Math 4310 Homework 12 Solutions

Problem 1. (a) We start off by computing the characteristic polynomial; since A is a block matrix we can split up the determinant we get as a product of 2×2 determinants.

$$\det(A - xI) = \det \begin{bmatrix} -x & 1 & 1 & 0 \\ -1 & -x & 0 & 1 \\ 0 & 0 & -x & 1 \\ 0 & 0 & -1 & -x \end{bmatrix} = (x^2 + 1)^2.$$

So the characteristic polynomial factors as $(x + i)^2(x - i)^2$.

The next step in finding the Jordan form is to look at the eigenspaces. The eigenspace for $\lambda = i$ is 2-dimensional, and we can work out that

$$A - iI = \det \begin{bmatrix} -i & 1 & 1 & 0 \\ -1 & -i & 0 & 1 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix} \qquad (A - iI)^2 = \det \begin{bmatrix} -2 & -2i & -2i & 2 \\ 2i & -2 & -2 & -2i \\ 0 & 0 & -2 & -2i \\ 0 & 0 & 2i & -2 \end{bmatrix}.$$

We can see that $(A - iI)^2$ has a 2-dimensional kernel spanned by

$$v_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}.$$

Moreover, v_1 is not in the kernel of $(A - iI)$, and we have $v_2 = (A - iI)v_1$. So we get a 2×2 Jordan block with eigenvalue i, corresponding to the generalized eigenspace spanned by these two vectors.

An identical computation gives that there is also a 2×2 Jordan block with eigenvalue $-i$, spanned by the vectors

$$v_3 = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -i \end{bmatrix},$$

which satisfy $(A + iI)v_3 = v_4$. Thus the Jordan canonical form for A is the matrix

$$J = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Actually we can say a bit more (to be useful in the second part) - in particular we have $A = PJP^{-1}$ for

$$P = [v_1 \ v_2 \ v_3 \ v_4] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{bmatrix}.$$

(b) By what we did in part (a), we can find a square root for A by taking a square root of J and conjugating. In particular for our J we need to find square roots of the Jordan blocks

$$J_1 = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} \qquad J_2 = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}.$$

The Jordan block J_1 can be written as a (commuting) product of a scalar matrix times a nilpotent matrix:

$$J_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}.$$

It's then easy to find (commuting) square roots of these individual pieces, namely

$$\sqrt{J_1} = \begin{bmatrix} \zeta_8 & 0 \\ 0 & \zeta_8 \end{bmatrix} \begin{bmatrix} 1 & -i/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \zeta_8 & \zeta_8^7/2 \\ 0 & \zeta_8 \end{bmatrix}.$$

where $\zeta_8 = \exp(\pi i/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ is the usual 8th root of unity (which satisfies $\zeta_8^2 = i$, $\zeta_8^4 = -1$, $\zeta_8^6 = -i$, and $\zeta_8^8 = 1$). Similarly ζ_8^3 is a square root of $-i$ so we can find

$$\sqrt{J_2} = \begin{bmatrix} \zeta_8^3 & \zeta_8^5/2 \\ 0 & \zeta_8^3 \end{bmatrix}.$$

Putting these together, a square root for J is

$$\sqrt{J} = \begin{bmatrix} \zeta_8 & \zeta_8^7/2 & 0 & 0 \\ 0 & \zeta_8 & 0 & 0 \\ 0 & 0 & \zeta_8^3 & \zeta_8^5/2 \\ 0 & 0 & 0 & \zeta_8^3 \end{bmatrix}.$$

Finally, we get a square root of A as $\sqrt{A} = P\sqrt{J}P^{-1}$:

$$\sqrt{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{bmatrix} \cdot \begin{bmatrix} \zeta_8 & \zeta_8^7/2 & 0 & 0 \\ 0 & \zeta_8 & 0 & 0 \\ 0 & 0 & \zeta_8^3 & \zeta_8^5/2 \\ 0 & 0 & 0 & \zeta_8^3 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{bmatrix}.$$

