a G action and $p \in X$, then the canonical map $\mathcal{D}eg(X/G,p) \to \mathcal{D}eg(*/G,*)$ is a closed embedding [HL1, Proposition 2.24]. More explicitly, for any homomorphism with finite kernel $\phi : \mathbb{G}_m^n \to G$ we associate a rational simplex $\Delta_\phi \hookrightarrow \mathcal{D}eg(*/G,*)$. Let $\mathbb{G}_m^n$ act diagonally on $\mathbb{A}^n \times X$ via the homomorphism ϕ and the G action on X, and let $X_{\phi,p}$ be the normalization of the $\mathbb{G}_m^n$ orbit closure of $(1,\ldots,1,p) \in \mathbb{A}^n \times X$. Then $X_{\phi,p}$ is a normal toric variety with a toric map to $\mathbb{A}^n$, so the support of the fan $\Sigma_{X_{\phi,p}}$ of $X_{\phi,p}$ is canonically a union of rational polyhedral cones in $(\mathbb{R}_{\geq 0})^n$. The closed subspace $\mathcal{D}eg(X/G,p) \hookrightarrow \mathcal{D}eg(*/G,*)$ is determined uniquely by the fact that

$$\mathcal{D}eg(X/G,p) \cap \Delta_\phi = (\mathrm{Supp}(\Sigma_{X_{\phi,p}}) - \{0\})/\mathbb{R}_{>0}^\times \subset \Delta_\phi$$

for each rational simplex $\Delta_\phi \subset \mathcal{D}eg(*/G,*)$.

2.1.1. *The space of components.* Instead of considering the fiber of $\mathrm{ev}_1 : \mathcal{M}ap(\Theta^n, \mathcal{X}) \to \mathcal{X}$ over a point as a set, for any map $\varphi : T \to \mathcal{X}$ we can consider the fiber product $\mathcal{M}ap(\Theta^n,\mathcal{X}) \times_{\mathcal{X}} T$ as an algebraic space over T.

Definition 2.14. Given $\varphi : T \to \mathcal{X}$, we define the set $\mathrm{Comp}(\mathcal{X},\varphi)_n \subset \pi_0(\mathcal{M}ap(\Theta^n,\mathcal{X}) \times_{\mathcal{X}} T)$ to consist of those connected components which contain at least one non-degenerate point. The *component space* $\mathcal{C}omp(\mathcal{X},\varphi)$ is the union of standard $(n-1)$-simplices $\Delta_{[f]}^{n-1}$, one copy for each $[f] \in \mathrm{Comp}(\mathcal{X},\varphi)_n$, glued in the same manner as in the construction of $\mathcal{D}eg(\mathcal{X},p)$ in Definition 2.10

Any non-degenerate map $f : \Theta^n \to \mathcal{X}$ with an isomorphism $f(1) \simeq p$ also defines an element $[f] \in \mathrm{Comp}(\mathcal{X},\mathrm{id}_{\mathcal{X}})_n$, and the identity maps $\Delta_f^{n-1} \subset \mathcal{D}eg(\mathcal{X},p) \to \Delta_{[f]}^{n-1} \to \mathcal{C}omp(\mathcal{X},\mathrm{id}_{\mathcal{X}})$ glue for different $f \in \mathcal{D}eg(\mathcal{X},p)_n$ to give a continuous map $\mathcal{D}eg(\mathcal{X},p) \to \mathcal{C}omp(\mathcal{X},\mathrm{id}_{\mathcal{X}})$. However, the component space tends to be much smaller that the degeneration space of any particular point.

Example 2.15. When $\mathcal{X} = X/G$ is a quotient of a G-quasi-projective scheme, then for a maximal torus $T \subset G$, the map $\mathrm{Comp}(\mathcal{X},\mathrm{id}_{\mathcal{X}})_n \to \mathrm{Comp}(X/T,\mathrm{id}_{X/T})_n$ is surjective for all n, hence $\mathcal{C}omp(\mathcal{X},\mathrm{id}_{\mathcal{X}}) \to \mathcal{C}omp(X/T,\mathrm{id}_{X/T})$ is surjective. One can show by explicit construction that $\mathcal{C}omp(X/T,\mathrm{id}_{X/T})$ can be covered by finitely many simplices, hence so can $\mathcal{C}omp(\mathcal{X},\mathrm{id}_{\mathcal{X}})$.

2.2. **A generalization of toric geometry?**

For a quotient stack $\mathcal{X} = X/T$, where X is a toric variety with generic point $p \in X$, we have seen in Example 2.11 that $\mathrm{Deg}(\mathcal{X},p)_\bullet$ remembers the information of the fan in $N_{\mathbb{R}}$ defining X, and hence remembers enough information to reconstruct X itself. For a reductive group G, define a *convex rational polytope* $\sigma \subset \mathcal{D}eg(*/G,*)$ to be a closed subset such that for any homomorphism with finite kernel $\phi : \mathbb{G}_m^n \to G$, the intersection of σ with the rational simplex Δ_ϕ as discussed in Example 2.12 is a convex polytope with rational vertices.

Now let X be a normal projective-over-affine scheme with an action of a reductive group G and a $p \in X$ such that $G \cdot p \subset X$ is dense. We have seen (Example 2.13) that $\mathcal{D}eg(X/G,p) \subset \mathcal{D}eg(*/G,*)$ is a closed subspace whose intersection with any rational simplex $\Delta_\phi \subset \mathcal{D}eg(*/G,*)$ is $(\mathrm{Supp}(\Sigma_{X_{\phi,p}}) - \{0\})/\mathbb{R}_{>0}^\times$ for some toric variety $X_{\phi,p}$ with a toric map to $\mathbb{A}^n$. The fan of $X_{\phi,p}$

therefore specifies a decomposition of $\Delta_\phi \cap \mathcal{D}eg(X/G,p)$ into a union of rational polytopes. One can define a collection of convex rational polytopes $\Sigma_X = \{\sigma \subset \mathcal{D}eg(*/G,*)\}$ characterized by the properties that

1. the intersection of any two polytopes in Σ_X is another polytope in Σ_X,
2. $\bigcup_{\sigma\in\Sigma_X} \sigma = \mathcal{D}eg(X/G,p) \subset \mathcal{D}eg(*/G,*)$, and
3. the intersections $\sigma \cap \Delta_\phi$ for $\sigma \in \Sigma_X$ are the polytopes in $\Delta_\phi \cap \mathcal{D}eg(X/G,p)$ induced by the fan of $X_{\phi,p}$.

Question 2.16. *To what extent does the resulting collection Σ_X of rational polytopes in $\mathcal{D}eg(*/G,*)$ remember the geometry of X?*

It can not be that Σ_X uniquely determines X: if $U \subset G$ is any unipotent subgroup group, then for $X = G/U$ one can check that $\mathcal{D}eg(X/G,p) \simeq \mathcal{D}eg(*/U,*) = \emptyset$ and hence $\Sigma_X = \emptyset$.

Let us formulate a more concrete question, assuming in addition that X is smooth: The stabilizer group $\mathrm{Stab}(p)$ acts on the space $\mathcal{D}eg(X/G,p)$. We use the notation $C(Y)$ to denote the cone over a topological space Y, and observe that for a rational polytope $\sigma \subset \Delta_\phi$, the cone $C(\sigma)$ is canonically a rational polyhedral cone in $(\mathbb{R}_{\geq 0})^n$, so we can consider polynomial functions on $C(\sigma)$. We define the ring of invariant piecewise polynomial functions on Σ_X,

$$PP(\Sigma_X)^{\mathrm{Stab}(p)} := \left\{ \begin{array}{c} \mathrm{Stab}(p)\text{-invariant continuous} \\ f : C(\mathcal{D}eg(X/G,p)) \to \mathbb{R} \end{array} \middle| \begin{array}{c} \forall \phi : \mathbb{G}_m^n \to G, \forall \sigma \in \Sigma_X, \\ f|_{C(\Delta_\phi \cap \sigma)} \text{ is a polynomial} \end{array} \right\}$$

In [HL1, Lemma 2.27] we construct a homomorphism $H^{even}_G(X) \to PP(\Sigma_X)^{\mathrm{Stab}(p)}$, where H^{2n} maps to functions which are locally homogeneous polynomials of degree n.

Question 2.17. *Under what conditions is the map $H^{even}_G(X) \to PP(\Sigma_X)^{\mathrm{Stab}(p)}$ an isomorphism?*

For smooth toric varieties, this is known to be an equivalence, and it remains an equivalence for all toric varieties after replacing singular cohomology with a suitable alternative cohomology theory [P1]. We have also verified that this map is an equivalence when $X = */G$ itself, and when $X = G/P$ is a generalized flag manifold.

2.3. The proof of Proposition 2.9.

We can now finish explaining the terminology of Proposition 2.9 and its proof.

Lemma 2.18. [HL1, Lemma 2.27] *Given a numerical invariant, the function $\mu = f^*\ell/\sqrt{f^*b}$ extends to a continuous function on $\mathcal{C}omp(\mathcal{X}, \mathrm{id}_\mathcal{X})$ and thus a continuous function on $\mathcal{D}eg(\mathcal{X},p)$ via the continuous map $\mathcal{D}eg(\mathcal{X},p) \to \mathcal{C}omp(\mathcal{X}, \mathrm{id}_\mathcal{X})$ for any $p \in \mathcal{X}$.*

Proof idea. Any rational simplex $\Delta^{n-1}_{[f]}$ corresponds to *some* non-degenerate map $f : \Theta^n \to \mathcal{X}$. The classes $f^*\ell \in H^*(\Theta^n;\mathbb{Q})$ and $f^*b \in H^4(\Theta^n;\mathbb{Q})$ can be identified with a linear and positive definite quadratic form on $(\mathbb{R}_{\geq 0})^n$ respectively. The function μ restricted to $\Delta^{n-1}_{[f]}$ is simply the quotient $f^*\ell/\sqrt{f^*b}$, which descends to a continuous function on $(\mathbb{R}^n_{\geq 0} - \{0\})/\mathbb{R}^\times_{>0}$. ☐

The proof of existence is essentially independent of the uniqueness proof. In light of the previous lemma, the existence of a maximizer for μ is an immediate consequence of the following property, which holds for *any* numerical invariant on a global quotient stack by Example 2.15:

Definition 2.19. Let μ be a numerical invariant on $\mathcal{X}$. We say that μ is *bounded* if for all $p \in \mathcal{X}(\mathbb{C})$ there is a finite collection of rational simplices in $\sigma_1, \ldots, \sigma_N \subset \mathcal{C}omp(\mathcal{X},p)$ such that for any point $x \in \mathcal{C}omp(\mathcal{X},p)$ there is a point $x' \in \sigma_i$ such that $\mu(x') \geq \mu(x)$.

