Math 4310 Homework 11 Solutions

Problem 1. We prove the cycle of implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) implies (2). Assume that the natural map $W_1 \times \cdots \times W_n \to V$ is an isomorphism. Then every element in V can be written as $w_1 + \cdots + w_n$ (by surjectivity of this map), so $\sum W_i = V$ by definition. Moreover, if $-w_k = \sum_{i \neq k} w_i$ is an element of $W_k \cap \sum_{i \neq k} W_i$ (written simultaneously as an element of W_k and an element of $\sum_{i \neq k} W_i$), then $(w_1, \ldots, w_k, \ldots, w_n)$ is in the kernel of our isomorphism; thus our element w_k is zero.

(2) implies (3). By definition, $v \in V = \sum W_i$ means that any element v can be written as a sum $w_1 + \cdots + w_n$. For uniqueness, if $v = w_1' + \cdots + w_n'$ then we have

$$0 = v - v = (w_1 - w_1') + \cdots + (w_n - w_n');$$

for each k, this sum lets us say that $w_k - w_k' \in W_k$ and also $w_k - w_k' \in \sum_{i \neq k} W_i$. Since the intersection is zero by (2), $w_k - w_k' = 0$ and thus $w_k = w_k'$. This proves uniqueness.

(3) implies (1). Saying that every element can be uniquely written as such a sum is saying that the linear transformation $(w_1, \ldots, w_n) \mapsto w_1 + \cdots + w_n$ is injective and surjective, and thus is an isomorphism.

Problem 2. (a) To see $W \cap W'$ is T-invariant, note that if $u \in W \cap W'$ then $u \in W$ and $u \in W'$ mean $T(u) \in W$ and $T(u) \in W'$; thus $T(u) \in W \cap W'$. Similarly, an element of $W + W'$ is of the form $w + w'$, and $T(w + w') = T(w) + T(w') \in W + W'$.

(b) If W is T-invariant, then any element of $S[W]$ is of the form $S(w)$ for some $W \in W$; we have $T(S(w)) = S(T(w)) \in S[W]$ because $T(w) \in W$.

Problem 3. We look dimension-by-dimension here at the possible subspaces. The only 0-dimensional subspace $W = 0$ is trivially invariant, and the only 3-dimensional subspace $W = \mathbb{R}^3$ is also trivially invariant. For intermediate dimensions we need to look a bit more carefully:

Dimension 1 subspaces. We know that a dimension-1 subspace W, spanned by a nonzero vector w, is invariant iff w is an eigenvector. Evidently the eigenvectors of A all have eigenvalue 1, and consist of all linear combinations of e_1 and e_3. So the 1-dimensional invariant subspaces are $W = L(e_1)$ and $W = L(ye_1 + e_3)$ for any $y \in \mathbb{R}$ (and when written this way everything on the list is distinct!).

Dimension 2 subspaces. There are two obvious dimension-2 invariant subspaces: $L(e_1, e_2)$ (corresponding to the upper 2×2 block) and $L(e_1, e_3)$ (the eigenspace for $\lambda = 1$). Trying things out you can find that there's actually a bunch more: anything of the form $L(e_1, ye_2 + ze_3)$ (with $ye_2 + ze_3 \neq 0$) is T-invariant because $T(e_1) = e_1$ and $T(ye_2 + ze_3) = ye_1 + (ye_2 + ze_3)$. We claim that no other 2-dimensional subspace can be invariant.

To prove this, it's sufficient to show that any 2-dimensional invariant subspace needs to contain e_1; in fact we've listed every 2-dimensional subspace W of V with $e_1 \in W$! To see this, suppose W is a 2-dimensional invariant subspace. If every vector

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

has $y = 0$, then W is necessarily $L(e_1, e_3)$. If not, then W contains $T(v)$ by assumption, and thus

$$\frac{1}{y}(T(v) - v) = \frac{1}{y}\left(\begin{bmatrix} x+y \\ y \\ z \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = e_1.$$

Problem 4. We know each nilpotent 6×6 matrix is similar to a Jordan canonical form with eigenvalues 0, and two such Jordan canonical forms are similar if and only if they have the same *unordered* list of Jordan blocks. Giving all distinct lists of Jordan blocks amounts to giving a *partition* of the integer 6: we just need to see how many different ways we can write 6 as a sum of positive integers, and each of these gives us a distinct Jordan canonical form. The correct number is eleven similarity classes (the value of the "partition function" for $n = 6$):