Computing this out in terms of ζ_8, and then simplifying (using that $\zeta_8 + \zeta_8^3 = \frac{1}{2}(\sqrt{2}+\sqrt{2}i) + \frac{1}{2}(-\sqrt{2}+\sqrt{2}i) = \sqrt{2}i$ and similarly $\zeta_8^5 + \zeta_8^7 = -\sqrt{2}i$) we get:

$$\sqrt{A} = \frac{1}{2} \begin{bmatrix} \zeta_8 + \zeta_8^3 & \zeta_8^5 + \zeta_8^7 & \frac{\zeta_8^5+\zeta_8^7}{2} & \frac{\zeta_8^5+\zeta_8^7}{2} \\ \zeta_8 + \zeta_8^3 & \zeta_8 + \zeta_8^3 & \frac{\zeta_8+\zeta_8^3}{2} & \frac{\zeta_8^5+\zeta_8^7}{2} \\ 0 & 0 & \zeta_8 + \zeta_8^3 & \zeta_8^5 + \zeta_8^7 \\ 0 & 0 & \zeta_8 + \zeta_8^3 & \zeta_8 + \zeta_8^3 \end{bmatrix} = \begin{bmatrix} i\sqrt{2}/2 & -i\sqrt{2}/2 & -i\sqrt{2}/4 & -i\sqrt{2}/4 \\ i\sqrt{2}/2 & i\sqrt{2}/2 & i\sqrt{2}/4 & -i\sqrt{2}/4 \\ 0 & 0 & i\sqrt{2}/2 & -i\sqrt{2}/2 \\ 0 & 0 & i\sqrt{2}/2 & i\sqrt{2}/2 \end{bmatrix}.$$

Problem 2. (a) We start by computing the characteristic polynomial:

$$c_A(x) = \det(A - xI) = \det \begin{bmatrix} -x & 0 & 1 \\ 0 & 2-x & 0 \\ -4 & 0 & 4-x \end{bmatrix} = -x(2-x)(4-x) - (2-x)\cdot(-4).$$

Simplifying this we find

$$c_A(x) = (x-2)(-x^2+4x-4) = -(x-2)^3.$$

So 2 is the only eigenvalue. To find the Jordan canonical form, then, we need to look at powers of $A - 2I$:

$$A - 2I = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}.$$

It's straightforward to see that $\ker(A - 2I)$ has dimension 2; this means A has two Jordan blocks with eigenvalue 2. Since A is 3×3 the only possibility is that one Jordan block is 1×1 and the other is 2×2; thus a JCF for A is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) We have $(A - 2I)^2 = 0$ (which we can see either from squaring the matrix $A - 2I$ above, or reading off from the Jordan blocks). Thus the minimal polynomial divides $(x-2)^2$. Furthermore it can't be $x - 2$ itself since we computed $A - 2I \neq 0$; so the only possibility is that the minimal polynomial is $(x - 2)^2$.

(c) The minimal polynomial being $(x - 2)^2 = x^2 - 4x + 4$ tells us that we have

$$A^2 - 4A + 4I = 0.$$

Multiplying through by A^{-1} (which we know exists because all of the eigenvalues are nonzero!) and rearranging we get

$$A^{-1} = \frac{1}{4}(4I - A) = \frac{1}{4}\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -4 & 0 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 3. (a) We can actually compute the characteristic polynomial of A over $\mathbb{Z}$ and then reduce modulo p to get the characteristic polynomial for any $\mathbb{F}_p$. In particular

$$\det(A - xI) = \det \begin{bmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{bmatrix} = -x^3 + 3x + 2 = -(x^3 - 3x - 2).$$

Thus the characteristic polynomial of A over $\mathbb{F}_2$ is the reduction of this modulo 2, and we can see

$$-(x^3 - 3x - 2) \equiv -(x^3 - x) \equiv -x(x^2 - 1) \equiv -x(x - 1)^2 \pmod{2}.$$

So the characteristic polynomial is what we specified, meaning that 0 is an eigenvalue of multiplicity 1 and 1 is an eigenvalue of multiplicity 2. We then know that the Jordan canonical form will have a 1×1 Jordan block with eigenvalue 0. For eigenvalue 1, we need to look at the kernel of

$$A - I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix};$$

this has rank 1 so $\ker(A - I)$ has dimension 2. This means there are two linearly independent eigenvectors for $\lambda = 1$, and thus A is diagonalizable. So $A \in M_3(\mathbb{F}_2)$ is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_3(\mathbb{F}_2).$$