The uniqueness part of Proposition 2.9, in contrast, uses the fact that $\mathcal{X}$ is Θ-reductive in an essential way. Given an algebraic stack $\mathcal{X}$ and $p \in \mathcal{X}(\mathbb{C})$, we say that two rational points in $\mathcal{D}eg(\mathcal{X},p)$ corresponding to $f, g \in \mathrm{Deg}(\mathcal{X},p)_1$ are *antipodal* if there is a group homomorphism $\mathbb{G}_m \to \mathrm{Aut}(p)$

corresponding to a map $\lambda : B\mathbb{G}_m \to \mathcal{X}$, such that both $f, g : \Theta \to \mathcal{X}$ can be factored as $f \simeq \lambda \circ \pi$ and $g \simeq \lambda^{-1} \circ \pi$, where $\pi : \Theta \to B\mathbb{G}_m$ is the canonical projection.

Lemma 2.20. [HL1, Proposition 2.47] *Let $\mathcal{X}$ be a Θ-reductive stack, and let $f, g \in \mathcal{D}eg(\mathcal{X}, p)$ be two distinct rational points which are not antipodal. Then there is a unique rational ray in $\mathcal{D}eg(\mathcal{X}, p)$ connecting f and g.*

Proof idea. A rational 1-simplex is represented by a map $\mathbb{A}^2/\mathbb{G}_m^2 = \Theta \times \Theta \to \mathcal{X}$, which is equivalent to a map $\Theta \to \mathcal{M}ap(\Theta, \mathcal{X})$. One can reduce the claim to the existence and uniqueness of a dotted arrow making the following diagram commute:

$$\begin{array}{ccc} \Theta - \{0\} \simeq * & \xrightarrow{\;f\;} & \mathcal{M}ap(\Theta, \mathcal{X}) \\ \big\downarrow 1 & \nearrow & \big\downarrow \mathrm{ev}_1 \\ \Theta & \xrightarrow{\;g\;} & \mathcal{X} \end{array} ,$$

and such that the resulting map $\gamma : \mathbb{A}^2/\mathbb{G}_m^2 \to \mathcal{X}$ is non-degenerate. The left vertical map is the open inclusion of the complement of a codimension 1 closed point, so this lifting property essentially follows from the valuative criterion for the map ev_1. The proof that γ is non-degenerate is not classical: it uses the Tannakian formalism [BHL] and the main structure theorem for equivariant derived categories established in [HL2] (See Theorem 3.9 for a more general version of this structure theorem) to show that if the homomorphism $\mathbb{G}_m^2 \to \mathrm{Aut}(\gamma(0,0))$ has a positive dimensional kernel then f and g must be either the same or antipodal. □

This lemma implies the uniqueness of a Harder-Narasimhan filtration for $p \in \mathcal{X}(\mathbb{C})$, assuming its existence. The sign of $\mu(f)$ must differ for antipodal points, so Lemma 2.20 implies that for two rational points $f, g \in \mathcal{D}eg(\mathcal{X}, p)$ such that $\mu(f), \mu(g) > 0$, there is a unique rational 1-simplex connecting f and g. Now we simply observe that the restriction of μ to this one simplex is the quotient of a linear form by the square root of a positive definite quadratic form. Such a function is strictly convex upward and therefore has a unique maximum on that interval.

2.4. **Θ-stratifications.** The Harder-Narasimhan problem is only the first step towards a classification theory for a stack $\mathcal{X}$ modelled after that of $\mathcal{C}oh(X)_P$. Ideally the locus of Θ-semistable points defines an open substack $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$, and the Harder-Narasimhan filtration varies upper-semi-continuously in families. Fix a totally ordered set Γ.

Definition 2.21. A Θ-*stratum* in an algebraic stack $\mathcal{X}$ is a closed substack which is identified with a union of connected component of $\mathcal{M}ap(\Theta, \mathcal{X})$ under the evaluation map $\mathrm{ev}_1 : \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{X}$. A Θ-*stratification* of $\mathcal{X}$ is a collection of open substacks $\mathcal{X}_{\leq \alpha} \subset \mathcal{X}$ for $\alpha \in \Gamma$ such that $\mathcal{X}_{\leq \alpha'} \subset \mathcal{X}_{\leq \alpha}$ for $\alpha' < \alpha$ and the closed subset $\mathcal{X}_{\leq \alpha} \setminus \bigcup_{\alpha' < \alpha} \mathcal{X}_{\leq \alpha'}$ is a Θ-stratum in $\mathcal{X}_{\leq \alpha}$.

The *center* of $\mathcal{S}_\alpha$ is defined to be the open substack $\mathcal{Z}_\alpha^{\mathrm{ss}} \subset \mathcal{M}ap(*/\mathbb{G}_m, \mathcal{X})$ which is the preimage of $\mathcal{S}_\alpha \subset \mathcal{M}ap(\Theta, \mathcal{X})$ under the map $\sigma : \mathcal{M}ap(*/\mathbb{G}_m, \mathcal{X}) \to \mathcal{M}ap(\Theta, \mathcal{X})$ induced by the projection $\Theta \to */\mathbb{G}_m$. The map $\mathrm{ev}_0 : \mathcal{S}_\alpha \subset \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{M}ap(*/\mathbb{G}_m, \mathcal{X})$ factors through $\mathcal{Z}_\alpha^{\mathrm{ss}}$ as well.

Remark 2.22. It is often convenient to assume that each $\mathcal{S}_\alpha$ is connected, and this can always be arranged by refining the indexing set Γ.

Consider a stack $\mathcal{X}$ and a subset consisting of rational points $S \subset \mathcal{C}omp(\mathcal{X}, \mathrm{id}_\mathcal{X})$ along with a map $\mu : S \to \Gamma$. We define the *stability function* for $p \in \mathcal{X}$:

$$M^\mu(p) := \sup \{ \mu([f]) \,|\, f : \Theta \to \mathcal{X} \text{ with } [f] \in S \text{ and } f(1) \simeq p \} .$$

The abuse of terminology here is justified, because typically $\Gamma = \mathbb{R}$, and the map μ will be a numerical invariant, restricted to the subset S. Given this set-up, we shall denote by $\mathcal{S} \subset \mathcal{M}ap(\Theta, \mathcal{X})$

the union of the connected components corresponding to S. The following is a simplified version of [HL1, Theorem 1.19].

Theorem 2.23. *Let $\mathcal{X}$ be a strongly Θ-reductive stack with $S \subset \mathcal{C}omp(\mathcal{X}, \mathrm{id}_{\mathcal{X}})$ and $\mu : S \to \Gamma$ as above. Assume that*

(1) ***HN-property:*** *For all finite type points $p \in \mathcal{X}(k)$, $\mathcal{S} \times_{\mathcal{X}} p$ is either empty or contains a unique point $f : \Theta \to \mathcal{X}$ along with $f(1) \simeq p$, up to ramified covers of Θ, such that $\mu([f]) = M^\mu(p)$, the* Harder-Narasimhan (HN) filtration.
(2) ***Local finiteness:*** *For any map $\varphi : T \to \mathcal{X}$, with T a finite type scheme, there there is a finite subset $\{s_1, \ldots, s_n\} \subset S$ such that the HN filtration of any finite type point in T corresponds to a point in s_i for some i.*
(3) ***Semi-continuity:*** *If $f : \Theta \to \mathcal{X}$ is a HN filtration for $f(1)$, then $M^\mu(f(0)) \le \mu([f])$.*

Then the subsets

$$|\mathcal{X}_{\le \alpha}| := \{p \in \mathcal{X} | M^\mu(p) \le \alpha\} \subset |\mathcal{X}|$$

are open, and the corresponding open substacks $\mathcal{X}_{\le\alpha}$ form a Θ-stratification of $\mathcal{X}$.

One non-trivial consequence of the theory of numerical invariants is that the semi-continuity property follows automatically from the first two in the following special case: Given a numerical invariant $\mu : \mathcal{C}omp(\mathcal{X}, \mathrm{id}_{\mathcal{X}}) \to \mathbb{R}$, we define the subset $S \subset \mathrm{Comp}(\mathcal{X}, \mathrm{id}_{\mathcal{X}})_1$ to consist of those connected components of $\mathcal{M}ap(\Theta, \mathcal{X})$ which contain a point f for which $\mu(f) = M^\mu(f(1))$. We restrict μ to define a function $S \to \Gamma = \mathbb{R}$.

Theorem 2.24. [HL1, Corollary 3.20] *Let μ be a numerical invariant on a strongly Θ-reductive stack. Assume that μ satisfies the HN property, and that for any finite type T with a map $T \to \mathcal{X}$, the HN filtrations of finite type points of T correspond to finitely many points of $\mathcal{C}omp(\mathcal{X}, T)$. Then the subsets*

$$|\mathcal{X}_{\le \alpha}| := \{p \in \mathcal{X} | M^\mu(p) \le \alpha\} \subset |\mathcal{X}|$$

are open, and the corresponding open substacks of $\mathcal{X}_{\le\alpha}$ form a Θ-stratification of $\mathcal{X}$.

We use this in [HL1] to construct a Θ-stratification of the stack of objects an abelian subcategory $\mathcal{A} \subset D^b(X)$ for a $K3$ surface X, corresponding to a Bridgeland stability condition on $D^b(X)$. This is the first example of a Θ-stratification which is not known to admit a description as a certain limit of Kempf-Ness stratifications of larger and larger GIT problems as in [Z, H2].

2.5. **The search for Θ-reductive stacks.** At the moment, the theory of Θ-stability is still in a nascent stage, with much yet to be developed. One of the most intriguing problems is to find new examples of Θ-reductive enlargements of classically studied moduli problems. For instance, we are not aware of a Θ-reductive enlargement of the moduli of principal G-bundles on a smooth curve C, except in the case where $G = \mathrm{GL}_n$ and one has the Θ-reductive enlargement $\mathcal{C}oh(X)$.

Another question, related to the theory of K-stability, is whether the moduli stack $\mathcal{X}$ of flat families of projective schemes along with a relatively ample invertible sheaf is Θ-reductive. Theorem 2.1 tells us that for abstract reasons there exists a locally finite type algebraic space $\mathcal{M}ap(\Theta, \mathcal{X})_{[(Y,L)]}$ classifying test configurations for a fixed polarized scheme $(Y, L) \in \mathcal{X}$, and it is a foundational open question whether or not the connected components of this space are proper.