1. One class for the JCF with a 6×6 Jordan block $J_6(0)$.
2. One class for the JCF with Jordan blocks $J_5(0)$ and $J_1(0)$.
3. One class for the JCF with Jordan blocks $J_4(0)$ and $J_2(0)$.
4. One class for the JCF with Jordan blocks $J_4(0)$ and $J_1(0)$, and $J_1(0)$.
5. One class for the JCF with two Jordan blocks $J_3(0)$.
6. One class for the JCF with Jordan blocks $J_3(0)$, $J_2(0)$, and $J_1(0)$.
7. One class for the JCF with the Jordan block $J_3(0)$ and three Jordan blocks $J_1(0)$.
8. One class for the JCF with three Jordan blocks $J_2(0)$.
9. One class for the JCF with two Jordan blocks $J_2(0)$ and two Jordan blocks $J_1(0)$.
10. One class for the JCF with one Jordan blocks $J_2(0)$ and four Jordan blocks $J_1(0)$.
11. One class for the JCF with six Jordan blocks $J_1(0)$ (i.e. the zero matrix).

Problem 5. (a) If $v \in W^\perp$, then by definition $\langle v, w\rangle = 0$ for any $w \in W$. Then we have $\langle T^*(v), w\rangle = \langle v, T(w)\rangle = 0$ for any $w \in W$, with the first equality by definition of the adjoint and the second by assumption that W is T-invariant (so $T(w) \in W$). So $T^*(v) \in W^\perp$ by definition, and thus $W^\perp$ is T^*-invariant.

(b) To check $(T - \lambda I)^* = T^* - \overline{\lambda} I$, we need to check that we have the identity

$$\langle (T - \lambda I)(v), w\rangle = \langle v, (T^* - \overline{\lambda} I)(w)\rangle$$

for every v, w; but indeed this holds because we can expand each expression and get

$$\langle (T - \lambda I)(v), w\rangle = \langle T(v) - \lambda v, w\rangle = \langle T(v), w\rangle - \lambda\langle v, w\rangle$$

and

$$\langle v, (T^* - \overline{\lambda} I)(w)\rangle = \langle v, T^*(w) - \overline{\lambda} w\rangle = \langle v, T^*(w)\rangle - \lambda\langle v, w\rangle$$

and these are equal by definition of T^*. Then to see $T - \lambda I$ is normal is just another comparison of two things we need to be equal:

$$(T - \lambda I) \circ (T - \lambda I)^* = (T - \lambda I) \circ (T^* - \overline{\lambda} I) = T \circ T^* - \lambda T^* - \overline{\lambda} T - \lambda\overline{\lambda} I,$$

$$(T - \lambda I)^* \circ (T - \lambda I) = (T^* - \overline{\lambda} I) \circ (T - \lambda I) = T^* \circ T - \lambda T^* - \overline{\lambda} T - \lambda\overline{\lambda} I,$$

and these are equal by normality of T.

(c) In general we can write

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle v, T^*(T(v))\rangle.$$

By the assumption that T is normal we can swap the order of $T^* \circ T$ here and continue

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle v, T^*(T(v))\rangle = \langle v, T(T^*(v))\rangle = \langle T^*(v), T^*(v)\rangle = \|T^*(v)\|^2.$$

Taking square roots gives the first statement we want. For the second statement, note $v \in \ker T$ iff $T(v) = 0$ iff $\|T(v)\| = 0$. By what we just showed this happens iff $\|T^*(v)\| = 0$, which is iff $v \in \ker T^*$ by the same logic. So $\ker T = \ker T^*$.

Then, if T is normal and v is an eigenvector with eigenvalue λ, we know v is in the kernel of the normal operator $T - \lambda I$. This means v is also in the kernel of $(T - \lambda I)^* = T^* - \overline{\lambda} I$, i.e. v is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

(d) As suggested, we prove the theorem by induction on the dimension of V; we can take the base case to be the vacuous one of $\dim V = 0$. For the inductive step, if $\dim V \geq 1$ then the characteristic polynomial $c_T(x)$ is a polynomial over $\mathbb{C}$ of degree at least 1, so has a root λ, and therefore T has an eigenvalue v with eigenvector λ (since $\det(T - \lambda I) = 0$ there must be something in the kernel). By part (c) we know that v is also an eigenvector of T^*. So the subspace $L(v)$ is both T-invariant and T^*-invariant; by part (a) we conclude that the orthogonal complement $L(v)^\perp$ is both T^*-invariant and $T^{**} = T$-invariant. Since $W = L(v)^\perp$ is T-invariant, we can restrict T to a normal operator $T|_W : W \to W$ and by induction conclude W has a basis of eigenvectors for T as well. Adding v to this basis gives a basis of V consisting of eigenvectors of T, which is what we wanted. (Note what was really important here was that $L(v)$ had a T-invariant complementary subspace, and that's what normality ultimately got for us! This is what may fail for non-diagonalizable matrices.)