(b) As in part (a), we can get the characteristic polynomial over $\mathbb{F}_3$ by reducing the characteristic polynomial over $\mathbb{Z}$ modulo 3:

$$-(x^3 - 3x - 2) \equiv -(x^3 - 2) \equiv -(x^3 - 6x^2 + 12x - 8) = -(x - 2)^3.$$

So there is just one eigenvalue, $\lambda = 2$, with multiplicity 3. We thus need to look at the kernel of $A - 2I$:

$$B - I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

This matrix has 2-dimensional kernel again, so we can conclude that A will have two Jordan blocks. Since it's a 3×3 matrix this means we have to have one 2×2 Jordan block and one 1×1 one, so $B \in M_3(\mathbb{F}_3)$ is similar to

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{F}_3).$$

Problem 4. (a) If A satisfies $A^3 - I = 0$, then that means it satisfies the polynomial $x^3 - 1$, and thus the minimal polynomial divides $x^3 - 1 = (x-1)(x-\zeta)(x-\zeta^2)$. The minimal polynomial having distinct roots is enough to force A to be diagonal: if $J = J_p(\lambda)$ is a $p \times p$ Jordan block for $p > 1$ then $J - \lambda I$ is nonzero and $J - \lambda' I$ is invertible for any $\lambda' \neq \lambda$, so $(J-I)(J-\zeta I)(J-\zeta^2 I)$ will be nonzero (and this will be a diagonal block inside the product $(A-I)(A-\zeta I)(A-\zeta^2 I)$).

(b) By the previous part, every Jordan canonical form is diagonal, and thus A is similar to some diagonal matrix with diagonal entries $1, \zeta, \zeta^2$. There's 27 such matrices, but ones with the same lists in different order are similar to each other. Accounting for this our list is as follows, using the notation

$$\Delta(x,y,z) = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}:$$

- $\Delta(1,1,1)$, with minimal polynomial $x - 1$.
- $\Delta(\zeta,\zeta,\zeta)$, with minimal polynomial $x - \zeta$.
- $\Delta(\zeta^2,\zeta^2,\zeta^2)$, with minimal polynomial $x - \zeta^2$.
- $\Delta(1,1,\zeta)$, with minimal polynomial $(x-1)(x-\zeta)$.
- $\Delta(1,1,\zeta^2)$, with minimal polynomial $(x-1)(x-\zeta^2)$.
- $\Delta(\zeta,\zeta,1)$, with minimal polynomial $(x-1)(x-\zeta)$.
- $\Delta(\zeta,\zeta,\zeta^2)$, with minimal polynomial $(x-\zeta)(x-\zeta^2)$.
- $\Delta(\zeta^2,\zeta^2,1)$, with minimal polynomial $(x-1)(x-\zeta^2)$.
- $\Delta(\zeta^2,\zeta^2,\zeta)$, with minimal polynomial $(x-\zeta)(x-\zeta^2)$.
- $\Delta(1,\zeta,\zeta^2)$, with minimal polynomial $(x-1)(x-\zeta)(x-\zeta^2)$.

Problem 5. We know that A is similar to some Jordan canonical form, i.e. $A = PJP^{-1}$; transposing this equation we get $A^\top = (P^\top)^{-1} J^\top P^\top$, so $A^\top$ is similar to $J^\top$. Since similarity is an equivalence relation, if we can prove J and $J^\top$ are similar to each other then we're done.

To see this, we first note that any Jordan block $J_p(\lambda)$ is similar to its transpose $J_p(\lambda)^\top$; if $J_p(\lambda)$ represents the transformation in terms of a basis $\{b_1, b_2, \dots, b_p\}$ then $J_p'(\lambda)$ represents it in terms of the reversed basis $\{b_p, b_{p-1}, \dots, b_1\}$. Then if J is a Jordan canonical form, we can see that it is similar to $J^\top$ by a change-of-basis that does this individually for each Jordan block.