3. Derived Kirwan surjectivity

We now discuss applications of Θ-stratifications. The original applications were for computing the topology of the stack $\mathcal{X}$, in the case when $\mathcal{X}$ is the moduli stack of vector bundles on a curve [AB] and the stratification is the Harder-Narasimhan-Shatz stratification, or when $\mathcal{X} = X/G$ is a quotient of a smooth projective variety by a reductive group and the stratification is the Kempf-Ness stratification [K3]. In general for a smooth stack $\mathcal{X}$, the mapping stack $\mathcal{M}ap(\Theta, \mathcal{X})$ and hence the strata $\mathcal{S}_\alpha$ will also be smooth [HL1, Section 1.2]. In our language, the theorem states

Theorem 3.1 (Kirwan surjectivity)**.** *If $\mathcal{X}$ is a* smooth *locally finite type algebraic stack with a Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_\alpha \mathcal{S}_\alpha$, then there is an isomorphism*

$$H^*(\mathcal{X};\mathbb{Q}) \simeq H^*(\mathcal{X}^{\mathrm{ss}};\mathbb{Q}) \oplus \bigoplus_\alpha H^{*-c_\alpha}(\mathcal{Z}_\alpha;\mathbb{Q})$$

where $c_\alpha = \mathrm{codim}(\mathcal{S}_\alpha, \mathcal{X})$, assuming $c_\alpha \to \infty$ for large α.

The isomorphism in this theorem is not canonical: the theorem amounts to the fact that the canonical restriction map $H^*(\mathcal{X};\mathbb{Q}) \to H^*(\mathcal{X}_{\leq\alpha};\mathbb{Q})$ is surjective for any c, and the direct sum decomposition depends on a choice of splitting for this surjection.

3.1. **Categories and cohomology theories.** Our goal is to categorify Kirwan's surjectivity theorem, Theorem 3.1, to say even more about the geometry of $\mathcal{X}$. We adopt the perspective of non-commutative algebraic geometry. We regard the functor which assigns a scheme X its dg-category of perfect complexes $\mathrm{Perf}(X)$ as a sort of enhanced cohomology theory. It can be formalized as a functor of ∞-categories from schemes to the ∞-category of dg-categories dgCat. Alternatively, one can regard $\mathrm{Perf}(X)$ as an object of the ∞-category of non-commutative motives, which is constructed as a localization of the ∞-category of dg-categories (see [BGT] for the spectral case).

In [B5], Anthony Blanc constructs a "topological K-theory" spectrum $K^{top}(\mathcal{C})$ associated to any dg-category over $\mathbb{C}$ and a natural Chern character map to the periodic cyclic homology $K^{top}(\mathcal{C}) \otimes \mathbb{C} \to HP(\mathcal{C})$ which is an equivalence when $\mathcal{C} = \mathrm{Perf}(X)$ for a scheme X. Let $G_c \subset G$ be a maximal compact subgroup, and let $K_{G_c}(X)$ denote the equivariant topological K-theory [S1, S2] of (the analytification of) X. With Daniel Pomerleano, we have shown that

Theorem 3.2 (See [HLP1, Theorems 3.9 & 3.20] for more general statements)**.** *Let X be smooth and projective-over-affine G-variety. Then there is a canonical equivalence $K^{top}(\mathrm{Perf}(X/G)) \xrightarrow{\simeq} K_{G_c}(X)$, and combined with the Chern character of [B5] this gives a canonical equivalence*

$$K_{G_c}(X) \otimes \mathbb{C} \simeq K^{top}(\mathrm{Perf}(X/G)) \otimes \mathbb{C} \xrightarrow{\simeq} HP(\mathrm{Perf}(X/G))$$

The method of proof actually uses Theorem 3.9 to "chop up" the noncommutative motive $\mathrm{Perf}(X/G)$ and realize it as a direct summand of a direct sum of countably infinitely many copies of the noncommutative motive of a quasi-projective Deligne-Mumford stack (or even a smooth quasi-projective scheme, after [BLS]). Roughly speaking, the Deligne-Mumford stack appearing in this construction is a disjoint union of various GIT quotients of X and closed subvarieties of X by certain reductive subgroups of G.

The noncommutative motivic statement underlying Theorem 3.2 has more applications, for instance constructing a functorial pure Hodge structure on the topological K-theory $K_{G_c}(X)$ when X is smooth and projective-over-affine and $\Gamma(\mathcal{O}_X)^G$ is finite dimensional via the noncommutative Hodge-de Rham sequence. This intriguingly suggests the existence of a category of "stacky motives" which remembers the topological K-theory of $\mathcal{X}$ and also admits realization functors to the category of mixed Hodge structures.

3.1.1. *Borel-Moore homology theories.* We will also be interested in the (co)homology theory which de-categorifies the assignment $\mathcal{X} \to D^b(\mathcal{X})$, the derived category of coherent sheaves. When $\mathcal{X}$ is not smooth, then Perf and D^b do not agree, and in fact they have different functoriality properties. The assignment $\mathcal{X} \mapsto \mathrm{D}^b(\mathcal{X})$ is covariantly functorial for proper maps (because the derived pushforward of a bounded complex of coherent sheaves has bounded coherent cohomology) and contravariantly functorial for smooth maps $f : \mathcal{Y} \to \mathcal{X}$ (or more generally flat maps), because those are the maps for which $f^* : D_{qc}(\mathcal{X}) \to D_{qc}(\mathcal{Y})$ maps $D^b(\mathcal{X})$ to $D^b(\mathcal{Y})$. Furthermore the projection formula guarantees that proper pushforward commutes with smooth pullback in a cartesian diagram.

Classically, the Borel-Moore homology of locally compact spaces has a similar kind of functoriality: covariantly functorial for proper maps and contravariantly functorial for open immersions. In fact, we can associate a Borel-Moore homology theory E^{BM} to any cohomology theory (satisfying suitable axioms [HLP1, Definition 3.1]) on quotient stacks, regarded as a contravariant functor from the category of G-quasi-projective schemes and equivariant maps $E : (G\text{-quasi-proj})^{op} \to \mathcal{C}$ to some stable ∞-category $\mathcal{C}$, such as the category of spectra or chain complexes over a ring R, provided E satisfies certain axioms. The defining properties of E^{BM} are

- a version of Poincare duality $E^{BM}(X) = E(X)[2(\dim X - \dim G)]$ when X is smooth, and
- a localization exact triangle $E^{BM}(Z) \to E^{BM}(X) \to E^{BM}(U)$ for any closed subscheme $Z \hookrightarrow X$ with complement $U = X \setminus Z$.

For any G-quasi-projective scheme we choose a G-equivariant embedding into a smooth G-quasi-projective scheme $Z \hookrightarrow X$ and define

$$E^{BM}(Z) := \mathrm{fib}(E(X) \to E(X \setminus Z))[2(\dim X - \dim G)] \in \mathrm{Ho}(\mathcal{C}).$$

Provided the functor E satisfies certain axioms (Explained in [HLP1, Section 3.1]), $E^{BM}(Z)$ will be canonically independent, as an object of the homotopy category $\mathrm{Ho}(\mathcal{C})$, of the choice of closed embedding of $Z \hookrightarrow X$. This definition of E^{BM} is covariantly functorial for proper maps of G-quasi-projective schemes and contravariantly functorial for open immersions.

Example 3.3. Let $E : (G\text{-quasi-proj})^{op} \to \mathrm{Ch}$ corresponds to the usual theory of equivariant cohomology, for instance E could assign a G-scheme to the complex of singular cochains on the homotopy quotient $EG_c \times_{G_c} X$, then $E^{BM}(Z)$ is equivariant Borel-Moore homology of Z with integral coefficients, $H_i(E^{BM}(Z)) \simeq H^{BM}_{G_c,i}(Z)$.

Example 3.4. If $E(X) = K_{G_c}(X)$ is the topological K-theory spectrum (regarded as a naive G-spectrum) in the sense of [S2], then $K^{BM}_{G_c}(Z)$ is the Borel-Moore K-homology, sometimes referred to as "K-homology with locally compact supports" [T1]. $K^{BM}_{G_c,i}(X)$ is a module over the Grothendieck ring of complex representations $\mathrm{Rep}(G_c)$, as is $K^i_{G_c}(X)$.

The Atiyah-Segal completion theorem [AS] states that the Chern character $\mathrm{ch} : K^*_{G_c}(X;\mathbb{Q}) \to H^*_{G_c}(X;\mathbb{Q})$ induces an isomorphism between the 2-periodization $\prod_{n\in\mathbb{Z}} H^{i+2n}_{G_c}(X;\mathbb{Q})$ and the completion $K^i_{G_c}(X;\mathbb{Q})^\wedge$ with respect to the *augmentation ideal*, the kernel of the dimension homomorphism $\dim : \mathrm{Rep}(G_c) \to \mathbb{Z}$. It follows from the functoriality of ch and the definition of $K^{BM}_{G_c}$ that there is a Chern character for the Borel-Moore theory inducing an equivalence

$$\mathrm{ch}_i : K^{BM}_{G_c,i}(X;\mathbb{Q})^\wedge \to \prod_{n\in\mathbb{Z}} H^{BM}_{G_c,i+2n}(X;\mathbb{Q}).$$

So as in the case of ordinary equivariant K-theory, the equivariant Borel-Moore K-homology can be regarded as a "spreading out" of equivariant Borel-Moore homology to a finitely generated module over $\mathrm{Rep}(G_c)_\mathbb{Q}$.

Theorem 3.5. [HLP1, Theorem 3.9] *For any G-quasi-projective scheme X, there is a canonical equivalence*

$$\rho_{G,X} : K^{top}(\mathrm{D}^b(X/G)) \to K^{BM}_{G_c}(X)$$

which commutes up to homotopy with proper pushforward, restriction to an open subset, and restriction of equivariance to a reductive subgroup $H \subset G$ such that H is the complexification of $H \cap G_c$.

3.2. **Categorical Kirwan surjectivity.** In light of the categorification of cohomology theories discussed above, our categorification of Theorem 3.1 will amount to the following:

Theorem 3.6. *Let $\mathcal{X}$ be a smooth locally finite type algebraic stack $\mathcal{X}$ with a Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup \mathcal{S}_\alpha$. Then the dg-functor of restriction to the open substack* $\mathrm{res}_{\mathcal{X}^{\mathrm{ss}}} : \mathrm{Perf}(\mathcal{X}) \to \mathrm{Perf}(\mathcal{X}^{\mathrm{ss}})$ *admits a right inverse* $\mathrm{ext} : \mathrm{Perf}(\mathcal{X}^{\mathrm{ss}}) \to \mathrm{Perf}(\mathcal{X})$ *such that* $\mathrm{res}_{\mathcal{X}^{\mathrm{ss}}} \circ \mathrm{ext} \simeq \mathrm{id}_{\mathrm{Perf}(\mathcal{X}^{\mathrm{ss}})}$.

However, as stated Theorem 3.1 discusses the structure of the unstable locus as well. In order to categorify the full statement of Theorem 3.1, we must recall the notion of a semiorthogonal decomposition. We say that a pre-triangulated dg-category $\mathcal{C}$ admits a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}_i | i \in I\rangle$ indexed by a totally ordered set if the triangulated category $\mathrm{Ho}(\mathcal{C})$ admits a semiorthogonal decomposition $\langle \mathrm{Ho}(\mathcal{C}_i) | i \in I\rangle$. By definition this means that $\mathrm{RHom}(E_i, E_j) = 0$ for $i > j$ and every object in $\mathrm{Ho}(\mathcal{C})$ can be built as an iterated extension of objects of $\mathcal{C}_i$ for i in some finite subset of I. One of the basic results in the theory of semiorthogonal decompositions is that given a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}_i | i \in I\rangle$, every $F \in \mathcal{C}$ can be built from a unique and functorial sequence of extensions.

Example 3.7. A two term semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B}\rangle$ means that $\mathrm{RHom}(B, A) = 0$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and for every $F \in \mathcal{C}$ there is an exact triangle

$$B \to F \to A \to$$

with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. This triangle is uniquely and functorially associated to F, and the assignment $F \mapsto A$ (respectively $F \mapsto B$) is the left (respectively right) adjoint of the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ (respectively $\mathcal{B} \hookrightarrow \mathcal{C}$).

Let us return to the context of a smooth stack $\mathcal{X}$ with Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_N$. We shall assume that all of the stacks involved are finite type for simplicity, and that each $\mathcal{S}_\alpha$ is connected. Note that every point of each $\mathcal{Z}_\alpha^{\mathrm{ss}}$ has a canonical $\mathbb{G}_m$ mapping to its stabilizer. We can thus decompose the category $\mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}}) = \bigoplus_{w\in\mathbb{Z}} \mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^w$, where $\mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^w$ is the full subcategory of $\mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})$ consisting of complexes whose homology sheaves are all concentrated in weight w with respect to the canonical $\mathbb{G}_m$ stabilizer at each point. Given a complex $F \in \mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^w$, we say that F has weights *in the window* $[a, b)$ if it lies in the full subcategory $\bigoplus_{w=a}^{b-1} \mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^w$.

Definition 3.8. We define the following full subcategories of complexes $F \in \mathrm{Perf}(\mathcal{X})$ whose derived restriction $F|_{\mathcal{Z}_\alpha^{\mathrm{ss}}}$ satisfies certain constraints:

$$\begin{aligned} \mathcal{G}^w &:= \left\{ F | \forall \alpha, F|_{\mathcal{Z}_\alpha^{\mathrm{ss}}} \text{ has weights in the window } [w_\alpha, w_\alpha + \eta_\alpha) \right\} \\ \mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w} &:= \left\{ F | \operatorname{Supp}(F) \subset \mathcal{X}^{\mathrm{us}} \text{ and } \forall \alpha, F|_{\mathcal{Z}_\alpha^{\mathrm{ss}}} \text{ has weights } \geq w_\alpha \right\} \\ \mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{< w} &:= \left\{ F | \operatorname{Supp}(F) \subset \mathcal{X}^{\mathrm{us}} \text{ and } \forall \alpha, F|_{\mathcal{Z}_\alpha^{\mathrm{ss}}} \text{ has weights } < w_\alpha + \eta_\alpha \right\} \end{aligned}$$

The direct sum decomposition of Theorem 3.1 can be categorified to a semiorthogonal decomposition of $\mathrm{Perf}(\mathcal{X})$ involving the categories above.

Theorem 3.9 ([HL2, Theorem 2.10], see also [BFK])**.** *Let $\mathcal{X}$ be a smooth finite type algebraic stack with a Θ stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_N$. Then for any choice of weights $w = \{w_\alpha\}$ we have a semiorthogonal decomposition*

$$\mathrm{Perf}(\mathcal{X}) = \langle \mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}, \mathcal{G}^w, \mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w}\rangle.$$

The restriction to $\mathcal{X}^{\mathrm{ss}}$ defines an equivalence

$$\mathrm{res}_{\mathcal{X}^{\mathrm{ss}}} : \mathcal{G}^w \xrightarrow{\simeq} \mathrm{Perf}(\mathcal{X}^{\mathrm{ss}}),$$

and $\mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}$ (respectively $\mathrm{Perf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w}$) further admits an infinite semiorthogonal decomposition whose pieces are identified with $\mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^v$ for all α and $v < w_\alpha$ (respectively $\mathrm{Perf}(\mathcal{Z}_\alpha^{\mathrm{ss}})^v$ for $v \geq w_\alpha$).

When $\mathcal{X} = X/G$ is a smooth global quotient stack, then the notion of a Θ-stratification agrees with that of a "KN-stratification," and this theorem is a special case of [HL2, Theorem 2.10]. Some remarks on the case of local quotient stacks appear in [HL3], and the final version is a consequence of the general theorems below. The main theorem of [BFK] is not formulated as a structure theorem for the full equivariant derived category – instead it analyzes a certain kind of *elementary* (or *balanced*) variation of GIT quotient and describes how the category $\mathcal{G}^w$ (and thus the category $\mathrm{Perf}(\mathcal{X}^{\mathrm{ss}})$) changes under this wall crossing. Matthew Ballard has extended this analysis to elementary wall crossings for smooth stacks which are not explicitly presented as global quotient stacks in [B1]. The notion of "elementary stratum" introduced there is also a special case of a Θ-stratum.

In this note, however, we focus on the generalization of Theorem 3.9 to *singular* stacks. Although [HL2] contains some statements in the singular case, it became clear while finishing that paper that a much more general theorem holds, but expressing it most naturally requires the language of derived algebraic geometry.

3.3. **Derived Kirwan surjectivity.** In Theorem 3.17 below we shall extend Theorem 3.9 to arbitrary stacks with a Θ-stratification, but we must first establish some results and notation.

Our general structure theorem applies to the category of "almost perfect" or "pseudo-coherent" complexes $\mathrm{APerf}(\mathcal{X}) \subset \mathrm{QCoh}(\mathcal{X})$. Because $\mathcal{X}$ is locally Noetherian, $\mathrm{APerf}(\mathcal{X})$ coincides with the category $D^- \mathrm{Coh}(\mathcal{X})$, the full subcategory of the unbounded derived category of quasi-coherent complexes F such that the homology sheaves $H_i(F)$ are coherent and $= 0$ for $i \ll 0$.

We are forced to work with a larger category than $\mathrm{Perf}(\mathcal{X})$, because we have seen that the conclusion of Theorem 3.6 for a quotient stack implies that the restriction map $H^*(\mathcal{X};\mathbb{Q}) \to H^*(\mathcal{X}^{\mathrm{ss}};\mathbb{Q})$ is surjective. This surjectivity fails in simple singular examples, such as the quotient of the affine cone over a projective variety by $\mathbb{G}_m$. Under suitable hypotheses on $\mathcal{X}$ (See Theorem 3.22 below) our general structure theorem will specialize to a structure theorem for $\mathrm{DCoh}(\mathcal{X})$ as well, with implications for Borel-Moore homology, but in full generality it seems that the analog of Theorem 3.6 holds only for $\mathrm{APerf}(\mathcal{X})$. (See Remark 3.18 for further discussion).

The key idea will be to use the modular interpretation of a Θ-stratum $\mathcal{S} \hookrightarrow \mathcal{X}$ as an open substack of the mapping stack $\mathcal{M}ap(\Theta, \mathcal{X})$ in order to equip $\mathcal{S}$ with an alternative derived structure.

3.3.1. *The derived structure on a Θ-stratum.* First consider the example of an affine scheme $X = \mathrm{Spec}(R)$ with an action of $\mathbb{G}_m$, corresponding to a grading on R. The Θ-stratum associated to the tautological one parameter subgroup $\lambda(t) = t$ is $Y/\mathbb{G}_m \hookrightarrow X/\mathbb{G}_m$, where $Y = \mathrm{Spec}(R/R \cdot R_{>0})$ is the subscheme cut out by functions on X with positive weights. The center of the stratum is $Z \times (*/\mathbb{G}_m)$ where $Z = \mathrm{Spec}(R/R \cdot (R_{>0} + R_{<0}))$. When X is smooth we can identify the canonical short exact sequence

$$0 \to (N_Y^* X)|_Z \to (\Omega_X^1)|_Z \to (\Omega_Y^1)|_Z \to 0 \quad (3)$$

with the factorization of $(\Omega_X^1)|_Z$ into its weight eigensheaves $0 \to (\Omega_X^1)|_Z^{>0} \to (\Omega_X^1)|_Z \to (\Omega_X^1)|_Z^{\leq 0} \to 0$. The fact that the restriction $(N_{\mathcal{S}}^* \mathcal{X})|_{\mathcal{Z}^{\mathrm{ss}}}$ has positive weights with respect to the canonical $\mathbb{G}_m$ stabilizer of $\mathcal{Z}^{\mathrm{ss}}$, plays a key role in the proof of both Theorem 3.1 and its categorification Theorem 3.9.

When X is singular the sequence (3) is no longer exact. The natural replacement is the exact triangle of derived restrictions of cotangent complexes

$$L_{\mathcal{S}/\mathcal{X}}|_{\mathcal{Z}}[-1] \to L_{\mathcal{X}}|_{\mathcal{Z}} \to L_{\mathcal{S}}|_{\mathcal{Z}} \to, \quad (4)$$

Unfortunately the homology sheaves of $(L_{\mathcal{S}/\mathcal{X}})|_{\mathcal{Z}}$ can now fail to have positive weights.

Example 3.10. Let $X = \mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_m]/(f))$ where f is a non-zero polynomial in $(x_1,\ldots,x_n)$ that is homogeneous of weight d for the $\mathbb{G}_m$ action on $\mathrm{Spec}(\mathbb{C}[x_i,y_j])$ discussed above. Note that d can be either positive or negative. Then because $f \in (x_1,\ldots,x_n)$ we have $Y = \mathrm{Spec}(\mathbb{C}[x_i,y_j]/(f,x_i)) = \mathrm{Spec}(\mathbb{C}[y_j])$, so the cotangent complex of Y is $\mathcal{O}_Y \cdot dy_1 \oplus \cdots \mathcal{O}_Y \cdot dy_m$.

The cotangent complex of X is the two term complex in cohomological degrees $-1, 0$ given by $\mathcal{O}_X \cdot df \to \bigoplus_i \mathcal{O}_X \cdot dx_i \oplus \bigoplus_j \mathcal{O}_X \cdot dy_j$. Therefore applying the exact triangle (4) to the inclusion $\mathcal{S} = Y/\mathbb{G}_m \hookrightarrow \mathcal{X} = X/\mathbb{G}_m$ we have

$$L_{\mathcal{S}/\mathcal{X}} = \left[\mathcal{O}_Y \cdot df \to \bigoplus_i \mathcal{O}_Y \cdot dx_i \right]$$

in cohomological degrees $-2, -1$. Therefore, if f had weight $d < 0$ to begin with, the (derived) restriction of the relative cotangent complex $L_{\mathcal{S}/\mathcal{X}}|_{\mathcal{Z}}$ no longer has weights > 0.

To fix this problem we shall equip $\mathcal{S}$ with a different derived structure, such that $L_{\mathcal{S}/\mathcal{X}}|_{\mathcal{Z}}$ has positive weights. If $\mathcal{X}$ is a locally almost finitely presented algebraic derived stack with quasi-affine diagonal, then the *derived* mapping stack $\mathcal{M}ap(\Theta, \mathcal{X})$ is algebraic (Theorem 2.1) and in particular has a cotangent complex. If $\mathrm{ev} : \Theta \times \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{X}$ is the evaluation map and $p : \Theta \times \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{M}ap(\Theta, \mathcal{X})$ is the projection, then the cotangent complex of the derived mapping stack is $L_{\mathcal{M}ap(\Theta,\mathcal{X})} \simeq (p_* \mathrm{ev}^*(L_{\mathcal{X}}^\vee))^\vee$. This generalizes the classical observation that if X and Y are $\mathbb{C}$-schemes with X proper, then first order deformations of a map $f : X \to Y$ are classified by sections of f^*TY on X, i.e. the tangent space of $\mathcal{M}ap(X, Y)$ is $\Gamma(X, f^*TY)$.

Definition 3.11. Let $\mathcal{X}$ be a locally almost finitely presented derived algebraic stack $\mathcal{X}$ with quasi-affine diagonal. A *derived* Θ-stratum in $\mathcal{X}$ is a closed substack identified with a union of connected components of the *derived* mapping stack under $\mathrm{ev}_1 : \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{X}$.

Note that the underlying classical stack $\mathcal{M}ap(\Theta, \mathcal{X})^{cl}$ is the classical mapping stack to $\mathcal{X}^{cl}$, so if $\mathcal{S} \hookrightarrow \mathcal{X}$ is a derived Θ-stratum then $\mathcal{S}^{cl} \hookrightarrow \mathcal{X}^{cl}$ is a classical Θ-stratum, and conversely if $\mathcal{S}^{cl} \hookrightarrow \mathcal{X}^{cl}$ is a classical Θ-stratum, then it underlies a unique derived Θ-stratum. So we can refer to a derived Θ-stratum as a Θ-stratum without ambiguity. One can use the formula $L_{\mathcal{M}ap(\Theta,\mathcal{X})} \simeq (p_* \mathrm{ev}^*(L_{\mathcal{X}}^\vee))^\vee$ to prove the positivity of the weights of $L_{\mathcal{S}/\mathcal{X}}|_{\mathcal{Z}}$.

Lemma 3.12. *Let $\mathcal{S} \hookrightarrow \mathcal{X}$ be a derived Θ-stratum, then there is a canonical equivalence of exact triangles in* $\mathrm{APerf}(\mathcal{Z})$

$$\begin{array}{ccccccc} L_{\mathcal{X}}|_{\mathcal{Z}}^{\geq 0} & \longrightarrow & L_{\mathcal{X}}|_{\mathcal{Z}} & \longrightarrow & L_{\mathcal{X}}|_{\mathcal{Z}}^{\leq 0} & \longrightarrow & . \\ \downarrow & & \downarrow & & \downarrow & & \\ L_{\mathcal{S}/\mathcal{X}}[-1]|_{\mathcal{Z}} & \longrightarrow & L_{\mathcal{X}}|_{\mathcal{Z}} & \longrightarrow & L_{\mathcal{S}}|_{\mathcal{Z}} & \longrightarrow & \end{array}$$

3.3.2. *Baric structures.* In addition to the strata $\mathcal{S}$, we also equip the centers of the strata $\mathcal{Z}$ with an alternate derived structure coming from the modular interpretation $\mathcal{Z} \subset \mathcal{M}ap(B\mathbb{G}_m, \mathcal{X})$. A mentioned previously, points of the stack $\mathcal{Z}$ have a canonical $\mathbb{G}_m$ in their automorphism groups, both in the classical and derived context. As in the classical case, any object of $F \in \mathrm{QCoh}(\mathcal{Z})$ splits canonically as a direct sum $\bigoplus_{w \in \mathbb{Z}} F_w$, where the homology sheaves of F_w locally have weight w with respect to this canonical $\mathbb{G}_m$-action.

Definition 3.13. Let $\mathcal{C}$ be a pre-triangulated dg-category. Then a *baric decomposition* on $\mathcal{C}$ is a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{C}^{<w}, \mathcal{C}^{\geq w} \rangle$ for each $w \in \mathbb{Z}$, with $\mathcal{C}^{<w} \subset \mathcal{C}^{<w+1}$ and $\mathcal{C}^{\geq w} \subset \mathcal{C}^{\geq w-1}$. Given a baric decomposition of $\mathcal{C}$, we let $\beta^{\geq w}$ and $\beta^{<w}$ denote the projection functors onto $\mathcal{C}^{\geq w}$ and $\mathcal{C}^{<w}$ respectively (they are also called the *baric truncation functors*).

The category $\mathrm{QCoh}(\mathcal{Z})$ has a baric decomposition which splits any $F \in \mathrm{QCoh}(\mathcal{Z})$ into a direct sum of complexes whose homology has weight $< w$ and $\geq w$ respectively, but we shall see that baric decompositions appear in more general settings. Observe that the stack Θ is a monoidal object in the category of stacks where the monoidal product is given by the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, which is equivariant for the group homomorphism $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ which is also given in coordinates by $(z_1, z_2) \mapsto z_1 z_2$. Pre-composition $(t', f(t)) \mapsto f(t' \cdot t)$ defines an action of this monoidal object on the stack $\mathcal{M}ap(\Theta, \mathcal{X})$, giving an action map $a : \Theta \times \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{M}ap(\Theta, \mathcal{X})$ in addition to the projection map $\pi : \Theta \times \mathcal{M}ap(\Theta, \mathcal{X}) \to \mathcal{M}ap(\Theta, \mathcal{X})$. We will only use a very weak form of monoidal action, where we require the usual relations on the structure maps a, π, etc. to hold up to 2-isomorphism, without requiring any higher coherence data.

Lemma 3.14. *Let $\mathcal{S}$ be a connected component of $\mathcal{M}ap(\Theta, \mathcal{X})$. Then $\mathrm{QCoh}(\mathcal{S})$ admits a baric decomposition where the baric truncation functors are defined by the exact triangle*

$$\begin{array}{ccccccc}
\pi_*(\mathcal{O}_\Theta\langle w\rangle \otimes a^*(F)) & \xrightarrow{t^w} & \pi_*(\mathcal{O}_\Theta[t^{-1}] \otimes a^*(F)) & \longrightarrow & \pi_*((\mathcal{O}_\Theta[t^{-1}]/\mathcal{O}_\Theta \cdot t^w) \otimes a^*(F)) & \longrightarrow & , \\
\| & & \downarrow \simeq & & \| & & \\
\beta^{\geq w}(F) & \longrightarrow & F & \longrightarrow & \beta^{< w}(F) & \longrightarrow &
\end{array}$$

where t is the coordinate of weight -1 on $\mathbb{A}^1$, and $\mathcal{O}_\Theta[t^{-1}]$ corresponds to the graded $\mathbb{C}[t]$-module $\mathbb{C}[t^\pm]$. A complex $F \in \mathrm{APerf}(\mathcal{S})$ lies in $\mathrm{QCoh}(\mathcal{S})^{\geq w}$ (respectively $\mathrm{QCoh}(\mathcal{S})^{<w}$) if and only if the homology sheaves of $F|_{\mathcal{Z}^{\mathrm{ss}}}$ are concentrated in weight $\geq w$ (respectively $< w$), where $\mathcal{Z}^{\mathrm{ss}} \subset \mathcal{M}ap(B\mathbb{G}_m, \mathcal{X})$ is the center of $\mathcal{S}$.

Remark 3.15. Note that the baric truncation functors preserves the subcategories $\mathrm{Perf}(\mathcal{S})$ and $\mathrm{APerf}(\mathcal{S})$, inducing baric decompositions of these categories as well.

A version of this lemma is proved via concrete methods in [HL2] for a classical Θ-stratum in a classical quotient stack and in [HL3] for certain derived quotient stacks (using sheaves of CDGA's). In fact, Lemma 3.14 admits a purely formal proof which works for any stack $\mathcal{S}$ with an action (up to 2-isomorphism) of the monoidal object Θ, dramatically simplifying the previous proofs. Note that the monoidal structure on Θ equips the ∞-category $\mathrm{QCoh}(\Theta)$ with the structure of a co-monoidal object in the homotopy category of stable presentable ∞-categories. Then in fact any stable presentable ∞-category with a co-action of $\mathrm{QCoh}(\Theta)$ in the homotopy category of stable presentable ∞-categories, such as $\mathrm{QCoh}(\mathcal{S})$, admits a baric decomposition whose truncation functors are given by the formula of Lemma 3.14. This will be explained in [HL5].

Another important baric structure arises in the following

Lemma 3.16. *(See [HL3, Lemma 2.6] for a less general version) Let $i : \mathcal{S} \hookrightarrow \mathcal{X}$ be a Θ-stratum. Then there is a unique baric decomposition*

$$\mathrm{APerf}_{\mathcal{S}}(\mathcal{X}) = \left\langle \mathrm{APerf}_{\mathcal{S}}(\mathcal{X})^{<w}, \mathrm{APerf}_{\mathcal{S}}(\mathcal{X})^{\geq w} \right\rangle$$

such that $i_* : \mathrm{APerf}(\mathcal{S}) \to \mathrm{APerf}_{\mathcal{S}}(\mathcal{X})$ intertwines the baric truncation functors with the baric truncation functors on $\mathrm{APerf}(\mathcal{S})$ induced by Lemma 3.14.[4] Furthermore, we can identify $\mathrm{APerf}_{\mathcal{S}}(\mathcal{X})^w$ with the essential image of the fully faithful functor $i_*\pi^* : \mathrm{APerf}(\mathcal{Z})^w \to \mathrm{APerf}(\mathcal{X})$.

This lemma is where one needs the positivity of the weights of relative cotangent complex – more precisely, one needs $L_{\mathcal{S}/\mathcal{X}} \in \mathrm{APerf}(\mathcal{S})^{\geq 1}$, which holds for the derived Θ-stratum by Lemma 3.12.

3.3.3. *The main theorem.* Let $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \bigcup_\alpha \mathcal{S}_\alpha$ be a derived stack with a Θ-stratification, and denote the locally closed immersions $i_\alpha : \mathcal{S}_\alpha \hookrightarrow \mathcal{X}$. We choose $w_\alpha \in \mathbb{Z}$ for each α in the indexing set of the stratification. We define

$$\mathrm{APerf}(\mathcal{X})^{\geq w} := \left\{ F \in \mathrm{APerf}(\mathcal{X}) \,\middle|\, \forall \alpha, i_\alpha^* F \in \mathrm{APerf}(\mathcal{S}_\alpha)^{\geq w_\alpha} \right\}$$
$$\mathrm{APerf}(\mathcal{X})^{< w} := \left\{ F \in \mathrm{APerf}(\mathcal{X}) \,\middle|\, \forall \alpha, i_\alpha^{\mathrm{QCoh},!} F \in \mathrm{QCoh}(\mathcal{S}_\alpha)^{< w_\alpha} \right\}.$$

[4]An alternative description of these subcategories: $\mathrm{APerf}_{\mathcal{S}}(\mathcal{X})^{\geq w}$ consists of complexes in $\mathrm{APerf}_{\mathcal{S}}(\mathcal{X})$ such that $i^*(F) \in \mathrm{APerf}(\mathcal{S})^{\geq w}$ and $\mathrm{APerf}_{\mathcal{S}}(\mathcal{X})^{<w}$ consists of complexes which land in $\mathrm{QCoh}(\mathcal{S})^{<w}$ under the functor $i^{\mathrm{QCoh},!} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{S})$ right adjoint to i_*.

Then we further define $\mathcal{G}^w := \mathrm{APerf}(\mathcal{X})^{\geq w} \cap \mathrm{APerf}(\mathcal{X})^{<w}$, $\mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w} := \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X}) \cap \mathrm{APerf}(\mathcal{X})^{\geq w}$, and $\mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w} := \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X}) \cap \mathrm{APerf}(\mathcal{X})^{<w}$.

Theorem 3.17. *There are semiorthogonal decompositions*

$$\mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X}) = \left\langle \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}, \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w} \right\rangle, \text{ and}$$
$$\mathrm{APerf}(\mathcal{X}) = \left\langle \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}, \mathcal{G}^w, \mathrm{APerf}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w} \right\rangle.$$

Furthermore the restriction functor induces an equivalence $\mathcal{G}^w \simeq \mathrm{APerf}(\mathcal{X}^{\mathrm{ss}})$.

A version of this theorem was established for derived global quotient stacks in [HL3, Theorem 2.1], following a more concrete approach using CDGA's to define alternate derived structures on the strata. The theory of Θ-stratifications offers a much more natural approach and the slightly more general formulation above, which will appear in [HL5].

Theorem 3.17 is philosophically interesting for its lack of hypotheses: it shows that by working with the category APerf instead of the category Perf the conclusion of Theorem 3.6 holds for an arbitrary stack with a Θ-stratification. On the other hand, Theorem 3.17 does not de-categorify in a non-trivial way. To make this precise, we make the following

Remark 3.18. Let $H : \mathrm{Cat}_\infty^{ex} \to \mathcal{D}$ be an additive invariant of small stable ∞-categories with values in some stable presentable ∞-category $\mathcal{D}$ (using the terminology of [BGT]). For instance H could be connective or non-connective algebraic K-theory, Hochschild homology (if we are working over a field), etc.. Then $H(\mathrm{APerf}(\mathcal{X})) = 0$ for any algebraic stack $\mathcal{X}$.

Proof. Let $\mathcal{A} := \mathrm{APerf}(\mathcal{X})$. It suffices to show that $U_{add}(\mathcal{A}) = 0$, where U_{add} is the universal additive invariant constructed in [BGT]. Using the fact that $\mathcal{A}$ is idempotent complete, the fact that the ∞-category of stable idempotent complete ∞-categories Cat_∞^{ex} is compactly generated [BGT, Corollary 4.25], and the main theorem [BGT, Theorem 7.13], and it suffices to show that

$$\mathrm{Map}(U_{add}(\mathcal{B}), U_{add}(\mathcal{A})) = K(\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{A})) \simeq 0$$

for any compact idempotent complete stable ∞-category $\mathcal{B}$, where $K(-)$ denotes the connective K-theory. We now claim that the identity functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ induces the zero map $K(\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{A})) \to K(\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{A}))$, which implies the vanishing of this K-theory spectrum.

The proof of this last claim is a categorical version of Mazur's trick: define an exact functor $\Phi : \mathcal{A} \to \mathcal{A}$ by $\Phi(F) := F \oplus F[2] \oplus F[4] \oplus \cdots$, which exists because complexes in $\mathrm{APerf}(\mathcal{X})$ are allowed to be unbounded in one direction. Now consider the map $[\Phi] : K(\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{A})) \to K(\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{A}))$ induced by composition with Φ. The identity $\Phi \simeq \mathrm{id}_\mathcal{A} \oplus \Phi[2]$ as exact functors, combined with the fact that $[\Phi] = [\Phi[2]]$ because double suspension acts trivially on K-theory, implies that $[\Phi] = [\mathrm{id}_\mathcal{A}] + [\Phi]$ and hence $[\mathrm{id}_\mathcal{A}] = 0$. □

Theorem 3.17 has applications on a categorical level, but Remark 3.18 shows that we need a modified version in order to have implications for the topology of $\mathcal{X}$. This necessitates more restrictive hypotheses on $\mathcal{X}$.

Definition 3.19. A derived algebraic stack $\mathcal{X}$ is quasi-smooth if $L_\mathcal{X}$ is perfect and has tor-amplitude in $[-1, 1]$.

Example 3.20. If $\mathcal{X}_0, \mathcal{X}_1 \to \mathcal{Y}$ are maps between smooth stacks, then the derived fiber product $\mathcal{X}' := \mathcal{X}_0 \times_\mathcal{Y} \mathcal{X}_1$ is quasi-smooth. A special case is the "derived zero locus" of a section s of a locally free sheaf V on a smooth stack $\mathcal{X}$, which is by definition the derived intersection of s with the zero section of $\mathrm{Tot}_\mathcal{X}(V)$. Under the projection to $\mathcal{X}$ we can describe this more concretely as the relative derived Spec of the locally free sheaf of CDGA's $\mathrm{Sym}_{\mathcal{O}_\mathcal{X}}(V^\vee[1])$ over $\mathcal{X}$ with differential $V^\vee \to \mathcal{O}_\mathcal{X}$ given by the section s.

Example 3.21. If S is a smooth surface, then $\mathcal{X} = \mathcal{C}oh(S)$ is a quasi-smooth derived stack, because the fiber of the cotangent complex $L_\mathcal{X}$ at a point $[E] \in \mathcal{C}oh(S)$ is

$$L_{\mathcal{X},[E]} \simeq \mathrm{RHom}_S(E, E[1])^\vee \simeq \mathrm{RHom}_S(E, E)^\vee[-1].$$

RHom between coherent sheaves has homology in positive cohomological degree only, so this combined with Serre duality $\mathrm{RHom}_S(E, E)^\vee \simeq \mathrm{RHom}(E, E \otimes \omega_S[2])$ implies that $L_{\mathcal{X},[E]}$ has homology in degree $-1, 0$, and 1. Because $L_\mathcal{X} \in \mathrm{APerf}(\mathcal{C}oh(S))$, this implies that $L_\mathcal{X}$ is perfect with tor-amplitude in $[-1, 1]$, hence $\mathcal{C}oh(S)$ is quasi-smooth.

Theorem 3.22 ([HL5], see [HL3, Theorem 3.2] for the case of global quotient stacks)**.** *Let $\mathcal{X}$ be a quasi-smooth derived algebraic stack with a Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_\alpha \mathcal{S}_\alpha$, and assume that*

(†) *for every point $f : \mathrm{Spec}(\mathbb{C}) \to \mathcal{Z}_\alpha^{\mathrm{ss}}$, the weights of $H_1(f^* L_\mathcal{X})$ are ≥ 0 with respect to the canonical $\mathbb{G}_m$ acting on f.*

Then for any choice of weights $\{w_\alpha\}$ the semiorthogonal decomposition of Theorem 3.17 induces a semiorthogonal decomposition

$$\mathrm{DCoh}(\mathcal{X}) = \left\langle \mathrm{DCoh}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}, \mathcal{G}^w \cap \mathrm{DCoh}(\mathcal{X}), \mathrm{DCoh}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w} \right\rangle,$$

where the restriction functor induces an equivalence $\mathcal{G}^w \cap \mathrm{DCoh}(\mathcal{X}) \simeq \mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}})$. Furthermore, the baric decomposition of Lemma 3.16 induces an infinite semiorthogonal decomposition of $\mathrm{DCoh}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{<w}$ (respectively $\mathrm{DCoh}_{\mathcal{X}^{\mathrm{us}}}(\mathcal{X})^{\geq w}$) whose pieces are identified with $\mathrm{DCoh}(\mathcal{Z}_\alpha^{\mathrm{ss}})^v$ for all α and $v < w_\alpha$ (respectively $v \geq w_\alpha$).

If $\mathcal{X}$ is a derived stack such that $L_\mathcal{X} \simeq (L_\mathcal{X})^\vee$, then this implies that any Θ-stratification satisfies the property (†) and hence Theorem 3.22 applies to $\mathcal{X}$. In particular Theorem 3.22 applies to any Θ-stratification of a 0-shifted derived algebraic symplectic stack in the sense of [PTVV]. Simple examples of 0-shifted derived symplectic algebraic stacks arise as the derived Marsden-Weinstein quotient of an algebraic symplectic variety by a hamiltonian action of a reductive group [P2]. Another important class of 0-shifted derived symplectic stacks are the moduli stacks $\mathcal{X} = \mathcal{C}oh(S)_v^{H-\mathrm{ss}}$ for a $K3$ surface S [PTVV]. The equivalence $L_\mathcal{X} = (L_\mathcal{X})^\vee$ comes from the Serre duality equivalence $\mathrm{RHom}_S(E, E[1]) \simeq \mathrm{RHom}_S(E[1], E[2])^\vee \simeq \mathrm{RHom}_S(E, E[1])^\vee$ for $E \in \mathrm{Coh}(S)$, or more precisely from a version of this equivalence in families.

4. Applications of derived Kirwan surjectivity

Our first application of derived Kirwan surjectivity is an analog of Theorem 3.1 for Borel-Moore homology and the Borel-Moore Poincare polynomial $P_t^{BM}(\mathcal{X}) := \sum_{i \in \mathbb{Z}} t^i \dim H_i^{BM}(\mathcal{X})$.

Corollary 4.1. *Let $\mathcal{X}$ be a quasi-smooth derived quotient stack with a Θ-stratification satisfying (†). Then one has a direct sum decomposition*

$$H_*^{BM}(\mathcal{X}; \mathbb{Q}) \simeq H_*^{BM}(\mathcal{X}^{\mathrm{ss}}; \mathbb{Q}) \oplus \bigoplus_\alpha H_{*-2c_\alpha}^{BM}(\mathcal{Z}_\alpha^{\mathrm{ss}}; \mathbb{Q})$$

where $c_\alpha = \mathrm{rank}(L_\mathcal{X}|_{\mathcal{Z}_\alpha^{\mathrm{ss}}}^{<0})$ and hence $P_t^{BM}(\mathcal{X}) = P_t^{BM}(\mathcal{X}^{\mathrm{ss}}) + \sum_\alpha t^{2c_\alpha} P_t^{BM}(\mathcal{Z}_\alpha^{\mathrm{ss}})$.

Sketch of proof. It suffices to verify the claim in the case of a single closed Θ-stratum. The fact that the restriction functor $K_{G_c,i}^{BM}(X) \to K_{G_c,i}^{BM}(X^{\mathrm{ss}})$ is surjective for all i follows immediately from Theorem 3.22 and Theorem 3.5. The Atiyah-Segal completion theorem then implies that the restriction map $H_{G_c,i}^{BM}(X) \to H_{G_c,i}^{BM}(X^{\mathrm{ss}})$ is surjective for all i. It follows that the long exact "localization" sequence in Borel-Moore homology becomes a short exact sequence for all i

$$0 \to H_i^{BM}(S/G) \to H_i^{BM}(X/G) \to H_i^{BM}(X^{\mathrm{ss}}/G) \to 0,$$

which split non-canonically.

So to prove the theorem it suffices to show that $H^{BM}_{*+c}(S/G) = H^{BM}_*(\mathcal{Z}^{\mathrm{ss}})$ where $c = \mathrm{rank}(L_{\mathcal{X}}|^{<0}_{\mathcal{Z}^{\mathrm{ss}}})$. The map of derived stack $\pi : \mathcal{S} \to \mathcal{Z}^{\mathrm{ss}}$ is itself relatively quasi-smooth of relative virtual dimension c, hence of finite tor-amplitude. A bit of work is required to show that π induces a pullback map in Borel-Moore homology $\pi^* : H^{BM}_i(\mathcal{Z}^{\mathrm{ss}}) \to H^{BM}_i(\mathcal{S})$ which is compatible with the pullback functor $\pi^* : \mathrm{DCoh}(\mathcal{Z}^{\mathrm{ss}}) \to \mathrm{DCoh}(\mathcal{S})$ under the isomorphism of Theorem 3.5 and the Atiyah-Segal completion theorem. Once that has been established, however, the infinite semiorthogonal decomposition of $\mathrm{DCoh}(\mathcal{S})$ of Theorem 3.22 implies that $H^{BM}_{i+2c_\alpha}(\mathcal{S}) \simeq H^{BM}_i(\mathcal{Z}^{\mathrm{ss}})$ under this pullback map. □

Remark 4.2. These topological results are similar in spirit to those of [K2], where Kirwan proves that a version of Theorem 3.1 holds for the intersection cohomology of singular G-varieties.

4.1. **Topology of the moduli stack of sheaves on a $K3$ surface.** Let X be a smooth $K3$-surface. Then for Hilbert polynomials P corresponding to "primitive" numerical K-theory classes and for an ample class $H \in NS(X)_{\mathbb{R}}$ avoiding a locally finite set of real codimension 1 "walls", the stack $\mathcal{C}oh(X)^{H-\mathrm{ss}}_P$ is actually representable by a smooth hyperkähler variety [HL6, §4.C].[5] Assuming certain bounds on the numerical K-theory class, these varieties will be birational to each other, and so it follows that their Betti numbers agree [B2]. In fact we show that if one uses Borel-Moore homology of the stack, then a statement of this form continues to hold.

Theorem 4.3 ([HL5])**.** *For any two generic (i.e. avoiding an explicit locally finite collection of real hyperplanes) ample classes $H, H' \in NS(X)_{\mathbb{R}}$ we have an equality of Borel-Moore Poincare polynomials $P^{BM}_t(\mathcal{C}oh(X)^{H-\mathrm{ss}}_P) = P^{BM}_t(\mathcal{C}oh(X)^{H'-\mathrm{ss}}_P)$.*

This is proved by analyzing the Θ-stratification of $\mathcal{C}oh(X)_P$ as H varies. In particular, say H_+ and H_- lie on either side of a wall in $NS(X)_{\mathbb{R}}$, and let $H_0 \in NS(X)_{\mathbb{R}}$ be the point on the line segment joining $H_\pm$ which lies on this wall. Then $\mathcal{C}oh(X)^{H_\pm-\mathrm{ss}}_P \subset \mathcal{C}oh(X)^{H_0-\mathrm{ss}}_P$, and the complement of this open substack is a union of Harder-Narasimhan strata with respect to $H_\pm$:

$$\mathcal{C}oh(X)^{H_--\mathrm{ss}}_P \cup \bigcup_\alpha \mathcal{S}^{H_-}_\alpha = \mathcal{C}oh(X)^{H_0-\mathrm{ss}}_P = \mathcal{C}oh(X)^{H_+-\mathrm{ss}}_P \cup \bigcup_\alpha \mathcal{S}^{H_+}_\alpha \tag{5}$$

It turns out that the set of Harder-Narasimhan types appearing in either Θ-stratification are the same, and the c_α is the same for both $H_\pm$, so Corollary 4.1 implies an equality

$$P_t(\mathcal{C}oh(X)^{H_--\mathrm{ss}}_P) + \sum_\alpha t^{c_\alpha} P_t(\mathcal{C}oh(X)^{H_--\mathrm{ss}}_\alpha) = P_t(\mathcal{C}oh(X)^{H_+-\mathrm{ss}}_P) + \sum_\alpha t^{c_\alpha} P_t(\mathcal{C}oh(X)^{H_+-\mathrm{ss}}_\alpha),$$

where we are using the notation $\mathcal{C}oh(X)^{H-\mathrm{ss}}_\alpha = \mathcal{C}oh(X)^{H-\mathrm{ss}}_{P_0} \times \cdots \times \mathcal{C}oh(X)^{H-\mathrm{ss}}_{P_N}$ for a Harder-Narasimhan type $\alpha = (P_0, \ldots, P_N)$. This identity provides the basis for an inductive proof of the theorem.

4.1.1. *Speculations on the interaction with Hodge theory.* The Borel-Moore homology carries a mixed Hodge structure, and the fact that the restriction map of non-commutative motives "with compact support" $[\mathrm{DCoh}(\mathcal{X})] \to [\mathrm{DCoh}(\mathcal{X}^{\mathrm{ss}})]$ splits suggests that the restriction map $H^{BM}_i(\mathcal{X}) \to H^{BM}_i(\mathcal{X}^{\mathrm{ss}})$ splits as a map of mixed Hodge structures, so that the direct sum decomposition of Corollary 4.1 can be chosen compatibly with the Hodge structures. One might even suspect in this context that the restriction map of the motive with compact support $M^c(\mathcal{X}) \to M^c(\mathcal{X}^{\mathrm{ss}})$ in Voevodsky's big triangulated category of motives $DM(\mathbb{C}, \mathbb{Q})$ (as constructed for quotient stacks in [T2]) splits. We have neither a proof, nor a counterexample.

On the other hand, one can imagine a different proof of Theorem 4.3 along Hodge-theoretic lines. The virtual Hodge polynomial $E_{x,y}(\mathcal{X})$ of a stack [J] is additive for any stratification of $\mathcal{X}$, so except

[5] Technically the moduli functor as we have defined it is a $\mathbb{G}_m$-gerbe over a hyperkähler manifold, but that will not change any of the statements we make.

for the comparison of $E_{x,y}(\mathcal{Z}_\alpha^{\mathrm{ss}})$ with $E_{x,y}(\mathcal{S}_\alpha)$, a version of Corollary 4.1 would hold automatically for the virtual Hodge polynomial. If the Hodge structure on $H_i^{BM}(\mathcal{X})$ is pure, then one can recover $P_t^{BM}(\mathcal{X})$ from $E_{x,y}(\mathcal{X})$, which leads us to

Conjecture 4.4. *The Hodge structure on $H_i^{BM}(\mathcal{C}oh(X)_P^{H-\mathrm{ss}})$ is pure for any H, and in general, $H_i^{BM}(\mathcal{X})$ is pure for any algebraic-symplectic (derived) stack $\mathcal{X}$ which has a proper good moduli space. More ambitiously, one can ask if $H^i_{BM}(\mathcal{X})$ is pure for any algebraic-symplectic derived stack $\mathcal{X}$ which is cohomologically proper in the sense of* [HLP2].

This conjecture is also inspired by the work of Ben Davison, who shows that the moduli of representations of the pre-projective algebra of any quiver has pure Borel-Moore homology in [D1]. In fact, he has an entirely different approach to proving results akin to Corollary 4.1 using the theory of mixed Hodge modules and purity theorems.

4.2. **Other applications of Theorem 3.17.** One of the largest open problems in the theory of derived categories of coherent sheaves is the conjecture that any two birational Calabi-Yau manifolds have equivalent derived categories of coherent sheaves, and there is a long history of results by Bondal, Orlov, Kawamata, Bridgeland, Bezrukavnikov, Kaledin and others constructing equivalences in larger and larger classes of examples. One of the main applications of Theorem 3.9 is to establishing derived equivalences for flops between CY manifolds constructed explicitly via variation of GIT quotient. For instance, derived equivalences for the simplest kind of variation of GIT quotient were constructed in [HL2] and [BFK]. There has been recent progress in [HLS], where derived equivalences for a much larger class of variation of GIT quotient (and hyperkähler quotient) are established. The GIT problems studied in [HLS] serve as étale local models for the wall crossing (5), and in [HL5] we will use these methods to prove that if $H, H \in NS(X)_\mathbb{R}$ are two generic ample classes and P is a primitive Hilbert polynomial, then $\mathrm{DCoh}(\mathcal{C}oh(X)_P^{H-\mathrm{ss}}) \simeq \mathrm{DCoh}(\mathcal{C}oh(X)_P^{H'-\mathrm{ss}})$.

5. Non-abelian virtual localization theorem

In addition to the applications we have discussed, one can use the notion of derived Θ-stratifications to prove a "virtual non-abelian localization" formula in K-theory.

Definition 5.1. A complex $F \in \mathrm{Perf}(\mathcal{X})$ is said to be *integrable* if $\bigoplus_i H_i R\Gamma(\mathcal{X}, F \otimes G)$ is finite dimensional for any $G \in \mathrm{DCoh}(\mathcal{X})$.

The subcategory of integrable complexes is a stable idempotent complete $\otimes$-ideal of $\mathrm{Perf}(\mathcal{X})$, whose K-theory is a somewhat ad hoc version of algebraic K-theory "with compact supports."

Example 5.2. If $\mathcal{X}$ has a good moduli space $\mathcal{X} \to Y$ such that Y is proper – for instance the map $X/G \to X//G$ where X is a G-quasi-projective-scheme which admits a projective good quotient $X//G$ – then any $F \in \mathrm{Perf}(\mathcal{X})$ is integrable. The prototypical example is the moduli stack of semistable principal G-bundles on a smooth curve.

Example 5.3. More generally if $F \in \mathrm{Perf}(\mathcal{X})$ is set theoretically supported on a closed substack $\mathcal{Y} \subset \mathcal{X}$ which admits a proper good moduli space $\mathcal{Y} \to Y$ then F is integrable. Even more generally, $F \in \mathrm{Perf}(\mathcal{X})$ is integrable whenever the support of F is cohomologically proper in the sense of [HLP2].

If $\mathcal{X}$ is derived, then we shall assume that $H_i(\mathcal{O}_\mathcal{X}) = 0$ for $i \gg 0$ so that $\mathcal{O}_\mathcal{X} \in \mathrm{DCoh}$. For an integrable complex $F \in \mathrm{Perf}(\mathcal{X})$, one can define the K-theoretic "index" to be $\chi(\mathcal{X}, F)$, which is the K-theoretic analog of the integral of a compactly supported cohomology class. In fact when $\mathcal{X} = X$ is a scheme the two notions of integration are directly related via the Grothendieck-Riemann-Roch theorem. So we see that integration in K-theory, as opposed to integration in cohomology, generalizes easily from schemes to stacks.

When $\mathcal{X}$ is quasi-smooth derived stack, the index $\chi(\mathcal{X},F)$ of an admissible $F \in \mathrm{Perf}(\mathcal{X})$ is analogous to the integral of a compactly supported cohomology class on $\mathcal{X}^{cl}$ with respect to a *virtual fundamental class*. To make this precise, consider the surjective closed immersion $\iota : \mathcal{X}^{cl} \hookrightarrow \mathcal{X}$. In $K_0(\mathrm{DCoh}(\mathcal{X}))$ we have $[\mathcal{O}_\mathcal{X}] = \sum_i (-1)^i [H_i(\mathcal{O}_\mathcal{X})]$. The objects $H_i(\mathcal{O}_\mathcal{X})$ are canonically ι_* of coherent sheaves on $\mathcal{X}^{cl}$, which we denote $H_i(\mathcal{O}_\mathcal{X})$ as well, and we introduce the *virtual structure sheaf* $\mathcal{O}^{vir}_\mathcal{X} = \bigoplus_i H_i(\mathcal{O}_\mathcal{X})[i] \in \mathrm{DCoh}(\mathcal{X}^{cl})$. By the projection formula we have

$$\chi(\mathcal{X},F) = \chi(\mathcal{X}, F \otimes \iota_*(\bigoplus H_i(\mathcal{O}_\mathcal{X})[i])) = \chi(\mathcal{X}^{cl}, \mathcal{O}^{vir}_\mathcal{X} \otimes \iota^* F).$$

Virtual integrals of this form, along with their cohomological counterparts, play a central role in modern enumerative geometry.

Given a Θ-stratification of a quasi-smooth derived stack $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_\alpha \mathcal{S}_\alpha$, the virtual non-abelian localization formula relates the index of a K-theory class on $\mathcal{X}$ to the index of its restriction to $\mathcal{X}^{\mathrm{ss}}$ as well as $\mathcal{Z}^{\mathrm{ss}}_\alpha$ for all α. Let us define $L^+_\alpha := \beta^{\geq 1}(L_\mathcal{X}|_{\mathcal{Z}^{\mathrm{ss}}_\alpha}) \in \mathrm{APerf}(\mathcal{Z}^{\mathrm{ss}}_\alpha)^{\geq 1}$ and $L^-_\alpha := \beta^{<0}(L_\mathcal{X}|_{\mathcal{Z}^{\mathrm{ss}}_\alpha}) \in \mathrm{APerf}(\mathcal{Z}^{\mathrm{ss}}_\alpha)^{<0}$. Because $\mathcal{X}$ is quasi-smooth, $L_\mathcal{X}$ is actually perfect, and hence so are L^+_α and L^-_α. For each stratum, we define a complex

$$E_\alpha = \mathrm{Sym}(L^-_\alpha \oplus (L^+_\alpha)^\vee) \otimes (\det L^+_\alpha)^\vee[-\operatorname{rank} L^+_\alpha] \in \mathrm{QCoh}(\mathcal{Z}^{\mathrm{ss}}_\alpha),$$

which can be regarded as a K-theoretic reciprocal of the virtual normal bundles of $\mathcal{Z}_\alpha$ in $\mathcal{X}$. Note that $L^+_\alpha[1]$ is $L_{\mathcal{S}_\alpha/\mathcal{X}}|_{\mathcal{Z}^{\mathrm{ss}}_\alpha}$ once one equips $\mathcal{S}_\alpha$ with the "correct" derived structure discussed above. We also define the integer η_α to be the weight of $\det(L^+_\alpha)$.

Definition 5.4. We say that $F \in \mathrm{Perf}(\mathcal{X})$ is *almost admissible* if $F|_{\mathcal{Z}^{\mathrm{ss}}_\alpha} \in \mathrm{Perf}(\mathcal{Z}^{\mathrm{ss}}_\alpha)^{<\eta_\alpha}$ for all but finitely many α.

Example 5.5. Consider the moduli stack $\mathcal{X} = \mathrm{Bun}_G(C)$ of principal G-bundles on a smooth curve C. Then $\mathcal{X}$ is smooth, and the Shatz-Harder-Narasimhan stratification is a Θ-stratification. In [TW], Teleman and Woodward define the subspace of "admissible classes" in $K_0(\mathrm{Bun}_G(C))$ as the span of some explicit complexes and prove formulas for their index, $\chi(\mathrm{Bun}_G(C), F)$. These classes are almost admissible in the sense of Definition 5.4.[6]

Theorem 5.6 (Virtual non-abelian localization [HL3, Theorem 5.1])**.** *Let $\mathcal{X}$ be a quasi-smooth derived stack with a Θ-stratification $\mathcal{X} = \mathcal{X}^{\mathrm{ss}} \cup \bigcup_\alpha \mathcal{S}_\alpha$. Let $F \in \mathrm{Perf}(\mathcal{X}) \subset \mathrm{DCoh}(\mathcal{X})$ be an almost admissible complex such that $F|_{\mathcal{Z}_\alpha}$ is integrable for all α. Then $R\Gamma(\mathcal{X},F)$ is finite dimensional if and only if $R\Gamma(\mathcal{X}^{\mathrm{ss}},F)$ is, and in this case we have*

$$\chi(\mathcal{X},F) = \chi(\mathcal{X}^{\mathrm{ss}},F) + \sum_\alpha \chi(\mathcal{Z}^{\mathrm{ss}}_\alpha, F|_{\mathcal{Z}^{\mathrm{ss}}_\alpha} \otimes E_\alpha).$$

Example 5.7 (Abelian virtual localization)**.** Let X be a quasi-smooth scheme with an action of a torus T. For any one parameter subgroup $\lambda : \mathbb{G}_m \to T$, the Bialynicki-Birula stratification of X can be interpreted as a Θ-stratification of $X/\mathbb{G}_m$. If we choose λ to be suitably generic, then the centers of the strata will be the connected components Z_α of X^T. If we assume that the strata cover X, for instance if X is proper, then Theorem 5.6 implies that

$$\chi(X/\mathbb{G}_m, F) = \sum (-1)^i \dim((H_i R\Gamma(X,F))^{\mathbb{G}_m}) = \sum_\alpha \chi(Z_\alpha, (F|_{Z_\alpha} \otimes E_\alpha)^{\mathbb{G}_m}).$$

This is comparable to the virtual localization formula in cohomology [GP] via the equivariant Grothendieck-Riemann-Roch theorem.

[6]The "almost" in the terminology is to be consistent with [HL4]. There it was convenient to define for an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ the class of $\mathcal{L}$-admissible complexes to be those $F \in \mathrm{Perf}(\mathcal{X})$ for which $\mathcal{L} \otimes F^{\otimes m}$ is almost admissible for all $m > 0$. One can say that the Atiyah-Bott classes are $\mathcal{L}$-admissible for any $\mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_G(C))$ which has a positive level.

Example 5.8 (Equivariant Verlinde formula)**.** A version of the Verlinde formula on the moduli stack of semistable G-principal Higgs bundles $\mathcal{H}iggs_G^{\mathrm{ss}}$ on a fixed smooth curve Σ over $\mathbb{C}$ has recently been established in [HL4] and independently in [AGP]. It gives an explicit formula for the "graded dimension" of the space of sections of certain "positive" line bundles $\mathcal{L}$,

$$\dim_{\mathbb{C}^*}(H^0(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})) = \sum_n t^n \dim\left(H^0(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})_{\mathrm{weight}\, n}\right),$$

where $H^0(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})_{\mathrm{weight}\, n}$ denotes the weight n direct summand of $H^0(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})$ with respect to the $\mathbb{G}_m$ action on $\mathcal{H}iggs_G^{\mathrm{ss}}$ which scales the Higgs field. The proof reduces the formula to a computation on the stack of *all* Higgs bundles $\mathcal{H}iggs_G$, where one can use previous techniques [TW, FGT] to compute the K-theoretic graded index of $\mathcal{L}$, and prove that the cohomology vanishes there. In [HL4] we use the methods of derived Θ-stratifications and Theorem 5.6 and Theorem 3.17 above to go a bit further, identifying $R\Gamma(\mathcal{H}iggs_G, \mathcal{L}) \simeq R\Gamma(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})$. This establishes the vanishing $H^i(\mathcal{H}iggs_G^{\mathrm{ss}}, \mathcal{L})$ for $i > 0$ on the stack of semistable Higgs bundles, in addition to the formula for H^0 appearing in [AGP, HL4].

Example 5.9 (Wall-crossing)**.** Another thing that Theorem 5.6 is well suited for is comparing the index of tautological classes on $\mathcal{X}^{\mathrm{ss}}$ as one varies the notion of stability. For instance let X be a projective $K3$-surface and consider a variation of polarization $H \in NS(X)_{\mathbb{R}}$. For any class in $K^0(\mathcal{X})$ represented by some $F \in \mathrm{Perf}(\mathcal{C}oh(X)_P)$, one can apply Theorem 5.6 to the two stratifications of Equation 5 to obtain a wall-crossing formula

$$\begin{aligned} \chi(\mathcal{C}oh(X)_P^{H_- - \mathrm{ss}}, F) - \chi(\mathcal{C}oh(X)_P^{H_+ - \mathrm{ss}}, F) = \\ \sum_\alpha \left(\chi(\mathcal{C}oh(X)_\alpha^{H_+ - \mathrm{ss}}, F \otimes E_\alpha^{H_+}) - \chi(\mathcal{C}oh(X)_\alpha^{H_- - \mathrm{ss}}, F \otimes E_\alpha^{H_-})\right) \end{aligned} \quad (6)$$

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